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EXISTENCE, STABILITY AND NUMERICAL ANALYSIS OF A FRACTIONAL NEUTRAL IMPLICIT DELAY DIFFERENTIAL SYSTEM WITH AN EXPONENTIAL KERNEL*

Velusamy Kavitha† Ravi Deepa‡ Balakrishnan Ganesh Priya§ Seenith Sivasundaram¶ Mani Mallika Arjunan‖

Abstract

This article addresses the existence, uniqueness and stability analysis for various classes of implicit fractional neutral delay (finite and infinite) differential systems (IFNDDSs) employing the Caputo-Fabrizio operator (CFO). The findings rely on the application of specific fixed-point theorems. Additionally, illustrative numerical examples are presented in the concluding section to clarify and demonstrate the derived results.

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†`kavi_velubagyam@yahoo.co.in`, Department of Mathematics, School of Sciences, Arts, Media & Management, Karunya Institute of Technology and Sciences, Karunya Nagar, Coimbatore-641114, Tamil Nadu, India

‡`deepa.maths1729@gmail.com`, Department of Mathematics, Panimalar Engineering College, Poonamallee, Chennai-600123, Tamil Nadu, India

§`primath85@gmail.com`, Department of Mathematics, Mepco Schlenk Engineering College, Sivakasi-626005, Tamil Nadu, India

¶`seenithi@gmail.com`, College of Engineering, Science and Mathematics, Daytona Beach, FL 32114, USA

‖`Corresponding author: arjunphd07@yahoo.co.in`, Department of Mathematics, School of Arts, Science, Humanities and Education, SASTRA Deemed to be University, Thanjavur-613401, Tamil Nadu, India

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1 Introduction

Fractional calculus, an advanced mathematical framework, introduces the concept of derivatives and integrals with non-integer orders, providing a nuanced perspective on the behaviour of dynamic systems. This unconventional approach has garnered attention for its capacity to model complex phenomena in diverse scientific and engineering domains [32, 42, 44]. Unlike classical calculus, fractional calculus offers a more inclusive representation of dynamic behaviours by accounting for non-local and hereditary effects. Its applications extend to physics, biology, control theory, and signal processing, where it excels in capturing intricate features like memory and long-term dependencies [13, 39]. The smooth transition it facilitates between derivatives and integrals contributes to its versatility in solving differential equations, making fractional calculus a pivotal tool in understanding and addressing real-world intricacies. Recently, researchers have extensively explored the application of fixed point theory to establish results related to the existence, uniqueness, and controllability of solutions for various fractional initial value problems [7, 19, 21–23, 31, 37].

Implicit fractional differential equations (IFDEs) constitute a specialized class of mathematical expressions where the fractional derivatives are applied implicitly. In these equations, the unknown function and its derivatives are intertwined in a non-explicit manner, often involving fractional orders. This distinctive formulation adds complexity but captures intricate relationships in various systems [1, 5, 18]. The advantages of IFDEs include their enhanced capacity to model phenomena with non-local dependencies and complex dynamics. The implicit approach allows for a more accurate representation of systems with memory effects and long-term dependencies. These equations provide a versatile framework for describing processes exhibiting anomalous diffusion, viscoelastic behaviours, and other intricate dynamics that traditional differential equations may struggle to capture. IFDEs have proven effective in modelling real-world scenarios across disciplines such as physics, engineering, and biology, making them valuable in understanding and analyzing complex systems [25, 30, 34, 41].

A neutral differential system is a differential system that incorporates the state variables' current and delayed values in its evolution. This delayed influence introduces an additional layer of complexity, distinguishing it from ordinary and delay differential systems. Mathematically, a neutral differential system can be represented as follows. Consider a system of first-order neutral differential equations involving a vector of state variables $\omega(\Upsilon)$ and their delayed values:

$$\omega'(\Upsilon) = F(\omega(\Upsilon), \omega(\Upsilon - \theta), \omega'(\Upsilon), \omega'(\Upsilon - \theta)),$$

where $\omega(\Upsilon)$ represents the current state variables, $\omega(\Upsilon - \theta)$ denotes the delayed state variables, $\omega'(\Upsilon)$ is the current derivative concerning time, $\omega'(\Upsilon-\theta)$ is the delayed derivative, θ is the time delay and F is a vector-valued function defining the dynamics of the system.

The literature about ordinary neutral differential equations (NDEs) is vast. For a comprehensive overview, we direct the reader to Hale and Lunel [27], along with the accompanying references. Their emergence is notable in the context of partial neutral differential equations featuring finite delays, particularly in transmission line theory. In a significant contribution, Wu and Xia [43] demonstrated that a ring array comprising identical resistively coupled loss-less transmission lines gives rise to a system of neutral functional differential equations. This system involves discrete diffusive coupling and manifests diverse types of discrete waves, showcasing the intriguing and varied nature of such equations. Recently, the authors have found some exciting existence results on fractional NDEs [12, 14, 18, 29, 33, 37, 38, 40].

Fractional delay differential equations (FDDEs) represent differential equations incorporating fractional derivatives and delays. In these equations, the evolution of a system is influenced not only by the current state but also by past states with fractional delays. This mathematical framework provides a more nuanced description of dynamic processes exhibiting delays, offering advantages in modelling various phenomena. The fractional aspect allows for a more flexible and accurate portrayal of systems with delays, contributing to a comprehensive understanding of complex dynamics. Additionally, FDDEs provide a versatile tool for analyzing and solving problems that involve delayed interactions and response dynamics in real-world applications [12, 20, 28, 33, 37].

In 2015, Fabrizio and Caputo introduced a novel CF derivative incorporating an exponential function. This derivative has garnered significant recognition and has been widely applied across diverse disciplines, including bio-medicine, dynamic systems, mechanics, signal processing, electromagnetism, and fluid dynamics. The CF derivative is defined by an integral operator without a singular kernel, as outlined in [16, 35]. Its unique properties distinguish it from alternative derivatives, augmenting its efficacy in accurately modelling real-world problems. Extensive research efforts have been devoted to exploring fractional differential systems, resulting in a substantial body of specialized research papers that address various applications of these derivatives [1–4, 9, 10, 15–17, 30, 34, 35].

Recently, the authors have found some interesting existence and stability results on ICFFDS [1, 11, 22, 25, 30, 33, 34, 41]. In [34], Krim et al. analyzed the existence theory for the subsequent system

$$\begin{cases} {}_0^{C\mathcal{F}}D_\omega^\theta \omega(\Upsilon) = f\left(\Upsilon, \omega(\Upsilon), {}_0^{CF}D_\omega^\theta \omega(\Upsilon)\right), \quad \Upsilon \in \mathcal{J} = [0,T] \\ u(0) = u_0, \quad u_0 \in \mathbb{R}, \end{cases} \tag{1.1}$$

where ${}_0^{CF}D_\omega^\theta$ is the CFFD of order $0 < \theta \leq 1, f : \mathcal{J} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$.

Later, Eiman et al. [22] examined the existence and stability results for the subsequent system

$$\left({}^{C\mathcal{F}}D_0^\theta \omega\right)(\Upsilon) = f\left(\Upsilon, \omega(\Upsilon), \left({}^{CF}D_0^\theta \omega\right)(\Upsilon)\right), \, \Upsilon \in I := [0,T],$$

with the boundary conditions

$$c\omega(0) + d\omega(T) = e,$$

where ${}_0^{C\mathcal{F}}D_0^\theta$ is the CFFD of order $0 < \theta \leq 1, f : \mathcal{J} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$.

In [25], Gul et al. generalized the results of the system (1.1) for $1 < \rho \leq 2$ under appropriate conditions and FP theorems. In [36], Lazreg et al. investigated the existence results for the subsequent system

$$\begin{cases} \left({}^{C\mathcal{F}}\mathfrak{D}_{\Upsilon_p}^\rho \omega\right)(\Upsilon) = f(\Upsilon, \omega(\Upsilon)), \quad \Upsilon \in \mathfrak{J}_p, \quad p = 0, \ldots, \ell, \\ \omega\left(\Upsilon_p^+\right) = \omega\left(\Upsilon_p^-\right) + L_p\left(\omega\left(\Upsilon_p^-\right)\right), \quad p = 1, \ldots, \ell, \\ \omega(0) = \omega_0, \end{cases} \tag{1.2}$$

where $\mathfrak{J}_0 = [0, \Upsilon_1], \mathfrak{J}_p = (\Upsilon_p, \Upsilon_{p+1}], p = 1, \ldots, \ell, 0 = \Upsilon_0 < \Upsilon_1 < \cdots < \Upsilon_\ell < \Upsilon_{\ell+1} = T, \omega_0 \in \mathbb{R}, f : \mathfrak{J}_p \times \mathbb{R} \to \mathbb{R}$, $p = 0, \ldots, \ell, L_p : \mathbb{R} \to \mathbb{R}, p = 1, \ldots, \ell$ are given continuous functions, ${}^{C\mathcal{F}}\mathfrak{D}_{\Upsilon_p}^\rho$ is the CFFD of order $\rho \in (0,1)$.

In [41], Sitthiwirattham et al. extends the results of (1.2) to the implicit system with $\omega(0) = g(\omega)$, where g is a continuous function under suitable FP theorems. Very recently, in [33], Krim et al. established the existence results of the system (1.1) with different types of delays under suitable FP theorems.

In the present study, we shall investigate the existence and uniqueness of IFNDDSs of the system

$$ {}^{C\mathcal{F}}\mathfrak{D}_0^{\rho}\left[\omega(\Upsilon)-h(\Upsilon,\omega_\Upsilon)\right]=\digamma\left(\Upsilon,\omega_\Upsilon,{}^{C\mathcal{F}}\mathfrak{D}_0^{\rho}\left[\omega(\Upsilon)-h(\Upsilon,\omega_\Upsilon)\right]\right),\quad \Upsilon\in\mathfrak{I}, \tag{1.3} $$

$$ \omega(\Upsilon)=\xi(\Upsilon),\quad \Upsilon\in[-d,0],\ d>0, \tag{1.4} $$

where $\mathfrak{I}=[0,T], T>0, \xi\in\mathcal{C}, \digamma:\mathfrak{I}\times\mathcal{C}\times\mathbb{R}\to\mathbb{R}, h:\mathfrak{I}\times\mathcal{C}\to\mathbb{R}$ are continuous functions, ${}^{C\mathcal{F}}\mathfrak{D}_0^{\rho}$ is the CFFD of order $\rho\in(0,1)$, and $\mathcal{C}:=\mathcal{C}([-d,0],\mathbb{R})$ is the space of continuous functions on $[-d,0]$. Here, for any $\Upsilon\in\mathfrak{I}$, we define ω_Υ by

$$ \omega_\Upsilon(\theta)=\omega(\Upsilon+\theta),\quad \text{for}\quad \theta\in[-d,0]. $$

In addition, we establish the stability results for the system (1.3)-(1.4) under Ulam-Hyers (U-H) and generalized U-H (G-U-H).

Furthermore, we also investigate the existence results with infinite delay for the system (1.3)-(1.4) . In particular, we examine the subsequent system

$$ {}^{C\mathcal{F}}\mathfrak{D}_0^{\rho}\left[\omega(\Upsilon)-h(\Upsilon,\omega_\Upsilon)\right]=\digamma\left(\Upsilon,\omega_\Upsilon,{}^{C\mathcal{F}}\mathfrak{D}_0^{\rho}\left[\omega(\Upsilon)-h(\Upsilon,\omega_\Upsilon)\right]\right),\quad \Upsilon\in\mathfrak{I}, \tag{1.5} $$

$$ \omega(\Upsilon)=\xi(\Upsilon),\quad \Upsilon\in(-\infty,0], \tag{1.6} $$

where $\xi:(-\infty,0]\to\mathbb{R}$, $\digamma:\mathfrak{I}\times\mathscr{B}\times\mathbb{R}\to\mathbb{R}$, $h:\mathfrak{I}\times\mathscr{B}\to\mathbb{R}$ are continuous functions, and $\mathscr{B}$ is denoted as a phase space, and its specifications will be detailed subsequently. In this situation, for any $\Upsilon\in\mathfrak{I}$, we designate $\omega_\Upsilon\in\mathscr{B}$ in such a way that

$$ \omega_\Upsilon(\theta)=\omega(\Upsilon+\theta),\quad \text{for}\quad \theta\in(-\infty,0]. $$

The remainder of this paper is structured as follows: Section 2 provides comprehensive background information utilized in this study, including the definition of CFFD and various properties of generalized Banach spaces and fixed-point theory. In Sections 3 and 5, we establish the existence and uniqueness of solutions for systems (1.3)-(1.4) and (1.5)-(1.6) with finite and infinite delay, respectively. Stability results for the system (1.3)-(1.4) is also discussed in Section 4. In the concluding section, we include illustrative numerical examples to help clarify and explain the findings of this work.

2 Preliminaries

This section will begin by presenting fundamental concepts, terminology, and preliminary information.

Consider the interval $\mathfrak{I} = [0, T]$ within the real numbers, where $T > 0$. Let $C(\mathfrak{I}, \mathbb{R})$ denote the space of continuous functions $\omega : \mathfrak{I} \longrightarrow \mathbb{R}$. This space, $C(\mathfrak{I}, \mathbb{R})$, is a Banach space equipped with the supremum norm $\|\cdot\|$, described as:

$$\|\omega\|_\infty = \sup\{|\omega(\curlyvee)| : \curlyvee \in \mathfrak{I}\}.$$

The functions designated by the notation $L^1([0, T], \omega)$ that are integrable in the Bochner concept with reference to the Lebesgue measure and come furnished with the notation

$$\|\omega\|_{L^1} = \int_0^T |\omega(\curlyvee)| d\curlyvee$$

is referred to as $\omega : \mathfrak{I} \to \mathbb{R}$.

Definition 2.1. *The CFF integral of order* $0 < \rho < 1$ *for a function* $g \in L^1(\mathfrak{I})$ *is defined as:*

$$({}^{C\mathcal{F}}I_0^\rho g)(\curlyvee) = \frac{2(1-\rho)}{N(\rho)(2-\rho)} g(\curlyvee) + \frac{2\rho}{N(\rho)(2-\rho)} \int_0^\curlyvee g(\omega) d\omega, \quad \curlyvee \geq 0. \quad (2.1)$$

Remark 2.1. *(i) According to the authors [35], the integral of order one for a function g and its average is the fractional integral of CF type of a function of order* $\rho \in (0, 1)$*. We get an explicit formula for* $N(\rho)$ *by imposing*

$$\frac{2(1-\rho)}{N(\rho)(2-\rho)} + \frac{2\rho}{N(\rho)(2-\rho)} = 1.$$

Then

$$N(\rho) = \frac{2}{2-\rho}, \quad 0 \leq \rho \leq 1.$$

(ii) If we take $N(\rho) = \dfrac{2}{2-\rho}$*, then becomes*

$$({}^{CF}I_0^\rho g)(\curlyvee) = (1-\rho) g(\curlyvee) + \rho \int_0^\curlyvee g(\sigma) d\sigma, \quad \curlyvee \geq 0.$$

Definition 2.2. *[4] The CFFD of order* $0 < \rho < 1$ *for a function* $g \in C^1(\mathfrak{I})$ *is defined as:*

$$({}^{C\mathcal{F}}\mathfrak{D}_0^\rho g)(\curlyvee) = \frac{(2-\rho)N(\rho)}{2(1-\rho)} \int_0^\curlyvee e^{\left(-\frac{\rho}{1-\rho}(\curlyvee - \sigma)\right)} g'(\sigma) d\sigma, \ \curlyvee \in \mathfrak{I}, \quad (2.2)$$

where a constant $N(\rho)$ is depending on ρ. Note that ${}^{C\mathcal{F}}\mathfrak{D}_0^{\rho} g = 0$ iff g is a constant function. For $N(\rho) = \dfrac{2}{2-\rho}$, one has

$$({}^{C\mathcal{F}}\mathfrak{D}_0^{\rho} g)(\Upsilon) = \frac{1}{(1-\rho)} \int_0^{\Upsilon} e^{\left(-\frac{\rho}{1-\rho}(\Upsilon-\sigma)\right)} g'(\sigma) d\sigma, \ \Upsilon \in \mathfrak{J}. \tag{2.3}$$

Lemma 2.1. *[6, Theorem 2] Let $g \in L^1(\mathfrak{J})$. Then, a function $\omega \in C(\mathfrak{J})$ is a solution of the following system:*

$$\begin{aligned} ({}^{C\mathcal{F}}\mathfrak{D}_0^{\rho} \omega)(\Upsilon) &= g(\Upsilon), \ \Upsilon \in \mathfrak{J}, \\ \omega(0) &= \omega_0, \end{aligned} \tag{2.4}$$

if and only if ω satisfies the following integral equation:

$$\omega(\Upsilon) = \omega_0 + \frac{2(1-\rho)}{(2-\rho)N(\rho)}[g(\Upsilon) - g(0)] + \frac{2\rho}{(2-\rho)N(\rho)} \int_0^{\Upsilon} g(\sigma)\, d\sigma. \tag{2.5}$$

From this point onwards, for simplicity, we take

$$\mathcal{A}_\rho = \frac{2(1-\rho)}{(2-\rho)N(\rho)} \quad \text{and} \quad \mathcal{B}_\rho = \frac{2\rho}{(2-\rho)N(\rho)}.$$

Then (2.5) can be written as

$$\omega(\Upsilon) = \omega_0 + \mathcal{A}_\rho(g(\Upsilon) - g(0)) + \mathcal{B}_\rho \int_0^{\Upsilon} g(\sigma) d\sigma. \tag{2.6}$$

Lemma 2.2. *Suppose $h : \mathfrak{J} \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $F \in L^1(\mathfrak{J})$. Then, a function $\omega \in C(\mathfrak{J})$ is a solution of the following system:*

$$ {}^{C\mathcal{F}}\mathfrak{D}_0^{\rho}[\omega(\Upsilon) - h(\Upsilon, \omega(\Upsilon))] = F(\Upsilon, \omega(\Upsilon)), \quad \Upsilon \in \mathfrak{J}, \quad 0 < \rho < 1, \tag{2.7}$$

$$\omega(0) = \omega_0, \tag{2.8}$$

if and only if ω satisfies the following integral equation:

$$\begin{aligned} \omega(\Upsilon) = \omega_0 - h(0, \omega_0) &+ h(\Upsilon, \omega(\Upsilon)) + \mathcal{A}_\rho F(\Upsilon, \omega(\Upsilon)) \\ &+ \mathcal{B}_\rho \int_0^{\Upsilon} F(s, \omega(s))\, ds, \end{aligned} \tag{2.9}$$

provided that $F(0, \omega_0) = 0$.

Note 1: To establish the aforementioned lemma, it is essential to revisit the Laplace transform of the CFO, as detailed in [16]. We have

$$L\{{}^{C\mathcal{F}}\mathfrak{D}_0^{\rho}\omega(\Upsilon)\}(s) = \frac{(2-\rho)N(\rho)}{2(s+\rho(1-s))}[sL\{\omega(\Upsilon)\}(s) - \omega(0)],$$
$$L\{{}^{C\mathcal{F}}\mathfrak{D}_0^{\rho}h(\Upsilon,\omega(\Upsilon))\}(s) = \frac{(2-\rho)N(\rho)}{2(s+\rho(1-s))}[sL\{h(\Upsilon,\omega(\Upsilon))\}(s) - h(0,\omega(0))].$$

Proof. We apply the Laplace transform to both sides of (2.7):

$$\begin{aligned}
&L\{{}^{C\mathcal{F}}\mathfrak{D}_0^{\rho}[\omega(\Upsilon) - h(\Upsilon,\omega(\Upsilon))]\}(s) = L\{F(\Upsilon,\omega(\Upsilon))\}(s)\\
&\frac{(2-\rho)N(\rho)}{2(s+\rho(1-s))}[sL\{\omega(\Upsilon)\}(s) - \omega(0)] - \frac{(2-\rho)N(\rho)}{2(s+\rho(1-s))}[sL\{h(\Upsilon,\omega(\Upsilon))\}(s)\\
&- h(0,\omega(0))] = L\{f(\Upsilon,\omega(\Upsilon))\}(s)\\
&\implies L\{\omega(\Upsilon)\}(s) - L\{h(\Upsilon,\omega(\Upsilon))\}(s) = \frac{1}{s}\omega(0) - \frac{1}{s}h(0,\omega(0))\\
&\quad + \frac{2\rho}{s(2-\rho)N(\rho)}L\{F(\Upsilon,\omega(\Upsilon))\}(s) + \frac{2(1-\rho)}{(2-\rho)N(\rho)}L\{F(\Upsilon,\omega(\Upsilon))\}(s).
\end{aligned} \tag{2.10}$$

Applying inverse Laplace transform on both sides of (2.10), we get

$$\omega(\Upsilon) = \omega_0 - h(0,\omega_0) + h(\Upsilon,\omega(\Upsilon)) + \mathcal{A}_\rho F(\Upsilon,\omega(\Upsilon)) + \mathcal{B}_\rho \int_0^{\Upsilon} F(s,\omega(s))\,ds.$$

We will now show that the solution (2.9) satisfies the given system (2.7). To do this, we rewrite the solution (2.9) as follows.

$$\begin{aligned}
\omega(\Upsilon) - h(\Upsilon,\omega(\Upsilon)) &= \omega_0 - h(0,\omega_0) + \frac{2(1-\rho)}{(2-\rho)N(\rho)}F(\Upsilon,\omega(\Upsilon))\\
&\quad + \frac{2\rho}{(2-\rho)N(\rho)}\int_0^{\Upsilon} F(s,\omega(s))\,ds.
\end{aligned}$$

Then

$$\begin{aligned}
\omega'(\Upsilon) - h'(\Upsilon,\omega(\Upsilon)) &= \frac{2(1-\rho)}{(2-\rho)N(\rho)}F'(\Upsilon,\omega(\Upsilon))\\
&\quad + \frac{2\rho}{(2-\rho)N(\rho)}F(\Upsilon,\omega(\Upsilon)), \quad \text{if} \quad F(0,\omega_0) = 0.
\end{aligned}$$

Multiply by $\dfrac{(2-\rho)N(\rho)}{2(1-\rho)}$ and integrate from 0 to $\curlyvee$, we have

$$\begin{aligned}&\frac{(2-\rho)N(\rho)}{2(1-\rho)}\int_0^{\curlyvee}\omega'(s)ds-\frac{(2-\rho)N(\rho)}{2(1-\rho)}\int_0^{\curlyvee}h'(s,\omega(s))ds\\&=\int_0^{\curlyvee}F'(s,\omega(s))ds+\int_0^{\curlyvee}\frac{\rho}{1-\rho}F(s,\omega(s))ds.\end{aligned}$$

Multiply the integrand by $e^{-\frac{\rho(\curlyvee-s)}{1-\rho}}$ in the above equation, we have

$$\begin{aligned}&\frac{(2-\rho)N(\rho)}{2(1-\rho)}\int_0^{\curlyvee}\omega'(s)e^{-\frac{\rho(\curlyvee-s)}{1-\rho}}ds-\frac{(2-\rho)N(\rho)}{2(1-\rho)}\int_0^{\curlyvee}h'(s,\omega(s))e^{-\frac{\rho(\curlyvee-s)}{1-\rho}}ds\\&=\int_0^{\curlyvee}F'(s,\omega(s))e^{-\frac{\rho(\curlyvee-s)}{1-\rho}}ds+\int_0^{\curlyvee}\frac{\rho}{1-\rho}F(s,\omega(s))e^{-\frac{\rho(\curlyvee-s)}{1-\rho}}ds\\&=\int_0^{\curlyvee}\frac{d}{ds}\left[F(s,p(s))e^{-\frac{\rho(\curlyvee-s)}{1-\rho}}\right]ds.\end{aligned}$$

Considering the definition of the CFFD, if $F(0,\omega_0)=0$, the aforementioned equation becomes:

$$\begin{aligned}{}^{C\mathcal{F}}\mathfrak{D}_0^{\rho}[\omega(\curlyvee)-h(\curlyvee,\omega(\curlyvee))]&=F(\curlyvee,\omega(\curlyvee)),\quad \curlyvee\in\mathfrak{I},\quad 0<\rho<1\\ \omega(0)&=\omega_0.\end{aligned}$$

□

Remark 2.2. *(i) The above Lemma 2.2 is true only when* $F(0,\omega_0)=0$.

(ii) If $F(0,\omega_0)\neq 0$, *then*

$$\begin{aligned}\omega(\curlyvee)=\omega_0-h(0,\omega_0)+h(\curlyvee,\omega(\curlyvee))-\mathcal{A}_\rho F(0,\omega_0)+\mathcal{A}_\rho F(\curlyvee,\omega(\curlyvee))\\+\mathcal{B}_\rho\int_0^{\curlyvee}F(s,\omega(s))\,ds\end{aligned}$$

is the solution of the following system

$$\begin{aligned}{}^{C\mathcal{F}}\mathfrak{D}_0^{\rho}[\omega(\curlyvee)-h(\curlyvee,\omega(\curlyvee))]&=F(\curlyvee,\omega(\curlyvee))-F(0,\omega_0)e^{-\frac{\rho}{1-\rho}\curlyvee},\quad \curlyvee\in\mathfrak{I},\\ \omega(0)&=\omega_0,\end{aligned}$$

where $0<\rho<1$.

Before describing the solution to the given system (1.3)-(1.4), we first define the following Banach space:

$$\Omega = \left\{ \omega : (-d, T] \to \mathbb{R}, \omega|_{[-d,0]} \equiv \xi, \omega|_{\mathfrak{J}} \in C(\mathfrak{J}) \right\}.$$

Thus, we have

$$\|\omega\|_{\Omega} = \max\{\|\xi\|_{\mathcal{C}}, \|\omega\|_{\infty}\}.$$

We are now ready to present the solution for the system (1.3)-(1.4). Considering Lemma 2.2, the following statement holds:

Definition 2.3. *A solution to the system* (1.3)-(1.4) *is described as a function $\omega \in \Omega$ such that*

$$\omega(\Upsilon) = \begin{cases} \xi(\Upsilon), & \text{if} \quad \Upsilon \in [-d, 0], \\ \xi(0) + h(\Upsilon, \omega_{\Upsilon}) + \mathcal{A}_{\rho} W(\Upsilon) + \mathcal{B}_{\rho} \int_0^{\Upsilon} W(s) \mathrm{d}s, & \text{if} \quad \Upsilon \in \mathfrak{J}, \end{cases} \tag{2.11}$$

where $W(\Upsilon) \in C(\mathfrak{J})$ and $W(\Upsilon) = \mathcal{F}(\Upsilon, \omega_{\Upsilon}, W(\Upsilon))$ with $W(0) = 0$.

3 Existence results: finite delay

In this section, we systematically present and establish the results regarding the existence and uniqueness of solutions to the system (1.3)-(1.4). The methodology employed involves the application of the Banach contraction principle [3] and Schauder's theorem [24], complemented by the use of Krasnoselskii's fixed-point theorem [12, 22].

To apply the aforementioned fixed-point theorems, it is crucial to specify the following conditions:

(A1) The function $\mathcal{F} : \mathfrak{J} \times \mathcal{C} \times \mathbb{R} \to \mathbb{R}$ is continuous and there exist $\mathfrak{M}_{\mathcal{F}} > 0$ and $0 < \widetilde{\mathfrak{M}}_{\mathcal{F}} < 1$ in a way that

$$|\mathcal{F}(\Upsilon, \omega_1, \omega_2) - \mathcal{F}(\Upsilon, \overline{\omega}_1, \overline{\omega}_2)| \leq \mathfrak{M}_{\mathcal{F}} \|\omega_1 - \overline{\omega}_1\|_{\mathcal{C}} + \widetilde{\mathfrak{M}}_{\mathcal{F}} |\omega_2 - \overline{\omega}_2|$$

for each $\Upsilon \in \mathfrak{J}$, $\omega_1, \overline{\omega}_1 \in \mathcal{C}, \omega_2, \overline{\omega}_2 \in \mathbb{R}$.

(A2) The function $h : \mathfrak{J} \times \mathcal{C} \to \mathbb{R}$ is continuous and there exists a constant $0 < \mathfrak{M}_h < 1$ in a way that

$$|h(\Upsilon, \omega_1) - h(\Upsilon, \omega_2)| \leq \mathfrak{M}_h \|\omega_1 - \omega_2\|_{\mathcal{C}}, \text{ for any } \omega_1, \omega_2 \in \mathcal{C}, \text{ for a.e., } \Upsilon \in \mathfrak{J}.$$

(A3) For any bounded set $\mathbb{E} \subset \Omega$, the set:

$$\left\{ \Upsilon \mapsto F\left(\Upsilon, \omega_\Upsilon, \left({}^{C\mathcal{F}}\mathfrak{D}_0^\rho \omega\right)(\Upsilon)\right) : \omega \in \mathbb{E} \right\}.$$

is equi-continuous in Ω.

Theorem 3.1. *If F and h are satisfy the conditions (A1)-(A2), and*

$$C_1 := \left(\mathfrak{M}_h + (\mathcal{A}_\rho + \mathcal{B}_\rho T) \frac{\mathfrak{M}_F}{1 - \widetilde{\mathfrak{M}}_F} \right) < 1, \tag{3.1}$$

then there exists a unique solution to system (1.3)-(1.4) *over the interval* $[-d, T]$.

Proof. Consider the operator $\Upsilon : \Omega \to \Omega$ by

$$(\Upsilon\omega)(\Upsilon) = \begin{cases} \xi(\Upsilon), & \text{if} \quad \Upsilon \in [-d, 0], \\ \xi(0) + h(\Upsilon, \omega_\Upsilon) + \mathcal{A}_\rho W(\Upsilon) + \mathcal{B}_\rho \displaystyle\int_0^\Upsilon W(s)\mathrm{d}s, & \text{if} \quad \Upsilon \in \mathfrak{I}, \end{cases} \tag{3.2}$$

where $W(\Upsilon) \in C(\mathfrak{I})$ and $W(\Upsilon) = F(\Upsilon, \omega_\Upsilon, W(\Upsilon))$ with $W(0) = 0$.

Now, we show that $\Upsilon B_Q \subset B_Q$. To do this, let $F(\cdot, 0, 0) = 0, \widetilde{\mathfrak{M}}_h = |h(\Upsilon, 0)|$ and let $B_Q = B(0, Q) = \{\omega \in \Omega(\mathfrak{I}, \mathbb{R}) : \|\omega\|_\Omega \leq Q\}$, be the ball centered at the origin with radius

$$Q > \max \left\{ \|\xi\|_{\mathcal{C}}, \frac{|\xi(0)| + \widetilde{\mathfrak{M}}_h}{1 - \left(\mathfrak{M}_h + (\mathcal{A}_\rho + \mathcal{B}_\rho T) \frac{\mathfrak{M}_F}{1 - \widetilde{\mathfrak{M}}_F} \right)} \right\}.$$

For each $\Upsilon \in [-d, 0]$ and $\omega \in B_Q$, then $|(\Upsilon\omega)(\Upsilon)| \leq \|\xi\|_{\mathcal{C}} \leq Q$. For any $\Upsilon \in \mathfrak{I}$, we have

$$\begin{aligned} |(\Upsilon\omega)(\Upsilon)| &= \left| \xi(0) + h(\Upsilon, \omega_\Upsilon) + \mathcal{A}_\rho W(\Upsilon) + \mathcal{B}_\rho \int_0^\Upsilon W(s)\mathrm{d}s \right| \\ &\leq |\xi(0)| + |h(\Upsilon, \omega_\Upsilon)| + \mathcal{A}_\rho |W(\Upsilon)| + \mathcal{B}_\rho \int_0^\Upsilon |W(s)|\mathrm{d}s. \end{aligned} \tag{3.3}$$

Since

$$\begin{aligned} |W(\Upsilon)| &= |F(\Upsilon, \omega_\Upsilon, W(\Upsilon))| \\ &\leq |F(\Upsilon, \omega_\Upsilon, W(\Upsilon)) - F(\Upsilon, 0, 0)| + |F(\Upsilon, 0, 0)| \\ &\leq \mathfrak{M}_F \|\omega_\Upsilon\|_{\mathcal{C}} + \widetilde{\mathfrak{M}}_F |W(\Upsilon)| \\ &\leq \mathfrak{M}_F \|\omega\|_\Omega + \widetilde{\mathfrak{M}}_F |W(\Upsilon)| \\ &\leq \frac{\mathfrak{M}_F Q}{1 - \widetilde{\mathfrak{M}}_F}, \end{aligned}$$

and

$$\begin{aligned}|h(\curlyvee,\omega_\curlyvee)| &\le |h(\curlyvee,\omega_\curlyvee)-h(\curlyvee,0)|+|h(\curlyvee,0)|\\ &\le \mathfrak{M}_h\|\omega_\curlyvee\|_{\mathcal{C}}+\widetilde{\mathfrak{M}}_h\\ &\le \mathfrak{M}_h\|\omega\|_\Omega+\widetilde{\mathfrak{M}}_h\\ &\le \mathfrak{M}_h Q+\widetilde{\mathfrak{M}}_h.\end{aligned}$$

Then (3.3) becomes

$$\begin{aligned}|(\Upsilon\omega)(\curlyvee)| &\le |\xi(0)|+\mathfrak{M}_h Q+\widetilde{\mathfrak{M}}_h+\mathcal{A}_\rho\frac{\mathfrak{M}_F Q}{1-\widetilde{\mathfrak{M}}_F}+\mathcal{B}_\rho T\frac{\mathfrak{M}_F Q}{1-\widetilde{\mathfrak{M}}_F}\\ &\le |\xi(0)|+\widetilde{\mathfrak{M}}_h+\left(\mathfrak{M}_h+(\mathcal{A}_\rho+\mathcal{B}_\rho T)\frac{\mathfrak{M}_F}{1-\widetilde{\mathfrak{M}}_F}\right)Q\\ &\le Q.\end{aligned}$$

Hence

$$\|\Upsilon(\omega)\|_\Omega \le Q.$$

This proves that Υ transforms the ball B_Q into itself. Next, for $\omega_1,\omega_2\in\Omega$ and for $\curlyvee\in[-d,0]$, we have

$$|(\Upsilon\omega_1)(\curlyvee)-(\Upsilon\omega_2)(\curlyvee)|=0,$$

and for $\curlyvee\in\mathfrak{I}$, we have

$$\begin{aligned}|(\Upsilon\omega_1)(\curlyvee)-(\Upsilon\omega_2)(\curlyvee)| &\le |h(\curlyvee,\omega_{1_\curlyvee})-h(\curlyvee,\omega_{2_\curlyvee})|+\mathcal{A}_\rho|W_{\omega_1}(\curlyvee)-W_{\omega_2}(\curlyvee)|\\ &\quad+\mathcal{B}_\rho\int_0^\curlyvee|W_{\omega_1}(s)-W_{\omega_2}(s)|\mathrm{d}s,\end{aligned}\tag{3.4}$$

where $W_{\omega_1}(\curlyvee),W_{\omega_2}(\curlyvee)\in C(\mathfrak{I},\mathbb{R})$ such that $W_{\omega_1}(\curlyvee)=F(\curlyvee,\omega_{1_\curlyvee},W_{\omega_1}(\curlyvee))$ and $W_{\omega_2}(\curlyvee)=F(\curlyvee,\omega_{2_\curlyvee},W_{\omega_2}(\curlyvee))$. From (A1)-(A2), we have

$$\begin{aligned}|W_{\omega_1}(\curlyvee)-W_{\omega_2}(\curlyvee)| &\le \mathfrak{M}_F\|\omega_{1_\curlyvee}-\omega_{2_\curlyvee}\|_{\mathcal{C}}+\widetilde{\mathfrak{M}}_F|W_{\omega_1}(\curlyvee)-W_{\omega_2}(\curlyvee)|\\ &\le \frac{\mathfrak{M}_F}{1-\widetilde{\mathfrak{M}}_F}\|\omega_1-\omega_2\|_\Omega,\end{aligned}\tag{3.5}$$

and

$$\begin{aligned}|h(\curlyvee,\omega_{1_\curlyvee})-h(\curlyvee,\omega_{2_\curlyvee})| &\le \mathfrak{M}_h\|\omega_{1_\curlyvee}-\omega_{2_\curlyvee}\|_{\mathcal{C}}\\ &\le \mathfrak{M}_h\|\omega_1-\omega_2\|_\Omega.\end{aligned}\tag{3.6}$$

Then (3.4) becomes

$$
\begin{aligned}
|(\Upsilon\omega_1)(\mathfrak{r}) - (\Upsilon\omega_2)(\mathfrak{r})| &\leq \mathfrak{M}_h\|\omega_1 - \omega_2\|_\Omega + \mathcal{A}_\rho \frac{\mathfrak{M}_F}{1 - \widetilde{\mathfrak{M}}_F}\|\omega_1 - \omega_2\|_\Omega \\
&\quad + \mathcal{B}_\rho T \frac{\mathfrak{M}_F}{1 - \widetilde{\mathfrak{M}}_F}\|\omega_1 - \omega_2\|_\Omega \\
&\leq \left(\mathfrak{M}_h + (\mathcal{A}_\rho + \mathcal{B}_\rho T)\frac{\mathfrak{M}_F}{1 - \widetilde{\mathfrak{M}}_F}\right)\|\omega_1 - \omega_2\|_\Omega \\
&\leq C_1\|\omega_1 - \omega_2\|_\Omega. \qquad (3.7)
\end{aligned}
$$

As a result, we have

$$\|\Upsilon(\omega_1) - \Upsilon(\omega_2)\|_\Omega \leq C_1\|\omega_1 - \omega_2\|_\Omega.$$

In light of the expression (3.1) and within the framework of Banach's fixed-point theorem [3], it is evident that the operator Υ has a unique fixed point. □

Remark 3.1. *(i) In the case where* $h = 0$ *in* F *from* (1.3)*, and accounting for the transition from Caputo-Fabrizio to Caputo fractional derivatives, the system* (1.3)-(1.4) *with impulsive conditions, as analyzed in [14] under the framework of Theorem 3.3, represents a specific instance that illustrates the principles outlined in Theorem 3.1.*

(ii) When $h = 0$ *in* (1.3)*, the system* (1.3)-(1.4) *explored in [33], within the context of Theorem 3.2 concerning Caputo-Fabrizio derivatives, serves as a particular case encompassed by Theorem 3.1.*

Next, utilizing Schauder's FP theorem [24], we can establish the existence of solutions for the system (1.3)-(1.4).

Theorem 3.2. *Suppose that the conditions (A1)-(A3) hold with*

$$C_1 := \left(\mathfrak{M}_h + (\mathcal{A}_\rho + \mathcal{B}_\rho T)\frac{\mathfrak{M}_F}{1 - \widetilde{\mathfrak{M}}_F}\right) < 1. \qquad (3.8)$$

As a result, the structure (1.3)-(1.4) *has at least one solution over the interval* $[-d, T]$*.*

Proof. Let us define the operator Υ as in (3.2). With reference to Theorem 3.1, we divide the proof of this theorem into three steps.

Step 1: Υ is continuous

Let $\{\omega_n\}_n$ be a sequence in such a way that $\omega_n \to \omega$ on B_Q, where B_Q is same as defined in Theorem 3.1. For any $\curlyvee \in [-d, 0]$, then we sustain

$$|(\Upsilon\omega_n)(\curlyvee) - (\Upsilon\omega)(\curlyvee)| = 0,$$

and for any $\curlyvee \in \mathfrak{I}$, one has

$$\begin{aligned}|(\Upsilon\omega_n)(\curlyvee) - (\Upsilon\omega)(\curlyvee)| \leq |h(\curlyvee, \omega_{n\curlyvee}) - h(\curlyvee, \omega_\curlyvee)| + \mathcal{A}_\rho |W_n(\curlyvee) - W(\curlyvee)| \\ + \mathcal{B}_\rho \int_0^\curlyvee |W_n(s) - W(s)| \mathrm{d}s,\end{aligned}$$

where $W_n(\curlyvee), W(\curlyvee) \in C(\mathfrak{I})$ in a way that $W_n(\curlyvee) = F(\curlyvee, \omega_{n\curlyvee}, W_n(\curlyvee))$ and $W(\curlyvee) = F(\curlyvee, \omega_\curlyvee, W(\curlyvee))$.

Due to $\|\omega_n - \omega\|_\Omega \to 0$ as $n \to \infty$ and the functions F, h, W and W_n are continuous, by utilizing Lebesgue dominated convergence theorem, it suggests

$$\|\Upsilon(\omega_n) - \Upsilon(\omega)\|_\Omega \to 0 \quad \text{as} \quad n \to \infty.$$

Consequently, the operator Υ is continuous.

Step 2: $\Upsilon(B_Q) \subset B_Q$.

The proof of this step is already given in the first part of Theorem 3.1, so we omit here.

Step 3: $\Upsilon(B_Q)$ is equi-continuous.

Now, for any $\upsilon_1, \upsilon_2 \in \mathfrak{I}$ with $\upsilon_1 < \upsilon_2$ and let $\omega \in B_Q$. Then, we get

$$\begin{aligned}|\Upsilon(\omega)(\upsilon_2) - \Upsilon(\omega)(\upsilon_1)| &\leq |h(\upsilon_2, \omega_{\upsilon_2}) - h(\upsilon_1, \omega_{\upsilon_1})| \\ &\quad + \left|\mathcal{B}_\rho \int_0^{\upsilon_2} W(s)ds - \mathcal{B}_\rho \int_0^{\upsilon_1} W(s)ds\right| \\ &\leq |h(\upsilon_2, \omega_{\upsilon_2}) - h(\upsilon_1, \omega_{\upsilon_1})| + \mathcal{B}_\rho \int_{\upsilon_1}^{\upsilon_2} |W(s)|ds \\ &\leq |h(\upsilon_2, \omega_{\upsilon_2}) - h(\upsilon_1, \omega_{\upsilon_1})| + \mathcal{B}_\rho \left(\frac{\mathfrak{M}_F Q}{1 - \widetilde{\mathfrak{M}}_F}\right)(\upsilon_2 - \upsilon_1).\end{aligned}$$

From the above, we see that if $\upsilon_2 \to \upsilon_1$, then the right-hand side of aforementioned equation goes to zero, so $\|(\Upsilon\omega)(\upsilon_2) - (\Upsilon\omega)(\upsilon_1)\| \to 0$ as $\upsilon_2 \to \upsilon_1$. Also $\Upsilon(B_Q) \subset B_Q$, therefore Υ is completely continuous, due to Arzela-Ascoli theorem. Consequently, from Schauder's FPT [24], Υ has at least one fixed point. □

Remark 3.2. *(i) In the case where $h = 0$ in $\digamma$ from* (1.3)*, and considering the transition from Caputo-Fabrizio to Caputo fractional derivatives, the system* (1.3)-(1.4) *with impulsive conditions, as investigated in [14] within the framework of Theorem 3.4, represents a particular instance exemplifying the principles laid out in Theorem 3.1.*

(ii) In the case where $h = 0$ in (1.3)*, the system* (1.3)-(1.4) *studied in [33] under the purview of Theorem 3.3, dealing with CF derivatives, constitutes a specific scenario covered by Theorem 3.1.*

Now, by utilizing Krasnoselskii's fixed-point theorem [22], we can conclusively establish the existence of solutions to the system (1.3)-(1.4).

Theorem 3.3. *Suppose that the conditions (A1)-(A3) hold with*

$$C_2 := \left(\mathfrak{M}_h + \mathcal{A}_\rho \frac{\mathfrak{M}_\digamma}{1 - \widetilde{\mathfrak{M}}_\digamma} \right) < 1. \tag{3.9}$$

As a result, the structure (1.3)-(1.4) *has at least one solution over the interval* $[-d, T]$.

Proof. Let us define the operators denoted by (3.2) as follows:

$$(\Upsilon_1 \omega)(\curlyvee) = \begin{cases} \xi(\curlyvee), & \text{if} \quad \curlyvee \in [-d, 0], \\ \xi(0) + h(\curlyvee, \omega_\curlyvee) + \mathcal{A}_\rho W(\curlyvee) & \text{if} \quad \curlyvee \in \mathfrak{I}, \end{cases}$$

and

$$(\Upsilon_2 \omega)(\curlyvee) = \mathcal{B}_\rho \int_0^\curlyvee W(s) \mathrm{d}s, \quad \text{if} \quad \curlyvee \in \mathfrak{I},$$

where $W(\curlyvee) = \digamma(\curlyvee, \omega_\curlyvee, W(\curlyvee))$ with $W(0) = 0$.

In view of Theorem 3.1, we easily prove that $\Upsilon_1(\omega) + \Upsilon_2(\omega) \in B_Q$. Subsequently, we establish the contraction property of Υ_1. Given the continuity of $\digamma$ and h, and considering $\omega_1, \omega_2 \in B_Q$ based on (3.5)-(3.7), for each $\curlyvee \in [-d, 0]$, we sustain

$$|(\Upsilon_1 \omega_1)(\curlyvee) - (\Upsilon_1 \omega_2)(\curlyvee)| = 0,$$

and for any $\curlyvee \in \mathfrak{I}$, we sustain

$$|(\Upsilon_1 \omega_1)(\curlyvee) - (\Upsilon_1 \omega_2)(\curlyvee)| \leq |h(\curlyvee, \omega_{1\curlyvee}) - h(\curlyvee, \omega_{2\curlyvee})| + \mathcal{A}_\rho |W_{\omega_1}(\curlyvee) - W_{\omega_2}(\curlyvee)|,$$

where $W_{\omega_1}(\curlyvee)$ and $W_{\omega_2}(\curlyvee)$ are same as defined in Theorem 3.1. Thus, we have

$$\|\Upsilon_1\omega_1 - \Upsilon_1\omega_2\|_\Omega \leq \left(\mathfrak{M}_h + \mathcal{A}_\rho \frac{\mathfrak{M}_F}{1-\widetilde{\mathfrak{M}}_F}\right) \|\omega_1 - \omega_2\|_\Omega.$$

By using (3.9), Υ_1 is a contraction. Next, we establish that Υ_2 is completely continuous. Since the function F is continuous, so the operator Υ_2 is also continuous (refer step 1 of Theorem 3.1). Also Υ_2 is uniformly bounded on B_Q as

$$\begin{aligned} |(\Upsilon_2\omega)(\curlyvee)| &\leq \left|\mathcal{B}_\rho \int_0^\curlyvee W(s)ds\right| \\ &\leq \mathcal{B}_\rho T \left(\frac{\mathfrak{M}_F Q}{1-\widetilde{\mathfrak{M}}_F}\right) = A, \end{aligned}$$

which suggest that $\|\Upsilon_2\omega\| \leq A$, demonstrating the uniform boundedness of Υ_2. To establish the compactness of the operator Υ_2, we need to demonstrate that Υ_2 is equi-continuous. From Step 3 of Theorem 3.2, we notice that the operator Υ_2 is equi-continuous. Consequently, from KFPT [22], Υ has at least one fixed point. □

4 Stability analysis

In this section, we examine the stability aspects related to U-H and G-U-H for the specified model (1.3)-(1.4).

Definition 4.1. *The problem represented by equation* (1.3) *is deemed U-H stable if, for any* $\varepsilon > 0$*, the following inequality holds under the given conditions:*

$$\left|{}^{C\mathcal{F}}\mathfrak{D}_0^\rho\left[\omega(\curlyvee) - h(\curlyvee,\omega_\curlyvee)\right] - F\left(\curlyvee, \omega_\curlyvee, {}^{C\mathcal{F}}\mathfrak{D}_0^\rho\left[\omega(\curlyvee) - h(\curlyvee,\omega_\curlyvee)\right]\right)\right| \leq \varepsilon \quad \forall\ \curlyvee \in \mathfrak{I}.$$

In such a case, there exists a unique solution $\overline{\omega}(\curlyvee)$ *accompanied by a constant* K*, satisfying the condition:*

$$|\omega(\curlyvee) - \overline{\omega}(\curlyvee)| \leq K\varepsilon \quad \forall\ \curlyvee \in \mathfrak{I}.$$

Definition 4.2. *The problem represented by equation* (1.3)-(1.4) *is deemed G-U-H stable if there exists non-decreasing function* $\psi : (0,T) \to (0,\infty)$ *such that*

$$|\omega(\curlyvee) - \overline{\omega}(\curlyvee)| \leq K\psi(\varepsilon), \quad \forall \curlyvee \in \mathfrak{I}$$

with $\psi(0) = 0, \psi(T) = 0.$

Remark 4.1. *Assuming the presence of a function* $\Phi(\curlyvee)$*, dependent on* $\omega \in \Omega$*, where* $\Phi(0) = 0$ *and* $\Phi(T) = 0$*,*

(1) $|\Phi(\curlyvee)| \leq \epsilon, \ \forall \, \curlyvee \in \mathfrak{J}$*,*

(2) ${}^{C\mathcal{F}}\mathfrak{D}_0^\rho \left[\omega(\curlyvee) - h(\curlyvee, \omega_\curlyvee)\right] = \digamma\left(\curlyvee, \omega_\curlyvee, {}^{C\mathcal{F}}\mathfrak{D}_0^\rho \left[\omega(\curlyvee) - h(\curlyvee, \omega_\curlyvee)\right]\right) + \Phi(\curlyvee)$*,* $\forall \ \curlyvee \in \mathfrak{J}$*.*

Lemma 4.1. *The solution of the presented model*

$$ {}^{C\mathcal{F}}\mathfrak{D}_0^\rho \left[\omega(\curlyvee) - h(\curlyvee, \omega_\curlyvee)\right] = \digamma\left(\curlyvee, \omega_\curlyvee, {}^{C\mathcal{F}}\mathfrak{D}_0^\rho \left[\omega(\curlyvee) - h(\curlyvee, \omega_\curlyvee)\right]\right) + \Phi(\curlyvee), \quad \curlyvee \in \mathfrak{J}, \tag{4.1} $$

$$ \omega(\curlyvee) = \xi(\curlyvee), \quad \curlyvee \in [-d, 0], \, d > 0 \tag{4.2} $$

is

$$ \omega(\curlyvee) = \begin{cases} \xi(\curlyvee), \quad \text{if} \quad \curlyvee \in [-d, 0], \\ \xi(0) + h(\curlyvee, \omega_\curlyvee) + \mathcal{A}_\rho W(\curlyvee) + \mathcal{B}_\rho \displaystyle\int_0^\curlyvee W(s) \mathrm{d}s + \mathcal{A}_\rho \Phi(\curlyvee) \\ +\mathcal{B}_\rho \displaystyle\int_0^\curlyvee \Phi(s) ds, \quad \text{if} \ \ \curlyvee \in \mathfrak{J}, \end{cases} \tag{4.3} $$

where $W(\curlyvee) \in C(\mathfrak{J})$ *and* $W(\curlyvee) = \digamma(\curlyvee, \omega_\curlyvee, W(\curlyvee))$ *with* $W(0) = 0$*.*

Further, for each $\curlyvee \in \mathfrak{J}$ *and* $\omega \in B_Q$*, we find*

$$ \left| \omega(\curlyvee) - \left[\xi(0) + h(\curlyvee, \omega_\curlyvee) + \mathcal{A}_\rho W(\curlyvee) + \mathcal{B}_\rho \int_0^\curlyvee W(s) \mathrm{d}s \right] \right| \leq (\mathcal{A}_\rho + \mathcal{B}_\rho T)\varepsilon. \tag{4.4} $$

Theorem 4.1. *According to Lemma 4.1, the solution to the model* (1.3)*-*(1.4) *is stable in both U-H and G-U-H senses if*

$$ K = \frac{(\mathcal{A}_\rho + \mathcal{B}_\rho T)}{1 - \left(\mathfrak{M}_h + (\mathcal{A}_\rho + \mathcal{B}_\rho T) \frac{\mathfrak{M}_\digamma}{1 - \widetilde{\mathfrak{M}}_\digamma} \right)} < 1. \tag{4.5} $$

Proof. Suppose $\omega(\curlyvee) \in C(\mathfrak{J})$ is any solution of (1.3)-(1.4) and $\overline{\omega}(\curlyvee) \in C(\mathfrak{J})$ is a unique solution of (1.3)-(1.4), then we need to consider for each $\curlyvee \in \mathfrak{J}$ and $\omega \in B_Q$,

$$ |\omega(\curlyvee) - \overline{\omega}(\curlyvee)| = \left| \omega(\curlyvee) - \left[\xi(0) + h(\curlyvee, \overline{\omega}_\curlyvee) + \mathcal{A}_\rho \overline{W}(\curlyvee) + \mathcal{B}_\rho \int_0^\curlyvee \overline{W}(s) \mathrm{d}s \right] \right|, $$

where $\overline{W}(\Upsilon) = \digamma(\Upsilon, \overline{\omega}_\Upsilon, \overline{W}(\Upsilon))$. Thus, we have

$$\begin{aligned}
|\omega(\Upsilon) - \overline{\omega}(\Upsilon)| = &\Bigg|\omega(\Upsilon) - \left[\xi(0) + h(\Upsilon, \omega_\Upsilon) + \mathcal{A}_\rho W(\Upsilon) + \mathcal{B}_\rho \int_0^\Upsilon W(s)\mathrm{d}s\right] \\
&+ \left[\xi(0) + h(\Upsilon, \omega_\Upsilon) + \mathcal{A}_\rho W(\Upsilon) + \mathcal{B}_\rho \int_0^\Upsilon W(s)\mathrm{d}s\right] \\
&- \left[\xi(0) + h(\Upsilon, \overline{\omega}_\Upsilon) + \mathcal{A}_\rho \overline{W}(\Upsilon) + \mathcal{B}_\rho \int_0^\Upsilon \overline{W}(s)\mathrm{d}s\right]\Bigg| \\
\leq &\left|\omega(\Upsilon) - \left[\xi(0) + h(\Upsilon, \omega_\Upsilon) + \mathcal{A}_\rho W(\Upsilon) + \mathcal{B}_\rho \int_0^\Upsilon W(s)\mathrm{d}s\right]\right| \\
&+ |h(\Upsilon, \omega_\Upsilon) - h(\Upsilon, \overline{\omega}_\Upsilon)| + \mathcal{A}_\rho |W(\Upsilon) - \overline{W}(\Upsilon)| \\
&+ \mathcal{B}_\rho \int_0^\Upsilon |W(s) - \overline{W}(s)| ds.
\end{aligned}$$

By utilizing (4.4) and assumptions (A1)-(A2), we sustain

$$|\omega(\Upsilon) - \overline{\omega}(\Upsilon)| \leq (\mathcal{A}_\rho + \mathcal{B}_\rho T)\varepsilon + \left(\mathfrak{M}_h + (\mathcal{A}_\rho + \mathcal{B}_\rho T)\frac{\mathfrak{M}_\digamma}{1 - \widetilde{\mathfrak{M}}_\digamma}\right)\|\omega - \overline{\omega}\|_{\mathcal{C}}.$$

Hence

$$\|\omega - \overline{\omega}\|_\Omega \leq (\mathcal{A}_\rho + \mathcal{B}_\rho T)\varepsilon + \left(\mathfrak{M}_h + (\mathcal{A}_\rho + \mathcal{B}_\rho T)\frac{\mathfrak{M}_\digamma}{1 - \widetilde{\mathfrak{M}}_\digamma}\right)\|\omega - \overline{\omega}\|_\Omega.$$

This implies that

$$\begin{aligned}
\|\omega - \overline{\omega}\|_\Omega &\leq \frac{(\mathcal{A}_\rho + \mathcal{B}_\rho T)\varepsilon}{1 - \left(\mathfrak{M}_h + (\mathcal{A}_\rho + \mathcal{B}_\rho T)\frac{\mathfrak{M}_\digamma}{1 - \widetilde{\mathfrak{M}}_\digamma}\right)} \\
&\leq K\varepsilon. \qquad (4.6)
\end{aligned}$$

Therefore, the solution is U-H stable. Furthermore, there exists a non-decreasing function $\psi \in C(\mathfrak{I})$, then from (4.6), we have

$$\|\omega - \overline{\omega}\|_{\mathcal{PC}} \leq K\psi(\varepsilon),$$

with $\psi(0) = \psi(T) = 0$. Thus, we conclude that the solution of (1.3)-(1.4) is G-U-H. □

Remark 4.2. *For the case where* $h = 0$ *in* $\digamma$ *of* (1.3), *and considering the transition from Caputo-Fabrizio to Caputo fractional derivatives, the system* (1.3)-(1.4) *with impulsive conditions, as explored in [14] under the framework of Theorem 4.2, stands out as a particular case exemplifying the principles laid out in Theorem 4.1.*

5 Existence results: infinite delay

To analyze a system with infinite delay, it is necessary to formulate the phase space axioms. Consider the space denoted as $(\mathscr{B}, \|\cdot\|_{\mathscr{B}})$, a semi-normed linear space comprising functions that map the interval $(-\infty, T]$ to the real numbers $\mathbb{R}$. These functions adhere to fundamental axioms derived and modified from the principles initially proposed by Hale and Kato [26].

(C1) If $\omega : (-\infty, T] \to \mathbb{R}$ and $\omega_0 = \xi(0) \in \mathscr{B}$, there exist positive constants $\Theta, \overline{\Theta}, \widetilde{\Theta}$ such that, for every $\curlyvee \in \mathfrak{I}$, the subsequent inequalities hold:

- (a) $\omega_{\curlyvee} \in \mathscr{B}$,
- (b) $\|\omega_{\curlyvee}\|_{\mathscr{B}} \leq \Theta \|\omega_0\|_{\mathscr{B}} + \overline{\Theta} \sup_{\theta \in [0,\curlyvee]} |\omega(\theta)|$,
- (c) $|\omega(t)| \leq \widetilde{\Theta} \|\omega_{\curlyvee}\|_{\mathscr{B}}$. Furthermore, we have $|\xi(0)| \leq \widetilde{\Theta} \|\xi\|_{\mathscr{B}}$.

(C2) In the context of condition $C1$, the function $\omega(\cdot)$, implies that $\omega_{\curlyvee}$ is a continuous function mapping to the space $\mathscr{B}$ over the interval $\mathfrak{I}$.

(C3) The completeness of the space s is assured.

Before describing the solution to the given system (1.5)-(1.6), we first define the following Banach space:

$$\Omega_1 = \left\{ \omega : (-\infty, T] \to \mathbb{R}, \omega|_{(-\infty,0]} \in \mathscr{B}, \omega|_{\mathfrak{I}} \in C(\mathfrak{I}) \right\}.$$

In consideration of Lemma 2.2, the ensuing statement stands:

Definition 5.1. *A solution to the system* (1.5)-(1.6) *is defined as a function* $\omega \in \Omega_1$ *such that*

$$\omega(\curlyvee) = \begin{cases} \xi(\curlyvee), & if \quad \curlyvee \in (-\infty, 0], \\ \xi(0) + h(\curlyvee, \omega_{\curlyvee}) + \mathcal{A}_\rho W(\curlyvee) + \mathcal{B}_\rho \int_0^{\curlyvee} W(s)\mathrm{d}s, & if \quad \curlyvee \in \mathfrak{I}, \end{cases} \tag{5.1}$$

where $W(\curlyvee) \in C(\mathfrak{I})$ *and* $W(\curlyvee) = \digamma(\curlyvee, \omega_{\curlyvee}, W(\curlyvee))$ *with* $W(0) = 0$.

To utilize the stated fixed-point theorems, it is crucial to specify the subsequent conditions:

(A4) The function $\digamma : \mathfrak{I} \times \mathscr{B} \times \mathbb{R} \to \mathbb{R}$ is continuous and there exist $\mathfrak{M}_{\digamma_1} > 0$ and $0 < \widetilde{\mathfrak{M}}_{\digamma_1} < 1$ in a way that

$$|\digamma(\curlyvee, \omega_1, \omega_2) - \digamma(\curlyvee, \overline{\omega}_1, \overline{\omega}_2)| \leq \mathfrak{M}_{\digamma_1} \|\omega_1 - \overline{\omega}_1\|_{\mathscr{B}} + \widetilde{\mathfrak{M}}_{\digamma_1} |\omega_2 - \overline{\omega}_2|$$

for each $\quad \curlyvee \in \mathfrak{I}$, $\omega_1, \overline{\omega}_1 \in \mathscr{B}, \omega_2, \overline{\omega}_2 \in \mathbb{R}$.

(A5) The function $h : \mathfrak{I} \times \mathscr{B} \to \mathbb{R}$ is continuous and there exists a constant $0 < \mathfrak{M}_{h_1} < 1$ in a way that

$$|h(\curlyvee, \omega_1) - h(\curlyvee, \omega_2)| \leq \mathfrak{M}_{h_1} \|\omega_1 - \omega_2\|_{\mathscr{B}}, \text{ for any } \omega_1, \omega_2 \in \mathscr{B}, \text{ for a.e., } \curlyvee \in \mathfrak{I}.$$

(A6) For any bounded set $\mathbb{E} \subset \Omega_1$, the set:

$$\left\{ \curlyvee \mapsto F\left(\curlyvee, \omega_{\curlyvee}, \left({}^{C\mathcal{F}}\mathfrak{D}_0^{\rho} \omega\right)(\curlyvee)\right) : \omega \in \mathbb{E} \right\}$$

is equi-continuous in Ω_1.

At this point, we are able to derive the outcomes of existence and uniqueness for the system represented by equations (1.5)-(1.6) through the application of Banach's FP theorem.

Theorem 5.1. *Let F and h satisfy conditions (A4)-(A5). If the following inequality holds:*

$$\widetilde{C}_1 := \overline{\Theta}\left(\mathfrak{M}_{h_1} + (\mathcal{A}_\rho + \mathcal{B}_\rho T) \frac{\mathfrak{M}_{F_1}}{1 - \widetilde{\mathfrak{M}}_{F_1}} \right) < 1, \tag{5.2}$$

then the system (1.5)-(1.6) *admits a unique solution on the interval* $(-\infty, T]$.

Proof. Consider the operator $\widetilde{\Upsilon} : \Omega_1 \to \Omega_1$ by

$$(\widetilde{\Upsilon}\omega)(\curlyvee) = \begin{cases} \xi(\curlyvee), & \text{if } \curlyvee \in (-\infty, 0], \\ \xi(0) + h(\curlyvee, \omega_{\curlyvee}) + \mathcal{A}_\rho W(\curlyvee) + \mathcal{B}_\rho \displaystyle\int_0^{\curlyvee} W(s) \mathrm{d}s, & \text{if } \curlyvee \in \mathfrak{I}, \end{cases} \tag{5.3}$$

where $W(\curlyvee) \in C(\mathfrak{I})$ and $W(\curlyvee) = F(\curlyvee, \omega_{\curlyvee}, W(\curlyvee))$ with $W(0) = 0$.

Consider the function $u(\cdot) : (-\infty, T] \to \mathbb{R}$ defined as follows:

$$u(\curlyvee) = \begin{cases} \xi(\curlyvee); & \text{for } \curlyvee \in (-\infty, 0], \\ \xi(0); & \text{for } \curlyvee \in \mathfrak{I}. \end{cases}$$

Then $u_0 = \xi$. For any $\upsilon \in C(\mathfrak{I})$ satisfying $\upsilon(0) = 0$, let $\overline{\upsilon}$ be the function described as follows:

$$\overline{\upsilon}(\curlyvee) = \begin{cases} 0; & \text{for } \curlyvee \in (-\infty, 0], \\ \upsilon(\curlyvee); & \text{for } \curlyvee \in \mathfrak{I}. \end{cases}$$

We assume that $\omega(\cdot)$ fulfills the integral equation:

$$\omega(\Upsilon) = \xi(0) + h(\Upsilon, \omega_\Upsilon) + \mathcal{A}_\rho W(\Upsilon) + \mathcal{B}_\rho \int_0^\Upsilon W(s)\mathrm{d}s.$$

We represent $\omega(\cdot)$ as $\omega(\Upsilon) = \overline{\upsilon}(\Upsilon) + u(\Upsilon)$ for $\Upsilon \in \mathfrak{I}$. Consequently, it follows that $\omega_\Upsilon = \overline{\upsilon}_\Upsilon + u_\Upsilon$ holds for all $\Upsilon \in \mathfrak{I}$. Furthermore, the function $\upsilon(\cdot)$ satisfies the following condition:

$$\upsilon(\Upsilon) = h(\Upsilon, \overline{\upsilon}_\Upsilon + u_\Upsilon) + \mathcal{A}_\rho W(\Upsilon) + \mathcal{B}_\rho \int_0^\Upsilon W(s)\mathrm{d}s,$$

where

$$W(\Upsilon) = \digamma\left(\Upsilon, \overline{\upsilon}_\Upsilon + u_\Upsilon, W(\Upsilon)\right) \text{ for } \Upsilon \in \mathfrak{I}.$$

Define $\Omega_0 = \{\upsilon \in C(\mathfrak{I}) : \upsilon_0 = 0 \in \mathscr{B}\}$. Let $\upsilon \in \Omega_0$, then

$$\|\upsilon\|_T = \|\upsilon_0\|_{\mathscr{B}} + \sup_{\Upsilon \in \mathfrak{I}} |\upsilon(\Upsilon)| = \sup_{\Upsilon \in \mathfrak{I}} |\upsilon(\Upsilon)|.$$

As a result, $(\Omega_0, \|\cdot\|_T)$ is a Banach space. Next, $\overline{\Upsilon} : \Omega_0 \to \Omega_0$ is defined by:

$$(\overline{\Upsilon}\upsilon)(\Upsilon) = \xi(0) + h(\Upsilon, \overline{\upsilon}_\Upsilon + u_\Upsilon) + \mathcal{A}_\rho W(\Upsilon) + \mathcal{B}_\rho \int_0^\Upsilon W(s)\mathrm{d}s. \tag{5.4}$$

The operator $\widetilde{\Upsilon}$ possesses a fixed point denoted as $\overline{\Upsilon}$. Now, it is time to demonstrate that $\overline{\Upsilon}$ also has a fixed point.

We will show that the mapping $\overline{\Upsilon} : \Omega_0 \to \Omega_0$ possesses contraction properties. Let υ and υ' be any elements of Ω_0. For each $\Upsilon \in \mathfrak{I}$, the following inequality holds:

$$\begin{aligned} |(\overline{\Upsilon}\upsilon)(\Upsilon) - (\overline{\Upsilon}\upsilon')(\Upsilon)| \le{}& |h(\Upsilon, \overline{\upsilon}_\Upsilon + u_\Upsilon) - h(\Upsilon, \overline{\upsilon}'_\Upsilon + u_\Upsilon)| + \mathcal{A}_\rho |W_\upsilon(\Upsilon) - W_{\upsilon'}(\Upsilon)| \\ &+ \mathcal{B}_\rho \int_0^\Upsilon |W_\upsilon(s) - W_{\upsilon'}(s)|\mathrm{d}s, \end{aligned} \tag{5.5}$$

$W_\upsilon(\Upsilon), W_{\upsilon'}(\Upsilon) \in C(\mathfrak{I}, \mathbb{R})$ such that $W_\upsilon(\Upsilon) = \digamma(\Upsilon, \overline{\upsilon}_\Upsilon + u_\Upsilon, W_\upsilon(\Upsilon))$ and $W_{\upsilon'}(\Upsilon) = \digamma\left(\Upsilon, \overline{\upsilon'}_\Upsilon + u_\Upsilon, W_{\upsilon'}(\Upsilon)\right)$. From (A4)-(A5), we have

$$\begin{aligned} |W_\upsilon(\Upsilon) - W_{\upsilon'}(\Upsilon)| &\le \mathfrak{M}_{\digamma_1} \|\overline{\upsilon}_\Upsilon - \overline{\upsilon'}_\Upsilon\|_{\mathscr{B}} + \widetilde{\mathfrak{M}}_{\digamma_1} |W_\upsilon(\Upsilon) - W_{\upsilon'}(\Upsilon)| \\ \implies |W_\upsilon(\Upsilon) - W_{\upsilon'}(\Upsilon)| &\le \frac{\mathfrak{M}_{\digamma_1}}{1 - \widetilde{\mathfrak{M}}_{\digamma_1}} \|\overline{\upsilon}_\Upsilon - \overline{\upsilon'}_\Upsilon\|_{\mathscr{B}}, \end{aligned}$$

where

$$\|\overline{\upsilon}_\Upsilon - \overline{\upsilon'}_\Upsilon\|_{\mathscr{B}} \le \Theta \|\overline{\upsilon}_0 - \overline{\upsilon'}_0\|_{\mathscr{B}} + \overline{\Theta} \sup_{\Upsilon \in [0,T]} |\overline{\upsilon}(\Upsilon) - \overline{\upsilon'}(\Upsilon)| = \overline{\Theta} \sup_{\Upsilon \in [0,T]} |\overline{\upsilon}(\Upsilon) - \overline{\upsilon'}(\Upsilon)|,$$

and

$$\begin{aligned}|h(\curlyvee, \overline{\upsilon}_{\curlyvee} + u_{\curlyvee}) - h(\curlyvee, \overline{\upsilon}'_{\curlyvee} + u_{\curlyvee})| &\leq \mathfrak{M}_{h_1} \|\overline{\upsilon}_{\curlyvee} - \overline{\upsilon'}_{\curlyvee}\|_{\mathscr{B}} \\ &\leq \mathfrak{M}_{h_1} \overline{\Theta} \sup_{\curlyvee \in [0,T]} |\overline{\upsilon}(\curlyvee) - \overline{\upsilon'}(\curlyvee)|.\end{aligned}$$

Then (5.5) becomes

$$|(\overline{\Upsilon}\upsilon)(\curlyvee) - (\overline{\Upsilon}\upsilon')(\curlyvee)| \leq \overline{\Theta} \left(\mathfrak{M}_{h_1} + (\mathcal{A}_\rho + \mathcal{B}_\rho T) \frac{\mathfrak{M}_{F_1}}{1 - \widetilde{\mathfrak{M}}_{F_1}} \right) \|\overline{\upsilon} - \overline{\upsilon'}\|_T.$$

Consequently, we have

$$\|(\overline{\Upsilon}\upsilon) - (\overline{\Upsilon}\upsilon')\|_T \leq \widetilde{C}_1 \|\overline{\upsilon} - \overline{\upsilon'}\|_T.$$

In light of the expression (5.2) and in the framework of Banach's FP Theorem [3], it becomes evident that the operator Υ possesses a unique fixed point. □

Remark 5.1. *In the case where* $h = 0$ *in* (1.3)*, the system* (1.5)-(1.6)*, as studied in [33] under Theorem 4.2 concerning CF derivatives, represents a special case encompassed by Theorem 5.1.*

Next, utilizing Schauder's FP theorem [24], we can establish the existence of solutions for the system (1.5)-(1.6).

Theorem 5.2. *Suppose that conditions (A4)-(A6) hold. Then the system* (1.5)-(1.6) *has at least one solution over the interval* $(-\infty, T]$.

Proof. Let us define operator $\overline{\Upsilon}$ as in (5.4). Furthermore, we can easily establish the continuity and equicontinuity of the operator $\overline{\Upsilon}$ by applying the principles outlined in Theorem 3.2. To complete the proof of this theorem, the following two steps must be established:

Step 1: Boundedness of the operator $\overline{\Upsilon}$.

Indeed, it is enough to show that there exists a positive constant Λ such that for each $\upsilon \in B_Q = \{\upsilon \in \Omega_0 : \|\upsilon\|_T \leq Q\}$, one has $\|\overline{\Upsilon}\upsilon\|_T \leq \Lambda$. Thus from (A4)-(A5), we have

$$|(\overline{\Upsilon}\upsilon)(\curlyvee)| \leq |\xi(0)| + |h(\curlyvee, \overline{\upsilon}_{\curlyvee} + u_{\curlyvee})| + \mathcal{A}_\rho |W(\curlyvee)| + \mathcal{B}_\rho \int_0^{\curlyvee} |W(s)| \mathrm{d}s, \quad (5.6)$$

where

$$\begin{aligned} |W(\curlyvee)| &\leq |F(\curlyvee, \overline{\upsilon}_\curlyvee + u_\curlyvee, W(\curlyvee)) - F(\curlyvee, 0, 0)| + |F(\curlyvee, 0, 0)| \\ &\leq \mathfrak{M}_{F_1} \|\overline{\upsilon}_\curlyvee + u_\curlyvee\|_{\mathscr{B}} + \widetilde{\mathfrak{M}}_{F_1} |W(\curlyvee)| \quad (\because F(\curlyvee, 0, 0) = 0) \\ \implies |W(\curlyvee)| &\leq \frac{\mathfrak{M}_{F_1}}{1 - \widetilde{\mathfrak{M}}_{F_1}} \|\overline{\upsilon}_\curlyvee + u_\curlyvee\|_{\mathscr{B}}, \end{aligned}$$

and

$$\begin{aligned} \|\overline{\upsilon}_\curlyvee + u_\curlyvee\|_{\mathscr{B}} &\leq \Theta \|\overline{\upsilon}_0\|_{\mathscr{B}} + \overline{\Theta} \sup_{\curlyvee \in [0,T]} |\overline{\upsilon}(\curlyvee)| + \Theta \|u_0\|_{\mathscr{B}} + \overline{\Theta} \sup_{\curlyvee \in [0,T]} |u(\curlyvee)| \\ &\leq \overline{\Theta} Q + \Theta \|\xi\|_{\mathscr{B}} + \overline{\Theta} |\xi(0)| \\ &\leq \overline{\Theta} Q + \left(\Theta + \overline{\Theta} \widetilde{\Theta} \right) \|\xi\|_{\mathscr{B}}. \end{aligned}$$

Furthermore

$$\begin{aligned} |h(\curlyvee, \overline{\upsilon}_\curlyvee + u_\curlyvee)| &\leq |h(\curlyvee, \overline{\upsilon}_\curlyvee + u_\curlyvee) - h(\curlyvee, 0)| + |h(\curlyvee, 0)| \\ &\leq \mathfrak{M}_{h_1} \|\overline{\upsilon}_\curlyvee + u_\curlyvee\|_{\mathscr{B}} + \widetilde{\mathfrak{M}}_{h_1}, \end{aligned}$$

where $\widetilde{\mathfrak{M}}_{h_1} = \sup_{\curlyvee \in \mathfrak{I}} |h(\curlyvee, 0)|$.

Thus (5.6) becomes

$$\begin{aligned} &|(\overline{\Upsilon} \upsilon)(\curlyvee)| \\ &\leq |\xi(0)| + |h(\curlyvee, \overline{\upsilon}_\curlyvee + u_\curlyvee)| + \mathcal{A}_\rho |W(\curlyvee)| + \mathcal{B}_\rho \int_0^\curlyvee |W(s)| \mathrm{d}s \\ &\leq \widetilde{\Theta} \|\xi\|_{\mathscr{B}} + \left(\mathfrak{M}_{h_1} + (\mathcal{A}_\rho + \mathcal{B}_\rho T) \frac{\mathfrak{M}_{F_1}}{1 - \widetilde{\mathfrak{M}}_{F_1}} \right) \|\overline{\upsilon}_\curlyvee + u_\curlyvee\|_{\mathscr{B}} + \widetilde{\mathfrak{M}}_{h_1} \\ &\leq \widetilde{\Theta} \|\xi\|_{\mathscr{B}} + \left(\mathfrak{M}_{h_1} + (\mathcal{A}_\rho + \mathcal{B}_\rho T) \frac{\mathfrak{M}_{F_1}}{1 - \widetilde{\mathfrak{M}}_{F_1}} \right) \left(\overline{\Theta} Q + \left(\Theta + \overline{\Theta} \widetilde{\Theta} \right) \|\xi\|_{\mathscr{B}} \right) \\ &\quad + \widetilde{\mathfrak{M}}_{h_1} \\ &:= \Lambda. \end{aligned}$$

Step 2: The set

$$\mathcal{S} = \{ \omega \in \Omega_0 : \upsilon = \nu \overline{\Upsilon}(\omega) \quad \text{for some} \quad \nu \in (0,1) \}$$

is bounded.

Let $\upsilon \in \Omega_0$ and for any $\curlyvee \in \mathfrak{I}$, we have

$$\upsilon(\curlyvee) = \nu(\overline{\Upsilon}\omega)(\curlyvee) = \xi(0) + h(\curlyvee, \overline{\upsilon}_\curlyvee + u_\curlyvee) + \mathcal{A}_\rho W(\curlyvee) + \mathcal{B}_\rho \int_0^\curlyvee W(s)\mathrm{d}s, \quad (5.7)$$

where

$$|W(\curlyvee)| \leq \frac{\mathfrak{M}_{F_1}\left(\overline{\Theta}\|\upsilon\|_T + \left(\Theta + \overline{\Theta}\widetilde{\Theta}\right)\|\xi\|_{\mathscr{B}}\right)}{1 - \widetilde{\mathfrak{M}}_{F_1}} := \mu.$$

Then (5.7) becomes

$$\begin{aligned}
|\upsilon(\curlyvee)| &\leq |\xi(0)| + |h(\curlyvee, \overline{\upsilon}_\curlyvee + u_\curlyvee)| + \mathcal{A}_\rho |W(\curlyvee)| + \mathcal{B}_\rho \int_0^\curlyvee |W(s)|\mathrm{d}s \\
&\leq \widetilde{\Theta}\|\xi\|_{\mathscr{B}} + \mathfrak{M}_{h_1}\mu' + (\mathcal{A}_\rho + \mathcal{B}_\rho T)\mu + \widetilde{\mathfrak{M}}_{h_1} \\
&\leq \overline{\mu},
\end{aligned}$$

where $\mu' = \left(\overline{\Theta}\|\upsilon\|_T + \left(\Theta + \overline{\Theta}\widetilde{\Theta}\right)\|\xi\|_{\mathscr{B}}\right)$. Thus the set $\mathcal{S}$ is bounded. Consequently, from Schauder's FPT [24], Υ has at least one fixed point for the system (1.5)-(1.6). □

Remark 5.2. *When* $h = 0$ *in* (1.3)*, the system* (1.5)-(1.6)*, as studied in [33] under Theorem 4.3 concerning CF derivatives, represents a specific case encompassed by Theorem 5.1.*

Remark 5.3. *Numerous researchers have explored the existence of solutions for models similar to equations* (1.3)-(1.6)*, involving conditions with or without neutral, impulses, delays, implicitness, and the Caputo-Fabrizio (CF) framework in Banach spaces [14, 29, 33, 34, 40]. For instance, Ren et al. [40] investigated the existence of solutions for fractional integro-differential equations with infinite delay under certain conditions. Similarly, [29] established results on the existence and controllability of fractional neutral integro-differential equations with state-dependent delay in Banach spaces. In another study, the authors in [14] examined the existence, uniqueness, and stability of implicit neutral fractional differential equations with impulsive and finite delay conditions in Banach spaces. More recently, Krim et al. [34] focused on the existence of solutions for a system under CF derivatives in Banach spaces, and further extended their work to various delay types under CF derivatives [33]. Despite these advances, the specific system described by equations* (1.3)-(1.6) *has not been thoroughly investigated in the existing literature, which serves as the primary motivation for this study. This work builds upon and generalizes the findings of earlier studies [14, 33, 34].*

6 Applications

Example 6.1. *Academic Example*

Consider the subsequent CF fractional system

$$\left({}^{C\mathcal{F}}\mathfrak{D}_0^{\frac{1}{4}}\right)\left[\omega(\Upsilon)-\left(\frac{\Upsilon^{\frac{1}{4}}}{2}+\frac{1}{9(1+\|\omega_\Upsilon\|)}\right)\right]=\frac{\Upsilon^{\frac{1}{4}}}{2}+\frac{1}{16(1+\|\omega_\Upsilon\|)} \tag{6.1}$$

$$+\frac{1}{36\left(1+\left|\left({}^{C\mathcal{F}}\mathfrak{D}_0^{\frac{1}{4}}\right)\left[\omega(\Upsilon)-\left(\frac{\Upsilon^{\frac{1}{4}}}{2}+\frac{1}{9(1+\|\omega_\Upsilon\|)}\right)\right]\right|\right)},\quad \Upsilon\in[0,2],$$

$$\omega(\Upsilon)=1+\Upsilon^2,\quad \Upsilon\in[-1,0].$$

Set $\rho=\frac{1}{4}, T=2, \mathcal{A}_\rho=\frac{3}{4}, \mathcal{B}_\rho=\frac{1}{4}, N\left(\frac{1}{4}\right)=\frac{8}{7}$, and

$$F\left(\Upsilon,\omega_\Upsilon,{}^{C\mathcal{F}}\mathfrak{D}_0^{\frac{1}{4}}\left[\omega(\Upsilon)-h(\Upsilon,\omega_\Upsilon)\right]\right)=\frac{\Upsilon^{\frac{1}{4}}}{2}+\frac{1}{16(1+\|\omega_\Upsilon\|)}$$
$$+\frac{1}{36\left(1+\left|\left({}^{C\mathcal{F}}\mathfrak{D}_0^{\frac{1}{4}}\right)\left[\omega(\Upsilon)-\left(\frac{\Upsilon^{\frac{1}{4}}}{2}+\frac{1}{9(1+\|\omega_\Upsilon\|)}\right)\right]\right|\right)}$$
$$h(\Upsilon,\omega_\Upsilon)=\frac{\Upsilon^{\frac{1}{4}}}{2}+\frac{1}{9(1+\|\omega_\Upsilon\|)}$$

are continuous for all $\Upsilon\in[0,2]$. Moreover, let $u,\overline{u}\in\mathcal{C}; v,\overline{v}\in\mathbb{R}$ and $\Upsilon\in[0,1]$. Then one has

$$F(\Upsilon,u,v)=\frac{\Upsilon^{\frac{1}{4}}}{2}+\frac{1}{16(1+\|u\|)}+\frac{1}{36(1+|v|)},$$
$$h(\Upsilon,u)=\frac{\Upsilon^{\frac{1}{4}}}{2}+\frac{1}{9(1+\|u\|)}.$$

Then, we have

$$|F(\Upsilon,u,v)-F(\Upsilon,\overline{u},\overline{v})|\leq\frac{1}{16}\|u-\overline{u}\|_{[-1,0]}+\frac{1}{36}|v-\overline{v}|.$$

Thus, assumption (A1) holds with $\mathfrak{M}_F=\frac{1}{16}$ and $\widetilde{\mathfrak{M}}_F=\frac{1}{36}$. We also have

$$|h(t,u)-h(t,\overline{u})|\leq\frac{1}{9}\|u-\overline{u}\|_{[-1,0]}.$$

Thus assumption (A2) holds with $\mathfrak{M}_h = \dfrac{1}{9}$.

Furthermore

$$\left[\mathfrak{M}_h + (\mathcal{A}_\rho + \mathcal{B}_\rho T)\frac{\mathfrak{M}_F}{1-\widetilde{\mathfrak{M}}_F}\right] = \frac{1}{9} + (1.25)\frac{\frac{1}{16}}{1-\frac{1}{36}} = 0.19 < 1.$$

Therefore, all the conditions stipulated in Theorem 3.1 are fulfilled. Hence, the system (6.1) has a unique solution on $[-1, 2]$.

Example 6.2. *Numerical Examples*

To illustrate the proposed theory, two numerical examples are presented in this section. To approximate the integration in (2.11), we employ the Haar wavelet method, as described in Remark 6.1. Here is a rephrased version of the remark with a more formal mathematical tone:

Remark 6.1. *Let $f(x)$ be a function defined over the interval $[a, b]$. The numerical integration of $f(x)$ using the Haar wavelet method is expressed as follows [8]:*

$$\int_a^b f(x)\,dx = \frac{b-a}{2M}\sum_{k=0}^{2M-1} f\left(a + (b-a)\frac{k+0.5}{2M}\right).$$

Example 6.3. *Consider the Caputo-Fabrizio fractional-order neutral delay shown below:*

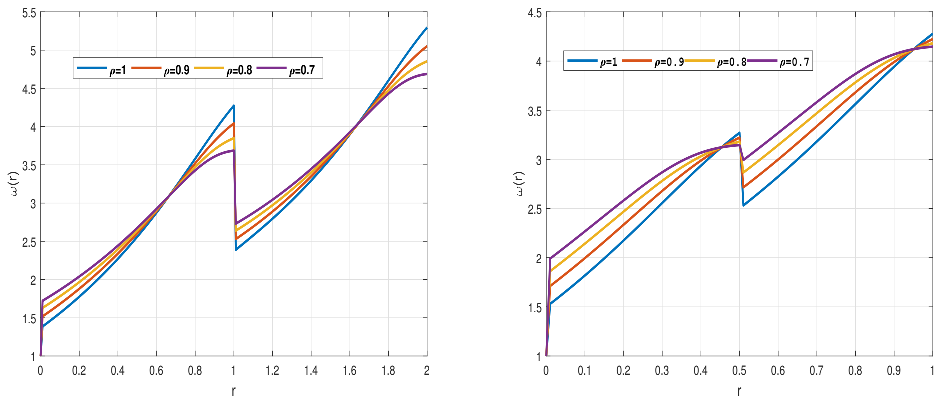
$$^{C\mathcal{F}}\mathfrak{D}_0^\rho\left[\omega(\curlyvee) - \omega(\curlyvee-\theta)\right] = \frac{2\omega(\curlyvee-\theta)}{1+[\omega(\curlyvee-\theta)]^{9.65}}, \quad \curlyvee \in [0, T]. \tag{6.2}$$

From the solution equation defined in (2.11), the solution trajectory for the system (6.2) is displayed in Figure (1).

Example 6.4. *Consider the following Caputo-Fabrizio fractional-order neutral delay system*

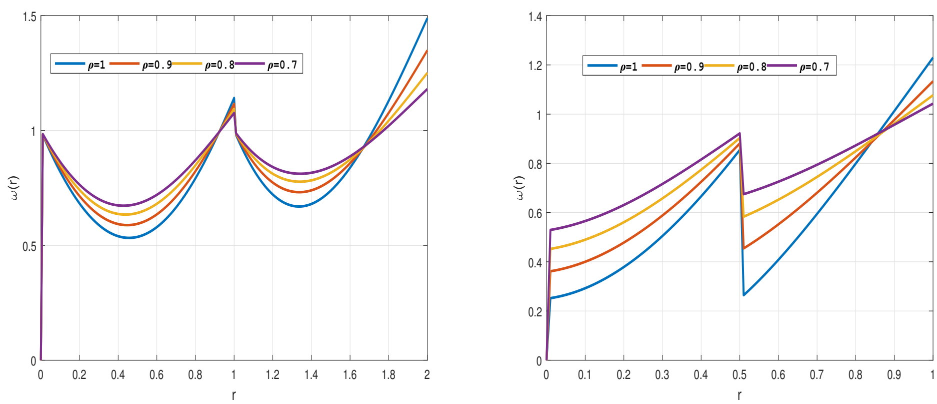
$$^{C\mathcal{F}}\mathfrak{D}_0^\rho\left[\omega(\curlyvee) - \omega(\curlyvee-\theta)\right] = \frac{1-\omega(\curlyvee-\theta)}{1+\omega(\curlyvee-\theta)}, \quad \curlyvee \in [0, T]. \tag{6.3}$$

Figure (2) represents the numerical solution of the system (6.3) by using the derived integral equation given in (2.11).



(a) (b)

Figure 1: The graph of the numerical solutions of Example (6.3) of various fractional order with $M = 10$: (a) $\theta = 1; \omega(\curlyvee) = e^{\curlyvee}, \curlyvee \in [-1, 0], T = 2$; (b) $\theta = 0.5, \omega(\curlyvee) = 1 + \curlyvee, \curlyvee \in [-0.5, 0], T = 1$.



(a) (b)

Figure 2: The graph of the numerical solutions of Example (6.4) of various fractional order with $M = 10$: (a) $\theta = 1; \omega(\curlyvee) = \curlyvee^2, \curlyvee \in [-1, 0], T = 2$; (b) $\theta = 0.5, \omega(\curlyvee) = \curlyvee^2, \curlyvee \in [-0.5, 0], T = 1$.

7 Conclusion

In recent years, Caputo and Fabrizio [16] introduced the CF operator, a novel fractional derivative approach characterized by an exponentially decaying kernel. This innovative concept has unlocked new avenues for research, particularly in exploring the qualitative and quantitative behaviors of various systems. Building on their pioneering work from [16], we applied this operator to our system (1.3)-(1.6). Using Banach's, Schaefer's, and Krasnoselskii's fixed-point theorems, we successfully established Theorems 3.1-3.3 and 5.1-5.2 for cases involving finite and infinite delays, respectively. Additionally, we demonstrated the stability of the system (1.3)-(1.4) in terms of U-H and G-U-H stability as outlined in Theorem 4.1. These results pave the way for future research, potentially applying similar fixed-point methods to demonstrate controllability with non-instantaneous impulses across a range of models.

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