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THE BEREZIN TRANSFORMATION ON $L^2(\mathbb{U}_+)$*

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Dedicated to Dr. Dan Tiba on the occasion of his 70th anniversary

Abstract

Let $L_a^2(\mathbb{U}_+)$ be the Bergman space of the upper half plane $\mathbb{U}_+$. In this paper, we consider the integral operator H from $L^2(\mathbb{U}_+)$ into $L^2(\mathbb{U}_+)$ defined by $(Hf)(w) = \widetilde{f}(w) = \int_{\mathbb{U}_+} f(s)|d_{\overline{w}}(s)|^2 d\widetilde{A}(s), w \in \mathbb{U}_+$, where $d_{\overline{w}}(s) = \frac{1}{\sqrt{\pi}}\frac{w+i}{\overline{w}-i}\frac{(-2i)\text{Im } w}{(s+w)^2}$ and $d\widetilde{A}$ is the area measure on $\mathbb{U}_+$. We refer the map H as the Berezin transformation defined on $L^2(\mathbb{U}_+)$. We have derived various algebraic properties of the operator and showed that $||H|| \leq \frac{3\pi}{4}$ considered as an operator on $L_a^2(\mathbb{U}_+)$.

Keywords: Bergman space, upper half plane, integral operators, Berezin transformation, reproducing kernel.

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1 Introduction

The Berezin transform was first introduced by F.A. Berezin [1] as a tool in quantization [2]. It has since found applications in many areas of mathematics and mathematical physics [3]. The Berezin transform was studied

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systematically [11], [13] for a number of reproducing kernel Hilbert spaces. It has become an indispensable tool in the study of operators in function spaces, including Toeplitz, Hankel and composition operators.

The Berezin transform is the analog of the Poisson transform in the Bergman space theory. On $\mathbb{D}$, the only measure left invariant by all Mobius transformations $z \to e^{i\theta}\frac{z-a}{1-\bar{a}z}, \theta \in \mathbb{R}, a \in \mathbb{D}$, is the pseudo-hyperbolic measure $d\eta(z) = \frac{dA(z)}{(1-|z|^2)^2}$. It turns out that the Berezin transform behaves well with respect to invariant measure. Berezin exhibited an explicit formula for the Berezin transform B on $L^2(\mathbb{D}, d\eta)$ in terms of the Laplace-Beltrami operator $\triangle_h := (1-|z|^2)^2\frac{\partial^2}{\partial z\partial\bar{z}}$ on $\mathbb{D}$.

In the setting of Fock spaces (or Segal-Bargmann spaces) and when parametrized appropriately, the Berezin transform is nothing but the heat transform [11]. This connection with the heat equation makes the Berezin transform on Fock space particularly useful. The Lebesgue measure dz on $\mathbb{C}^N$ is (up to multiplication by a constant factor) the only measure invariant with respect to the group of the rigid motion of $\mathbb{C}^N$. An explicit formula for Berezin transform has been established by Berger and Coburn [4] on $L^2(\mathbb{C}^N, dz)$. They showed that $Bf = \widetilde{f} = e^{\frac{\triangle}{2}}f$, where $\widetilde{f}$ is the solution of the heat equation with the initial condition f at the time $\frac{1}{2}, \triangle := \prod_{j=1}^{N} 4\frac{\partial^2}{\partial z_j\partial\bar{z}_j}$ on $\mathbb{C}^N$. There are several natural and successful applications of the Berezin transform in operator theory, but the Berezin transform can also be studied as an operator itself.

The Berezin transform is a contractive linear operator on certain L^p spaces provided they are taken with respect to an appropriate measure – the measure which is intrinsic for the Riemannian geometry of the domain namely, the space $L^p(\mathbb{C}^N, d\mu)$, where $d\mu(z) = \frac{1}{(2\pi)^N}e^{-\frac{|z|^2}{2}}dz$ and dz denotes the Lebesgue measure in $\mathbb{C}^N$, for all $N \geq 1$ or in $L^p(\mathbb{D}, dA)$, where $dA(z) = \frac{1}{\pi}dxdy$, respectively. It is known [6] that the Berezin transform is a contractive linear operator on the space $L^2(\mathbb{D}, d\eta)$ and in the space $L^2(\mathbb{C}, d\hat{A})$, where $d\hat{A}$ is the Euclidean area measure on [11] $\mathbb{C}, \|B\| = 1$. Further, Englis [6] has shown that the norm of the Berezin transform on the spaces $L^p(\mathbb{D}, dA), 1 < p < \infty$ is equal to $\frac{2p\sqrt{p}}{\sqrt{p^2-1}}$ and B is not a bounded operator on $L^1(\mathbb{D}, dA)$.

As an integral transform, one can certainly apply the Berezin transform B iterately many times to a 'reasonably good' function. In particular, $B^n f$ is well-defined for any $f \in L^\infty(\mathbb{D})$ and any positive integer n. Since for any $p, 1 < p < \infty$, the Berezin transform B is a bounded linear operator on

$L^p(\mathbb{D}, dA)$, then we can also consider $B^n f$ for $f \in L^p(\mathbb{D}, dA), p > 1$, and $n \geq 1$. A natural question one can ask at this point is the following: Is there anything we can say about $B^n f$ as $n \to \infty$? This describes various ergodicity properties of the Berezin transform [6]. In this paper, we shall investigate the boundeness of the Berezin transformation on the Bergman space $L_a^2(\mathbb{U}_+, d\tilde{A})$, where $\mathbb{U}_+$ is the upper half plane. Notice that in this case the domain is unbounded and does not have a rectifiable boundary. Moreover, the symmetry properties of the two domains $\mathbb{D}$ and $\mathbb{U}_+$ are different (although analogous).

Let $\mathbb{U}_+ = \{z = x + iy \in \mathbb{C} : \text{Im } z > 0\}$ be the upper half plane in $\mathbb{C}$, and let $d\widetilde{A} = dxdy$ be the area measure on $\mathbb{U}_+$. Let $L^2(\mathbb{U}_+, d\widetilde{A})$ denote the Hilbert space of complex valued, absolutely square integrable, Lebesgue measurable functions on $\mathbb{U}_+$ with the inner product

$$\langle f, g \rangle = \int_{\mathbb{U}_+} f(s)\overline{g(s)}d\widetilde{A}(s),$$

and

$$||f||_2 = \left(\int_{\mathbb{U}_+} |f(s)|^2 d\widetilde{A}(s) \right)^{\frac{1}{2}}.$$

Let $L_a^2(\mathbb{U}_+)$ be the closed subspace of $L^2(\mathbb{U}_+, d\widetilde{A})$ consisting of those functions in $L_a^2(\mathbb{U}_+)$ that are analytic. The space $L_a^2(\mathbb{U}_+)$ is called Bergman space of the upper half plane. The functions $K_w(z) = -\frac{1}{\pi(\overline{w}-z)^2}, w \in \mathbb{U}_+$, $z \in \mathbb{U}_+$ are the reproducing kernels [10] for $L_a^2(\mathbb{U}_+)$. The Bergman (orthogonal) projection P_+ from $L^2(\mathbb{U}_+, d\widetilde{A})$ onto $L_a^2(\mathbb{U}_+)$ is given by $(P_+ f)(w) = \langle f, K_w \rangle$. Let $L^\infty(\mathbb{U}_+)$ be the space of all complex valued, essentially bounded, Lebesgue measurable functions on $\mathbb{U}_+$. Define for $\varphi \in L^\infty(\mathbb{U}_+), ||\varphi||_\infty = ess \sup_{s \in \mathbb{U}_+} |\varphi(s)|$. The space $L^\infty(\mathbb{U}_+)$ is a Banach space with respect to the essential supremum norm. Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk and $dA(z)$ be the Lebesgue area measure on the open unit disk $\mathbb{D}$ normalized so that the measure of the disk $\mathbb{D}$ is 1. In rectangular and polar coordinates, we have $dA(z) = \frac{1}{\pi}dxdy = \frac{1}{\pi}rdrd\theta$. Let $L_a^2(\mathbb{D})$ be the space of all analytic functions that are in $L^2(\mathbb{D}, dA)$. The space $L_a^2(\mathbb{D})$ is called the Bergman space of the disk $\mathbb{D}$ and is a Hilbert space with respect to the inner product $\langle f, g \rangle = \int_{\mathbb{D}} f(z)\overline{g(z)}dA(z), f, g \in L_a^2(\mathbb{D})$. The sequence of functions $e_n(z) = \sqrt{n+1}\ z^n, n = 0, 1, 2, \cdots, z \in \mathbb{D}$ form an orthonormal basis for $L_a^2(\mathbb{D})$. The Bergman kernel or the reproducing kernel of $\mathbb{D}$ of $L^2_a(\mathbb{D})$ is given by $\mathcal{K}(z,w) = \frac{1}{(1-z\overline{w})^2}$ and the normalized reproducing kernels of $L^2_a(\mathbb{D})$ is given by $k_z(w) = \frac{1-|z|^2}{(1-\overline{z}w)^2}$. Let $L^\infty(\mathbb{D})$ be the space of all complex-valued, essentially bounded, Lebesgue measurable functions on $\mathbb{D}$. Let $\mathcal{L}(L^2_a(\mathbb{D}))$ be the space of all bounded linear operators from $L^2_a(\mathbb{D})$ into itself. For $f \in L^1(\mathbb{D}, dA)$, the Berezin transform of f is defined by $\widetilde{f}(w) = \langle fk_w, k_w\rangle = \int_{\mathbb{D}} \frac{(1-|w|^2)^2}{|1-\overline{w}z|^4} f(z)dA(z), w \in \mathbb{D}$. Notice that $k_w \in L^\infty(\mathbb{D})$ for all $w \in \mathbb{D}$, so the definition makes sense.

Define $M : \mathbb{U}_+ \to \mathbb{D}$ by $M(s) = \dfrac{i-s}{i+s} = z$. Then M is one-to-one, onto and $M^{-1} : \mathbb{D} \to \mathbb{U}_+$ is given by $M^{-1}(z) = i\dfrac{1-z}{1+z}$. Further $M^{'}(s) = \dfrac{-2i}{(i+s)^2}$ and $(M^{-1})^{'}(z) = \dfrac{-2i}{(1+z)^2}$. Let $W : L^2_a(\mathbb{D}) \to L^2_a(\mathbb{U}_+)$ be defined by $(Wg)(s) = g(Ms)\frac{(2i)}{\sqrt{\pi}(i+s)^2}$. The map W is one-to-one and onto. Hence W^{-1} exists and $W^{-1} : L^2_a(\mathbb{U}_+) \to L^2_a(\mathbb{D})$ is given by $(W^{-1}G)(z) = (2i)\sqrt{\pi}G(M^{-1}(z))\dfrac{1}{(1+z)^2}$.

The organization of the paper is as follows. In section 2, we introduce certain elementary functions $d_{\overline{w}}(s), D_w(s), D(s,w)$ which will be be used in defining the integral operator H. In section 3, we derive certain algebraic properties of the transformation H. In section 4, we establish that the operator H is not a bounded operator on $L^1(\mathbb{U}_+, d\widetilde{A})$. Further, we prove that the integral operator $\mathcal{D}$ given by $(\mathcal{D}f)(s) = \int_{\mathbb{U}_+} f(w)|d_{\overline{w}}(s)|^2 d\widetilde{A}(w)$, $s \in \mathbb{U}_+$ is a contraction on $L^1(\mathbb{U}_+, d\widetilde{A})$ which maps $L^\infty(\mathbb{U}_+)$ boundedly into $L^p(\mathbb{U}_+, d\widetilde{A})$ for $1 \leq p < \infty$. In section 5, we derive certain asymptotic properties of various related integral operators, using which we shall find the norm of H. In section 6, we establish that $||H|| = \dfrac{3\pi}{4}$ and the map L given by $Lf(w) = \dfrac{1}{4}\int_{\mathbb{U}_+} f(s)|d_{\overline{w}}(s)|^2 d\tilde{A}(s), w \in \mathbb{U}_+, f \in L^2(\mathbb{U}_+)$ is a strict contraction. Related maps are also considered in [12] and [7]. Here we consider $H \in \mathcal{L}(L^2(\mathbb{U}_+, d\widetilde{A}))$. Thus it follows from Theorem 6.1 that $||H|| \leq \frac{3\pi}{4}$ if it considered on the Bergman space $L^2_a(\mathbb{U}_+)$.

2 Preliminaries

In this section, we introduce certain elementary functions $d_{\overline{w}}(s), D_w(s), D(s,w)$ which will be used in defining the integral operator H.

For $a \in \mathbb{D}$, define the functions $\tau_a(s)$ from $\mathbb{U}_+$ onto $\mathbb{U}_+$ given by $\tau_a(s) = \frac{(c-1)+sd}{(1+c)s-d}$, where $a = c + id \in \mathbb{D}$ and $s \in \mathbb{U}_+$. It is not difficult to see that $\tau_a(s)$ are automorphisms of the upper half plane $\mathbb{U}_+$. Further $\tau_a^{'}(s) = \dfrac{1-|a|^2}{[(1+c)s-d]^2}$. Further for $s, w \in \mathbb{U}_+$, define the elementary functions $d_{\overline{w}}(s) = \dfrac{1}{\sqrt{\pi}}\dfrac{w+i}{\overline{w}-i}\dfrac{(-2i)\text{Im } w}{(s+w)^2}$. If $w = i\dfrac{1-\overline{a}}{1+\overline{a}} \in \mathbb{U}_+$, then $\overline{a} \in \mathbb{D}$ and $\overline{a} = \dfrac{i-w}{i+w} = Mw$. That is, $M^{-1}\overline{a} = w$. Define $D(s,w) = D_{\overline{w}}(s) = \frac{1}{\pi}\frac{(1+a)^2}{(i+s)^2}\frac{1}{(1-\overline{a}Ms)^2}$. Lemma 2.1 describes the relation between these elementary functions.

Lemma 2.1. *Let $s, w \in \mathbb{U}_+$. The following relations hold:*

(i) $(d_{\overline{w}}(-\overline{w}))^2 = D(\overline{w}, w)$.

(ii) $|d_{\overline{w}}(s)|\ ||D_{\overline{w}}|| = |D_{\overline{w}}(s)|$.

Proof. An easy calculation shows that

$$\begin{aligned}
d_{\overline{w}}(-\overline{w}) &= \frac{1}{\sqrt{\pi}}\frac{w+i}{\overline{w}-i}\frac{(-2i)(\text{Im } w)}{(-\overline{w}+w)^2} \\
&= \frac{(-2i)}{\sqrt{\pi}}\frac{M^{-1}\overline{a}+i}{\overline{M^{-1}\overline{a}}-i}\frac{\text{Im } w}{(w-\overline{w})^2} \\
&= \frac{(-2i)}{\sqrt{\pi}}\frac{i\frac{1-\overline{a}}{1+\overline{a}}+i}{\overline{\left(i\frac{1-\overline{a}}{1+\overline{a}}\right)}-i}\frac{w-\overline{w}}{(2i)(w-\overline{w})^2} \\
&= -\frac{1}{\sqrt{\pi}}\frac{i\left[\frac{1-\overline{a}}{1+\overline{a}}+1\right]}{[-i\frac{1-a}{1+a}-i]}\frac{1}{w-\overline{w}} \\
&= \frac{1}{\sqrt{\pi}}\frac{2}{1+\overline{a}}\frac{1+a}{2}\frac{1}{i\frac{1-\overline{a}}{1+\overline{a}}+i\frac{1-a}{1+a}} \\
&= \frac{1}{\sqrt{\pi}}\frac{1+a}{(1+\overline{a})}\frac{(1+\overline{a})(1+a)}{i[(1-\overline{a})(1+a)+(1-a)(1+\overline{a})]} \\
&= \frac{1}{i\sqrt{\pi}}\frac{(1+a)^2}{[1+a-\overline{a}-|a|^2+1+\overline{a}-a-|a|^2]} \\
&= \frac{1}{i\sqrt{\pi}}\frac{(1+a)^2}{2(1-|a|^2)}
\end{aligned}$$

$$= \frac{1}{(2i)\sqrt{\pi}} \frac{(1+a)^2}{(1-|a|^2)}.$$

Now

$$\begin{aligned}
d_{\overline{w}}(s)d_{\overline{w}}(-\overline{w}) &= \frac{(-2i)}{\sqrt{\pi}} \frac{w+i}{\overline{w}-i} \frac{\text{Im } w}{(s+w)^2} \frac{1}{(2i)\sqrt{\pi}} \frac{(1+a)^2}{1-|a|^2} \\
&= \frac{(-2i)}{\sqrt{\pi}} \left(\frac{i\frac{1-\overline{a}}{1+\overline{a}}+i}{-i\frac{1-a}{1+a}-i} \right) \frac{\left(\frac{w-\overline{w}}{2i}\right)}{(s+i\frac{1-\overline{a}}{1+\overline{a}})^2} \frac{1}{(2i)\sqrt{\pi}} \frac{(1+a)^2}{1-|a|^2} \\
&= \frac{(-2i)}{\sqrt{\pi}} \frac{\left(\frac{1-\overline{a}}{1+\overline{a}}+1\right)}{-\left(\frac{1-a}{1+a}+1\right)} \frac{\left[\left(i\frac{1-\overline{a}}{1+\overline{a}}\right)-\left(-i\frac{1-a}{1+a}\right)\right](1+\overline{a})^2}{(2i)[s(1+\overline{a})+i(1-\overline{a})]^2} \frac{1}{(2i)\sqrt{\pi}} \frac{(1+a)^2}{1-|a|^2} \\
&= \frac{1}{(2i)\pi} \left(\frac{\frac{1-\overline{a}+1+\overline{a}}{1+\overline{a}}}{\frac{1-a+1+a}{1+a}} \right) \frac{i\left[\frac{1-\overline{a}}{1+\overline{a}}+\frac{1-a}{1+a}\right]}{[s(1+\overline{a})+i(1-\overline{a})]^2} \frac{(1+a)^2}{1-|a|^2}(1+\overline{a})^2 \\
&= \frac{1}{2\pi} \frac{1+a}{1+\overline{a}} \frac{(1+a)^2}{(1-|a|^2)} \frac{2(1-|a|^2)}{(1+a)(1+\overline{a})} \frac{(1+\overline{a})^2}{[s(1+\overline{a})+i(1-\overline{a})]^2} \\
&= \frac{1}{\pi} \left(\frac{1+a}{1+\overline{a}}\right)^2 \frac{(1+\overline{a})^2}{[i+s+\overline{a}(s-i)]^2} \\
&= \frac{1}{\pi} \left(\frac{1+a}{1+\overline{a}}\right)^2 \frac{(1+\overline{a})^2}{[i+s-\overline{a}(i-s)]^2} \\
&= \frac{1}{\pi} \left(\frac{1+a}{1+\overline{a}}\right)^2 \frac{(1+\overline{a})^2}{(i+s)^2\left[1-\overline{a}\left(\frac{i-s}{i+s}\right)\right]^2} \\
&= \frac{1}{\pi} \frac{(1+a)^2}{(i+s)^2} \frac{1}{(1-\overline{a}Ms)^2} \\
&= D(s,w) \\
&= D_{\overline{w}}(s).
\end{aligned}$$

Hence

$$d_{\overline{w}}(s) = \frac{D(s,w)}{d_{\overline{w}}(-\overline{w})} \text{ and } (d_{\overline{w}}(-\overline{w}))^2 = D(\overline{w},w).$$

This proves (i). Now to prove (ii), notice that

$$\begin{aligned}
||D_{\overline{w}}||^2 &= \langle D_{\overline{w}}, D_{\overline{w}} \rangle \\
&= \int_{\mathbb{U}_+} |D_{\overline{w}}(s)|^2 d\widetilde{A}(s) \\
&= \int_{\mathbb{U}_+} |D(s,w)|^2 d\widetilde{A}(s)
\end{aligned}$$

$$\begin{aligned} &= \int_{\mathbb{U}_+} |d_{\overline{w}}(-\overline{w})|^2 |d_{\overline{w}}(s)|^2 d\widetilde{A}(s) \\ &= |d_{\overline{w}}(-\overline{w})|^2 \int_{\mathbb{U}_+} |d_{\overline{w}}(s)|^2 d\widetilde{A}(s) \\ &= |d_{\overline{w}}(-\overline{w})|^2 ||d_{\overline{w}}||_2^2 \\ &= |d_{\overline{w}}(-\overline{w})|^2 \text{ Since } ||d_{\overline{w}}||_2 = 1. \end{aligned}$$

Thus $||D_{\overline{w}}|| = |d_{\overline{w}}(-\overline{w})|$ and $|d_{\overline{w}}(s)|\ ||D_{\overline{w}}|| = |D_{\overline{w}}(s)|$. □

Lemma 2.2. *Let $l = 1 + c$, where $c \in \left(0, \frac{1}{2}\right)$ and $\theta_c(w) = \dfrac{(\mathrm{Im}\ w)^{\frac{c-1}{2}}}{(i+w)^l}$. Then $||\theta_c||_2 \sim \sqrt{\dfrac{\pi}{c}}$ as $c \to 0^+$.*

Proof. The proof is straightforward. □

3 The Berezin transformation

Let $w \in \mathbb{U}_+$ and $Mw = \overline{a}, a \in \mathbb{D}$. For $f \in L^1(\mathbb{U}_+, d\widetilde{A})$, define $(Hf)(w) = \widetilde{f}(w) = \int_{\mathbb{U}_+} f(s)|d_{\overline{w}}(s)|^2 d\widetilde{A}(s), w \in \mathbb{U}_+$, where $d_{\overline{w}}(s) = \dfrac{1}{\sqrt{\pi}} \dfrac{w+i}{\overline{w}-i} \dfrac{(-2i)\mathrm{Im}\ w}{(s+w)^2}$. Notice that $d_{\overline{w}} \in L^\infty(\mathbb{U}_+)$ for all $w \in \mathbb{U}_+$. Let $D(s,w) = D_{\overline{w}}(s) = \frac{1}{\pi} \frac{(1+a)^2}{(1-\overline{a}Ms)^2} \frac{1}{(i+s)^2}$ and $d\mu(w) = |D(\overline{w}, w)| d\widetilde{A}(w), w \in \mathbb{U}_+$. In the sequel we shall refer the map H as the Berezin map on $L_a^2(\mathbb{U}_+)$ for obvious reason. It is defined in the same way as the Berezin transformation defined on $L_a^2(\mathbb{D})$, the Bergman space of the disk $\mathbb{D}$ [13]. In this section we derive certain algebraic properties of the transformation H.

Theorem 3.1. *Let $f \in L^1(\mathbb{U}_+, d\widetilde{A})$. The following hold:*

(i) If f is bounded, then so is $Hf = \widetilde{f}$ and $||\widetilde{f}||_\infty \le ||f||_\infty$. In other words, H is a contraction in $L^\infty(\mathbb{U}_+)$.

(ii) The norm of H on $L^\infty(\mathbb{U}_+, d\widetilde{A})$, is equal to 1.

(iii) If $f \ge 0$, then $\widetilde{f} \ge 0$; if $f \ge g$, then $\widetilde{f} \ge \widetilde{g}$.

(iv) The mapping $H : f \to \widetilde{f}$ is a contractive linear operator on each of the spaces $L^p(\mathbb{U}_+, d\mu(z)), 1 \le p \le \infty$, where $d\mu(w) = |D(\overline{w}, w)| d\widetilde{A}(w)$.

(v) For arbitrary $f \in L^1(\mathbb{U}_+, d\widetilde{A}), \widetilde{f}(w) = \frac{1}{\pi} \int_{\mathbb{U}_+} (f \circ \tau_a \circ M)(s) d\widetilde{A}(s)$, where $a = M\overline{w}$.

Proof. (i) For proof of (i), assume $f \in L^\infty(\mathbb{U}_+)$. Then

$$|\widetilde{f}(w)| = \langle f d_{\overline{w}}, d_{\overline{w}} \rangle \leq ||f d_{\overline{w}}||_2 ||d_{\overline{w}}||_2 \leq ||f||_\infty ||d_{\overline{w}}||_2^2 = ||f||_\infty.$$

(ii) Since $f = \widetilde{f}$ when f is a constant function, hence the norm of H on $L^\infty(\mathbb{U}_+, d\widetilde{A})$ is equal to 1.

(iii) The operator H is an integral operator with positive kernel. Thus if $f \geq 0$, then $\widetilde{f} \geq 0$. If $f \geq g$, let $h = f - g$. Then $h \geq 0$ and therefore $\widetilde{h} \geq 0$. Hence $\widetilde{f} \geq \widetilde{g}$.

(iv) Since $L^1(\mathbb{U}_+, d\mu) \subset L^1(\mathbb{U}_+, d\widetilde{A})$, the operator H is defined on the former space, and

$$\big|\widetilde{f}(w)| = |\int_{\mathbb{U}_+} f(s)|d_{\overline{w}}(s)\big|^2 d\widetilde{A}(s)| \leq H(|f|)(s).$$

Hence since $D_{\overline{w}}(s) = \frac{-1}{2\pi i}\frac{(1+a)^2}{(1-\overline{a}Ms)^2}M'(s)$, therefore

$$\begin{aligned}
& \int_{\mathbb{U}_+} |\widetilde{f}(w)||D(\overline{w}, w)|d\widetilde{A}(w) \\
\leq \ & \int_{\mathbb{U}_+} \left(\int_{\mathbb{U}_+} |f(s)||d_{\overline{w}}(s)|^2 d\widetilde{A}(s) \right) |D(\overline{w}, w)|d\widetilde{A}(w) \\
= \ & \int_{\mathbb{U}_+} |f(s)| \int_{\mathbb{U}_+} |D_{\overline{w}}(s)|^2 d\widetilde{A}(w) d\widetilde{A}(s) \\
= \ & \int_{\mathbb{U}_+} |f(s)| \langle D_{\overline{w}}, D_{\overline{w}} \rangle d\widetilde{A}(s) \\
= \ & \int_{\mathbb{U}_+} |f(s)||D(\overline{s}, s)|d\widetilde{A}(s),
\end{aligned}$$

the change of order of integration being justified by the positivity of the integrand. It thus follows that H is a contraction on $L^1(\mathbb{U}_+, d\mu)$. The same is true for $L^\infty(\mathbb{U}_+)$, and so the result follows from the Marcinkiewicz interpolation theorem [8].

(v) $f \in L^1(\mathbb{U}_+, d\widetilde{A})$ and let $\overline{a} = Mw \in \mathbb{D}$. Then

$$\begin{aligned}
\widetilde{f}(w) &= \int_{\mathbb{U}_+} f(s)|d_{\overline{w}}(s)|^2 d\widetilde{A}(s) \\
&= \int_{\mathbb{U}_+} (f \circ \tau_a(s))|d_{\overline{w}}(\tau_a(s))|^2 |l_a(s)|^2 d\widetilde{A}(s) \\
&= \int_{\mathbb{U}_+} (f \circ \tau_a(s))|V_a d_{\overline{w}}(s)|^2 d\widetilde{A}(s)
\end{aligned}$$

$$\begin{aligned}
&= \int_{\mathbb{U}_+} (f\circ\tau_a(s))|\frac{(-1)}{\sqrt{\pi}}M^{'}(s)|^2 d\widetilde{A}(s)\\
&= \int_{\mathbb{D}} (f\circ\tau_a\circ M^{-1})(z)|\frac{(-1)}{\sqrt{\pi}}(M^{'}\circ M^{-1})(z)|^2|(M^{-1})^{'}(z)|^2 d\widetilde{A}(z)\\
&= \frac{1}{\pi}\int_{\mathbb{D}} (f\circ\tau_a\circ M^{-1})(z) d\widetilde{A}(z)\\
&= \int_{\mathbb{D}} (f\circ\tau_a\circ M^{-1})(z) dA(z).
\end{aligned}$$

□

4 Boundedness of the Berezin map

In this section we establish that the operator H is not a bounded operator on $L^1(\mathbb{U}_+, d\widetilde{A})$. Further, we prove that the integral operator $\mathcal{D}$ given by $(\mathcal{D}f)(s) = \int_{\mathbb{U}_+} f(w)|d_{\overline{w}}(s)|^2 d\widetilde{A}(w), s \in \mathbb{U}_+$ is a contraction on $L^1(\mathbb{U}_+, d\widetilde{A})$ which maps $L^\infty(\mathbb{U}_+)$ boundedly into $L^p(\mathbb{U}_+, d\widetilde{A})$ for $1 \le p < \infty$.

Proposition 4.1. *The operator H is not a bounded operator on $L^1(\mathbb{U}_+, d\widetilde{A})$.*

Proof. If it were, its adjoint $H^d \equiv \mathcal{D}$, where

$$(\mathcal{D}f)(s) = \int_{\mathbb{U}_+} f(w)|d_{\overline{w}}(s)|^2 d\widetilde{A}(w), s \in \mathbb{U}_+, \tag{1}$$

would be a bounded operator on $L^\infty(\mathbb{U}_+)$. Let $f \in L^\infty(\mathbb{U}_+)$. Now if $z = Ms$ and $\overline{a} = Mw$, then

$$\begin{aligned}
(\mathcal{D}f)(s) &= \int_{\mathbb{U}_+} f(w)|d_{\overline{w}}(s)|^2 d\widetilde{A}(w)\\
&= \int_{\mathbb{U}_+} f(w)|Wk_a(s)|^2 d\widetilde{A}(w)\\
&= \frac{1}{\pi}\int_{\mathbb{U}_+} f(w)|k_a(Ms)|^2|M^{'}(s)|^2 d\widetilde{A}(w)\\
&= \frac{|M^{'}(s)|^2}{\pi}\int_{\mathbb{D}} (f\circ M^{-1})(\overline{a})|k_a(z)|^2 d\widetilde{A}(M^{-1}\overline{a})\\
&= |M^{'}(s)|^2\int_{\mathbb{D}} (f\circ M^{-1})(a)|k_{\overline{a}}(z)|^2|(M^{-1})^{'}(a)|^2 dA(a).
\end{aligned}$$

Hence

$$\begin{aligned}(\mathcal{D}1)(s) &= \int_{\mathbb{U}_+} |d_{\overline{w}}(s)|^2 d\widetilde{A}(w) \\ &= |M^{'}(s)|^2 \int_{\mathbb{D}} |k_{\overline{a}}(z)|^2 |(M^{-1})^{'}(a)|^2 dA(a) \\ &\leq ||M^{'}||_\infty^4 \int_{\mathbb{D}} |k_{\overline{a}}(z)|^2 dA(a) \\ &= 2^4 \int_{\mathbb{D}} |k_{\overline{a}}(z)|^2 dA(a).\end{aligned}$$

Now

$$\begin{aligned}\int_{\mathbb{D}} |k_{\overline{a}}(z)|^2 dA(a) &= \int_{\mathbb{D}} \frac{(1-|a|^2)^2}{|1-az|^4} dA(a) \\ &= \int_0^1 (1-r^2)^2 \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|1-\overline{z}re^{it}|^4} dt \; 2r \; dr \\ &= \int_0^1 (1-r^2)^2 \sum_{n=0}^{\infty} (n+1)^2 r^{2n} |z|^{2n} \; 2r \; dr.\end{aligned}$$

Since

$$\begin{aligned}\int_0^{2\pi} \frac{1}{|1-\overline{z}re^{it}|^4} dt &= \frac{1+|z|^2 r^2}{(1-|z|^2 r^2)^3} \\ &= \sum_{n=0}^{\infty} (n+1)^2 r^{2n} |z|^{2n},\end{aligned}$$

for $z \in \mathbb{D}$ and $r \in (0,1)$. Thus

$$\begin{aligned}|(\mathcal{D}1)(s)| &\leq 2^4 \int_0^1 \sum_{n=0}^{\infty} (n+1)^2 (1-t)^2 t^n |z|^{2n} dt \\ &\leq 2^4 \sum_{n=0}^{\infty} \frac{2(n+1)}{(n+2)(n+3)} |z|^{2n},\end{aligned}$$

where $s = M^{-1} z$. As $|z| \to 1$, this expression behaves (asymptotically) like $-2^4 \log(1-|z|^2)$, hence $\mathcal{D}1 \notin L^\infty(\mathbb{U}_+)$, so $\mathcal{D} \equiv H^d$ cannot be a bounded operator on $L^\infty(\mathbb{U}_+)$. □

Lemma 4.1. *The integral operator $\mathcal{D}$ given by* (1) *is a contraction on $L^1(\mathbb{U}_+, d\widetilde{A})$ which maps $L^\infty(\mathbb{U}_+)$ boundedly into $L^p(\mathbb{U}_+, d\widetilde{A})$ for $1 \leq p < \infty$.*

Proof. For arbitrary $f \in L^1(\mathbb{U}_+, d\widetilde{A})$, by Fubini's theorem [9] it follows that

$$\begin{aligned}\int_{\mathbb{U}_+} |(\mathcal{D}f)(s)| d\widetilde{A}(s) &\leq \int_{\mathbb{U}_+}\int_{\mathbb{U}_+} |d_{\overline{w}}(s)|^2 |f(w)| d\widetilde{A}(w) d\widetilde{A}(s) \\ &= \int_{\mathbb{U}_+} |f(w)| \int_{\mathbb{U}_+} |d_{\overline{w}}(s)|^2 d\widetilde{A}(s) d\widetilde{A}(w) \\ &= \int_{\mathbb{U}_+} |f(w)| \langle d_{\overline{w}}, d_{\overline{w}} \rangle d\widetilde{A}(w) \\ &= \int_{\mathbb{U}_+} |f(w)| d\widetilde{A}(w),\end{aligned}$$

so $\mathcal{D}$ is a contraction on $L^1(\mathbb{U}_+, d\widetilde{A})$. If $f \in L^\infty(\mathbb{U}_+)$, then

$$|(\mathcal{D}f)(s)| \leq ||f||_\infty \int_{\mathbb{U}_+} |d_{\overline{w}}(s)|^2 d\widetilde{A}(w) = ||f||_\infty |(\mathcal{D}1)(s)|.$$

Hence, to prove the second assertion of the lemma, it suffices to check that $\mathcal{D}1$ belongs to $L^p(\mathbb{U}_+, d\widetilde{A})$ for each $p \in [1, \infty)$. We have already observed that $(\mathcal{D}1)(s)$ behaves like $-2^4 \log(1 - |Ms|^2)$ as $|Ms| \to 1$, so it is enough to show that $\log(1 - |z|^2) \in L^p(\mathbb{D}, dA)$ for all $p \in [1, \infty)$. Now, $\int_{\mathbb{D}} |\log(1 - |z|^2)|^p dA(z) = \int_0^1 |\log(1 - r^2)|^p 2r dr = \int_0^1 |\log(1 - t)|^p dt = \int_0^1 |\log t|^p dt$, and, changing the varriable to $y = -\log t$, this reduces to $\int_0^\infty y^p e^{-y} dy = \Gamma(p + 1) < \infty$. □

5 Asymptotic estimates

In this section, we derive certain asymptotic properties of various related integral operators, using which we shall find the norm of H.

Theorem 5.1. *Let* $l = 1 + c$*, where* $c \in (0, 1)$*. For fixed* $r > 0$*, let* $\psi_c(w) = \int_r^\infty \frac{y^{2+\frac{c-1}{2}}}{(y + \mathrm{Im}\ w)^3} \left(\frac{1}{(2iy + i + w)^l} - \frac{1}{(i + w)^l} \right) dy$*. Then* $\lim_{c \to 0} \sqrt{c}\ ||\psi_c||_2 = 0$.

Proof. Since $\mathrm{Im}\ w > 0$, we obtain $|2iy + i + w| \geq |i + w|$ and hence $|\psi_c(w)| \leq \frac{2}{|i + w|^l} \int_r^\infty \frac{y^{2+\frac{c-1}{2}}}{(y + \mathrm{Im}\ w)^3} dy = \frac{2}{|i + w|^l} (\mathrm{Im}\ w)^{\frac{c-1}{2}} F_c\left(\frac{r}{\mathrm{Im}\ w}\right)$, where $F_c(\delta) =$

$\int_\delta^\infty \frac{x^{2+\frac{c-1}{2}}}{(1+x)^3}dx$. Thus we have

$$\begin{aligned}
||\psi_c||_2^2 &= \int_{\mathbb{U}_+} |\psi_c(w)|^2 d\tilde{A}(s) \\
&\leq 4\int_{\mathbb{U}_+} \frac{(\text{Im } w)^{c-1}}{|i+w|^{2(1+c)}} F_c^2\left(\frac{r}{\text{Im } w}\right) d\tilde{A}(w) \\
&= 4\int_0^\infty \gamma^{c-1} F_c^2\left(\frac{r}{\gamma}\right) d\gamma \int_{-\infty}^\infty \frac{dx}{(x^2+(\gamma+1)^2)^{1+c}} \\
&= 8\int_0^\infty \frac{\gamma^{c-1}}{(\gamma+1)^{1+2c}} F_c^2\left(\frac{r}{\gamma}\right) d\gamma \int_0^\infty \frac{dl}{(1+l^2)^{1+c}} \\
&\leq \frac{8\pi}{2}\int_0^\infty \frac{\gamma^{c-1}}{(\gamma+1)^{1+2c}} F_c^2\left(\frac{r}{\gamma}\right) d\gamma \\
&= 4\pi\int_0^\infty \frac{\gamma^{c-1}}{(\gamma+1)^{1+2c}} F_c^2\left(\frac{r}{\gamma}\right) d\gamma \\
&= 4\pi r^c \int_0^\infty \frac{\rho^c}{(r+\rho)^{1+2c}} F_c^2(\rho) d\rho.
\end{aligned}$$

We shall now show that

$$\lim_{c\to 0} c\int_0^\infty \frac{\rho^c}{(r+\rho)^{1+2c}} F_c^2(\rho)d\rho = 0.$$

Notice that

$$\begin{aligned}
F_c(\rho) &\leq \int_0^\infty \frac{x^{2+\frac{c-1}{2}}}{(1+x)^3}dx \\
&= B\left(3+\frac{c-1}{2}, \frac{1-c}{2}\right) \leq \delta
\end{aligned}$$

for all $c \in (0, \frac{1}{2})$, where δ does not depend on $c \in (0, \frac{1}{2})$ and $B(.,.)$ is the Euler's Beta function. Hence we get the inequality

$$c\int_0^1 \frac{\rho^c}{(r+\rho)^{1+2c}} F_c^2(\rho)d\rho \leq \delta^2 c\int_0^1 \frac{\rho^c}{(r+\rho)^{1+2c}}d\rho.$$

Thus

$$\lim_{c\to 0} c\int_0^1 \frac{\rho^c}{(r+\rho)^{1+2c}} F_c^2(\rho)d\rho = 0.$$

We shall now show that

$$\lim_{c\to 0} \int_1^\infty \frac{\rho^c}{(r+\rho)^{1+2c}} F_c^2(\rho)d\rho = 0.$$

If $\rho \geq 1$, then, for $0 < c < \frac{1}{2}$, we have

$$\begin{aligned} F_c(\rho) &= \int_\rho^\infty \frac{x^{2+\frac{c-1}{2}}}{(1+x)^3} dx \leq \int_\rho^\infty \frac{x^{2-\frac{1}{4}}}{(1+x)^3} dx \\ &\leq \int_1^\infty \frac{x^{2-\frac{1}{4}}}{(1+x)^3} dx = R \end{aligned}$$

and for a given $\mu > 0$ there exists $\rho_0 > 1$ such that $\displaystyle\int_{\rho_0}^\infty \frac{x^{2-\frac{1}{4}}}{(1+x)^3} dx \leq \frac{\mu}{2}$.
Consequently, if $\rho > \rho_0$, then we obtain $F_c(\rho) < \frac{\mu}{2}$ for every $c \in \left(0, \frac{1}{2}\right)$.
Thus

$$\begin{aligned} c\int_1^\infty \frac{\rho^c}{(r+\rho)^{1+2c}} F_c^2(\rho) d\rho &= c\int_1^{\rho_0} \frac{\rho^c}{(r+\rho)^{1+2c}} F_c^2(\rho) d\rho + c\int_{\rho_0}^\infty \frac{\rho^c}{(r+\rho)^{1+2c}} F_c^2(\rho) d\rho \\ &< cR^2 \int_1^{\rho_0} \frac{\rho^c}{(r+\rho)^{1+2c}} d\rho + c\frac{\mu}{2} \int_{\rho_0}^\infty \frac{\rho^c}{(r+\rho)^{1+2c}} d\rho \\ &< cR^2 \int_1^{\rho_0} \frac{\rho^c}{(r+\rho)^{1+2c}} d\rho + c\frac{\mu}{2} \int_0^\infty \frac{\rho^c}{(r+\rho)^{1+2c}} d\rho \\ &= cR^2 \int_1^{\rho_0} \frac{\rho^c}{(r+\rho)^{1+2c}} d\rho + c\frac{\mu}{2} r^{-c} \frac{\Gamma(c)\Gamma(1+c)}{\Gamma(1+2c)} \\ &= cR^2 \int_1^{\rho_0} \frac{\rho^c}{(r+\rho)^{1+2c}} d\rho + \frac{\mu}{2} r^{-c} \frac{\Gamma^2(1+c)}{\Gamma(1+2c)} \to \frac{\mu}{2}, \end{aligned}$$

when $c \to 0$. It follows that for $\mu > 0$ there exists a $\epsilon \in \left(0, \frac{1}{2}\right)$ such that if $c \in (0, \epsilon)$, then

$$c\int_1^\infty \frac{\rho^c}{(r+\rho)^{1+2c}} F_c^2(\rho) d\rho < \mu.$$

Hence

$$\lim_{c\to 0^+} c\int_1^\infty \frac{\rho^c}{(r+\rho)^{1+2c}} F_c^2(\rho) d\rho = 0.$$

Further since

$$\lim_{c\to 0} c\int_0^1 \frac{\rho^c}{(r+\rho)^{1+2c}} F_c^2(\rho) d\rho = 0,$$

we obtain

$$\lim_{c\to 0} c\int_0^\infty \frac{\rho^c}{(r+\rho)^{1+2c}} F_c^2(\rho) d\rho = 0.$$

□

Theorem 5.2. *Let* $A_c(w) = \int_0^\infty \frac{y^{2+\frac{c-1}{2}}}{(y+\text{Im } w)^3}\left(\frac{1}{(2iy+i+w)^l} - \frac{1}{(i+w)^l}\right) dy,$ *where* $l = 1+c, 0 < c < \frac{1}{2}$. *Then* $\lim_{c\to 0} \frac{||A_c||_2}{||\theta_c||_2} = 0.$

Proof. From Lemma 2.2, it follows that $||\theta_c||_2 \sim \left(\frac{\pi}{c}\right)^{\frac{1}{2}}$ as $c \to 0$. Hence it is enough to prove that $\lim_{c\to 0} c^{\frac{1}{2}}||A_c||_2 = 0$. Let $f(t) = (2ity+i+w)^{-l}, 0 \le t \le 1$. Then since $|f(1)-f(0)| \le \int_0^1 |f^{'}(t)|dt \le 2yl\int_0^1 \frac{dt}{|2ity+i+w|^{l+1}}$, it follows that $\left|\frac{1}{(2iy+i+w)^l} - \frac{1}{(i+w)^l}\right| \le \frac{2yl}{|i+w|^l} < 3\frac{y}{|i+w|^l}$ as y, t, c are positive and $c \in \left(0, \frac{1}{2}\right)$. Now there exists a constant C such that $B\left(3+\frac{c-1}{2}, \frac{1-c}{2}\right) \le C$, for every $c \in \left(0, \frac{1}{2}\right)$. Since $c^{\frac{1}{2}}||\theta_c||_2 \to \sqrt{\pi}$ as $c \to 0$, there exists a constant N_0 independent of C such that $\sqrt{c}||\theta_c||_2 \le N_0$ for every $c \in \left(0, \frac{1}{2}\right)$. Let $\mu > 0$ be arbitrary and $l_0 = \frac{2\mu}{12CN_0}$ and

$$G_c(w) = \int_0^{l_0} \frac{y^{2+\frac{c-1}{2}}}{(y+\text{Im } w)^3}\left(\frac{1}{(2iy+i+w)^l} - \frac{1}{(i+w)^l}\right) dy.$$

Thus it follows that

$$\begin{aligned}
|G_c(w)| &\le 3\frac{1}{|i+w|^l}\int_0^{l_0} \frac{y^{2+\frac{c-1}{2}}y}{(y+\text{Im } w)^3}dy \\
&\le 3l_0\frac{1}{|i+w|^l}\int_0^{\infty} \frac{y^{2+\frac{c-1}{2}}}{(y+\text{Im } w)^3}dy \\
&< 3l_0\frac{1}{|i+w|^l}\int_0^{\infty} \frac{y^{2+\frac{c-1}{2}}}{(y+\text{Im } w)^3}dy \\
&= 3l_0\frac{1}{|i+w|^l}(\text{Im } w)^{\frac{c-1}{2}}B\left(3+\frac{c-1}{2}, \frac{1-c}{2}\right) \\
&< 3l_0C|\theta_c(w)|.
\end{aligned}$$

Hence we obtain

$$||G_c||_2 \le 3l_0C||\theta_c||_2.$$

Thus we obtain that for every $c \in \left(0, \frac{1}{2}\right), C^{\frac{1}{2}}||G_c||_2 < \frac{\mu}{2}$. Now consider the function

$$\Theta_c(w) = \int_{l_0}^{\infty} \frac{y^{2+\frac{c-1}{2}}}{(y+\text{Im } w)^3}\left(\frac{1}{(2iy+i+w)^l} - \frac{1}{(i+w)^l}\right) dy.$$

From Theorem 5.1, it follows that $\lim_{c\to 0} c^{\frac{1}{2}}||\Theta_c|| = 0$. This implies that there exists a $\epsilon \in \left(0, \frac{1}{2}\right)$ such that if $c \in (0, \epsilon)$, then $C^{\frac{1}{2}}||\Theta_c||_2 < \frac{\mu}{2}$.

Since $A_c = G_c + \Theta_c$ for $c \in (0, \epsilon)$, we obtain that $C^{\frac{1}{2}}||A_c||_2 < \mu$. That is, $C^{\frac{1}{2}}||A_c||_2 \to 0$ as $c \to 0$. The lemma follows. □

Theorem 5.3. *Let* $l = 1+c, c \in \left(0, \frac{1}{2}\right)$ *and* $\Phi_c(w) = \displaystyle\int_0^\infty \frac{y^{2+\frac{c-1}{2}}}{(y + \text{Im } w)^3} \frac{dy}{(2iy + i + w)^l}$. *Then* $\displaystyle\lim_{c\to 0} \frac{||\Phi_c||_2}{||\theta_c||_2} = \frac{\Gamma(3 - \frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(3)} = \frac{\Gamma(2\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(3)}$.

Proof. Since

$$
\begin{aligned}
|\Phi_c(w)| &= \int_0^\infty \frac{y^{2+\frac{c-1}{2}}}{(y + \text{Im } w)^3} \frac{dy}{|i + w|^l} \\
&= \frac{(\text{Im } w)^{\frac{c-1}{2}}}{|i + w|^l} B\left(3 + \frac{c-1}{2}, \frac{1-c}{2}\right),
\end{aligned}
$$

we obtain

$$
||\Phi_c||_2 \leq ||\theta_c||_2 = \frac{\Gamma(3 + \frac{c-1}{2})\Gamma(\frac{1-c}{2})}{\Gamma(3)}
$$

and hence

$$
\overline{\lim}_{c\to 0} \frac{||\Phi_c||_2}{||\theta_c||_2} \leq \frac{\Gamma(3 - \frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(3)}.
$$

Since $\Phi_c(w) = \theta_c(w) \dfrac{\Gamma(3 + \frac{c-1}{2})\Gamma(\frac{1-c}{2})}{\Gamma(3)} + A_c(w)$ we conclude that

$$
||\Phi_c||_2 \geq ||\theta_c||_2 \frac{\Gamma(3 + \frac{c-1}{2})\Gamma(\frac{1-c}{2})}{\Gamma(3)} - ||A_c||_2.
$$

That is,

$$
\frac{||\Phi_c||_2}{||\theta_c||_2} \geq \frac{\Gamma(3 + \frac{c-1}{2})\Gamma(\frac{1-c}{2})}{\Gamma(3)} - \frac{||A_c||_2}{||\theta_c||_2}.
$$

From Theorem 5.2, it follows that

$$
\underline{\lim}_{c\to 0} \frac{||\Phi_c||_2}{||\theta_c||_2} \geq \frac{\Gamma(3 - \frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(3)}.
$$

Hence the Theorem follows. □

Theorem 5.4. *Let* $l = 1 + c$, *where* $c \in \left(0, \frac{1}{2}\right), n \in \mathbb{N} \cup \{0\}$ *and* $\Xi_m^c(w) = \int_0^\infty \frac{y^{2n+\frac{c-1}{2}}}{(y + \text{Im } w)^{n+1+m}} \frac{dy}{(2iy + i + w)^{n+l-m}}$. *Then* $\lim_{c \to 0} \frac{||\Xi_m^c||_2}{||\theta_c||_2} = 0.$

Proof. From Theorem 5.1, it follows that $||\theta_c|| \sim \sqrt{\frac{\pi}{c}}$ as $c \to 0$. Hence it is enough to prove that for $m = 0, 1, \cdots, n-1$, we have $\lim_{c \to 0} c^{\frac{1}{2}} ||\Xi_m^c||_2 = 0$. Thus it is sufficient to show that there exists a constant K independent of c such that $||\Xi_m^c||_2 \leq K$ for every $c \in \left(0, \frac{1}{2}\right)$. Since

$$||\Xi_m^c||_2 \leq \left(\int_{\mathbb{U}_+} d\tilde{A}(w) \left| \int_0^\infty \frac{y^{2n+\frac{c-1}{2}}}{(y + \text{Im } w)^3} \frac{dy}{(2iy + y + w)^{n+l-m}} \right|^2 \right)^{\frac{1}{2}}$$

by applying Minkowski's integral inequality we get

$$||\Xi_m^c||_2 \leq \int_0^\infty y^{2n+\frac{c-1}{2}} Z(y)^{\frac{1}{2}} dy,$$

where $Z(y) = \int_{\mathbb{U}_+} \frac{d\tilde{A}(w)}{(y + \text{Im } w)^{(n+1+m)2} |2iy + i + w|^{2(n+l-m)}}$. A straightforward calculation gives

$$\begin{aligned} Z(y) &= \int_0^\infty \frac{d\gamma}{(y + \gamma)^{2(n+1+m)} (2y + 1 + \gamma)^{2(n+l-m)-1}} \int_{-\infty}^\infty \frac{dt}{(1 + t^2)^{n+l-m}} \\ &\leq \mu_0 X(y), \end{aligned}$$

where $\mu_0 = \int_{-\infty}^\infty \frac{dt}{(1 + t^2)^{\frac{3}{2}}}$ and $X(y) = \int_0^\infty \frac{d\gamma}{(y + \gamma)^{2(n+1+m)} (2y + 1 + \gamma)^{2(n+l-m)-1}}$.
If $y \in (0, 1)$, we have

$$\begin{aligned} X(y) &= \int_0^1 . \, dy + \int_1^\infty . \, dy \leq \int_0^1 \frac{d\gamma}{(y + \gamma)^{2(n+1+m)}} + \int_1^\infty \frac{d\gamma}{(\gamma)^{2(n+1+m)} (\gamma)^{2(n+l-m)-1}} \\ &\leq \tau_{n,m} y^{1-2(n+1+m)}, \end{aligned}$$

where $\tau_{n,m}$ is a constant which does not depend on y and c.
If $y \geq 1$, then

$$\begin{aligned} X(y) &\leq \int_0^\infty \frac{d\gamma}{(y + \gamma)^{2(n+1+m)} (y + \gamma)^{2(n+l-m)-1}} \\ &= \frac{1}{y^{2(2n+1)+2c}} \int_0^\infty \frac{d\gamma}{(1 + \rho)^{2(2n+1)+2c+1}} \\ &\leq \frac{\beta_n}{y^{2(2n+1)+2c}}, \end{aligned}$$

where β_n does not depend on y and c. Then

$$Z(y) \le \mu_0 X(y) \le \begin{cases} \mu_0 \tau_{n,m} y^{1-2(n+1+m)}, & y \in (0,1) \\ \mu_0 \beta_n y^{-2(2n+1)-2c}, & y \ge 1 \ . \end{cases}$$

That is,

$$Z(y)^{\frac{1}{2}} \le \begin{cases} (\mu_0 \tau_{n,m})^{\frac{1}{2}} y^{\frac{1}{2}-n-m-1}, & y \in (0,1) \\ (\mu_0 \beta_n)^{\frac{1}{2}} y^{-(2n+1)-c}, & y \ge 1 \ . \end{cases}$$

Thus we obtain

$$\begin{aligned} ||\Xi_m^c||_2 &\le \int_0^\infty y^{2n+\frac{c-1}{2}} (Z(y))^{\frac{1}{2}} dy \\ &\le (\mu_0 \tau_{n,m})^{\frac{1}{2}} \frac{1}{n-m+\frac{c}{2}} + (Q_0 \beta_n)^{\frac{1}{2}} \frac{2}{1+c} = R_{n,m}(c) \text{ (let)}. \end{aligned}$$

Now for $m = 0, 1, 2, \cdots, n-1$, the function $c \to R_{n,m}(c)$ is bounded on $(0, \frac{1}{2})$ and hence $||\Xi_m^c||_2 \le K$ where $K = \sup_{c \in (0,\frac{1}{2})} R_{n,m}(c)$. □

6 Norm of the Berezin transformation

Let $\mathcal{H}$ be a Hilbert space. Let $\mathcal{L}(\mathcal{H})$ be the set of all bounded linear operators from the Hilbert space $\mathcal{H}$ into itself. In this section we establish that $||H|| = \frac{3\pi}{4}$ and the map L given by $Lf(w) = \frac{1}{4}\int_{\mathbb{U}_+} f(s)|d_{\overline{w}}(s)|^2 d\widetilde{A}(s)$, $w \in \mathbb{U}_+, f \in L^2(\mathbb{U}_+)$ is a strict contraction. Related maps are also considered in [12] and [7]. Here we consider $H \in \mathcal{L}(L^2(\mathbb{U}_+, d\widetilde{A}))$. Thus it follows from Theorem 6.1 that $||H|| \le \frac{3\pi}{4}$ if it is considered on the Bergman space $L_a^2(\mathbb{U}_+)$.

Theorem 6.1. *Let* $Lf(w) = \frac{1}{4}\int_{\mathbb{U}_+} f(s)|d_{\overline{w}}(s)|^2 d\tilde{A}(s), w \in \mathbb{U}_+, f \in L^2(\mathbb{U}_+)$. *Then* $||L|| = \frac{3\pi}{16} < 1$ *and* L *is a strict contraction. Further* $||H|| = \frac{3\pi}{4}$.

Proof. Let $w \in \mathbb{U}_+$. Notice that $Lf(w) = \frac{1}{4}\int_{\mathbb{U}_+} f(s)|d_{\overline{w}}(s)|^2 d\tilde{A}(s)$ $= \frac{4}{4}\int_{\mathbb{U}_+} f(s)\frac{1}{\pi}\frac{(\text{Im } w)^2}{|s+w|^4} d\tilde{A}(s) = \frac{1}{\pi}\int_{\mathbb{U}_+} f(s)\frac{(\text{Im } w)^2}{|s+w|^4} d\tilde{A}(s)$. We shall show that $||L|| = \frac{3\pi}{16} < 1$. We shall establish this using Lemma 2.2, Theorem 5.3

and Theorem 5.4. Define E on $L^2(\mathbb{U}_+)$ by $Eg(z) = \dfrac{12}{\pi}\displaystyle\int_{\mathbb{U}_+} \frac{(\text{Im }\xi)^2}{|\overline{\xi} - z|^4} g(\xi) d\widetilde{A}(\xi)$. We shall show that the operator E is bounded on $L^2(\mathbb{U}_+)$. Notice that

$$E\theta_c(w) = \frac{12}{\pi}\int_0^\infty y^{2+\frac{c-1}{2}} dy \int_{-\infty}^\infty \frac{d\rho}{|\rho - iy - w|^4(\rho + i + iy)^l},$$

where $l = 1 + c$. The function

$$x \mapsto \frac{1}{(x - iy - w)^2(x + iy - \overline{w})^2(x + i + iy)^l}$$

is analytic in the upper half plane minus the point $x = y + iw$ (where it has a pole of order 2). Hence by Cauchy's residue theorem [5]

$$\int_{-\infty}^\infty \frac{d\rho}{|\rho - iy - w|^4(\rho + i + iy)^l} = 2\pi i(-1)\left[\frac{\Gamma(2)\Gamma(l+1)}{\Gamma(2)\Gamma(l)}(2iy + 2i\text{Im } w)^{-2}\right.$$
$$\left.(2iy + i + w)^{-l-1} + \frac{\Gamma(3)\Gamma(l)}{\Gamma(2)\Gamma(l)}(2iy + 2i\text{Im } w)^{-3}(2iy + i + w)^{-l}\right].$$

Hence

$$E\theta_c(w) = \frac{3\Gamma(3)}{\Gamma^2(2)}\Phi_c(w) + \frac{24\pi i}{\pi}\frac{\Gamma(2)\Gamma(l+1)}{\Gamma(2)\Gamma(l)}\Xi_0^c(w).$$

Thus

$$||\theta_c||_2||E|| \geq ||E\theta_c||_2 \geq \frac{3\Gamma(3)}{\Gamma^2(2)}||\Phi_c||_2 - 24\frac{\Gamma(2)\Gamma(l+1)}{\Gamma(2)\Gamma(l)}||\Xi_0^c||_2.$$

That is,

$$||E|| \geq \frac{3\Gamma(3)}{\Gamma^2(2)}\frac{||\Phi_c||_2}{||\theta_c||_2} - 24\frac{\Gamma(2)}{\Gamma(2)}\frac{\Gamma(l+1)}{\Gamma(l)}\frac{||\Xi_0^c||_2}{||\theta_c||_2}.$$

Using Theorem 5.3 and Theorem 5.4 when $c \to 0$, we obtain that

$$||E|| \geq \frac{3}{\Gamma^2(2)}\Gamma\left(3 - \frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right).$$

We shall now show that

$$||E|| \leq \frac{3}{\Gamma^2(2)}\Gamma\left(3 - \frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right).$$

Let $\eta(s) = (\text{Im } s)^{-\frac{1}{4}}$. Then

$$\begin{aligned} \int_{\mathbb{U}_+} \frac{(\text{Im } s)^2}{|s+w|^4} (\eta(s))^2 d\tilde{A}(s) &= B\left(\frac{1}{2}, \frac{3}{2}\right) B\left(3 - \frac{1}{2}, \frac{1}{2}\right) (\eta(w))^2 \\ \text{and} \int_{\mathbb{U}_+} \frac{(\text{Im } s)^2}{|s+w|^4} (\eta(w))^2 d\tilde{A}(w) &= B\left(\frac{1}{2}, \frac{3}{2}\right) B\left(3 - \frac{1}{2}, \frac{1}{2}\right) (\eta(s))^2. \end{aligned}$$

By Schur's theorem [13], the operator E is bounded on $L^2(\mathbb{U}_+)$ and

$$||E|| \leq \frac{12}{\pi} B\left(\frac{1}{2}, \frac{3}{2}\right) B\left(3 - \frac{1}{2}, \frac{1}{2}\right).$$

Notice that $L = \dfrac{1}{12} E^*$ and we obtain

$$\begin{aligned} ||L|| &= \frac{1}{12}||E^*|| = \frac{1}{12}||E|| = \frac{1}{12} 3\Gamma\left(3 - \frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) \\ &= \frac{1}{12} 3\Gamma\left(2 + \frac{1}{2}\right)\Gamma\left(1 - \frac{1}{2}\right) \\ &= \frac{1}{4}\Gamma\left(2 + \frac{1}{2}\right)\Gamma\left(1 - \frac{1}{2}\right) \\ &= \frac{1}{4}\frac{3}{4}\frac{\pi}{\sin \frac{\pi}{2}} \\ &= \frac{3\pi}{16} = \frac{66}{112} < 1 \end{aligned}$$

and

$$||E|| = \frac{12}{\pi} B\left(\frac{1}{2}, \frac{3}{2}\right) B\left(3 - \frac{1}{2}, \frac{1}{2}\right) = \frac{9\pi}{4}.$$

Since $4L = H$, hence $||H|| = 4||L|| = \dfrac{3\pi}{4}$. □

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