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ADMISSIBLE PERTURBATION OF SINGLE-VALUED OPERATORS IN VECTOR-VALUED METRIC SPACES*

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Abstract

In this paper, using the admissible perturbation technique, we will prove some data dependence and stability results for the fixed point equation in complete vector-valued metric spaces. Our approach generalizes some recent results in metric fixed point theory.

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1 Preliminary notions and results

Let (X, d) be a metric space and $P(X)$ be the set of all nonempty subsets of X. Let us recall now the following notions:
(1) the distance from a point $x \in X$ to a set $Y \in P(X)$:

$$D(x, Y) := \inf\{d(x, y) \mid y \in Y\}.$$

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(2) the excess of Y over Z (where $Y, Z \in P(X)$):

$$e(Y, Z) := \sup\{D(y, Z), y \in Y\}.$$

(3) the Hausdorff-Pompeiu distance between two sets $Y, Z \in P(X)$:

$$H(Y, Z) = \max\{e(Y, Z), e(Z, Y)\}.$$

Let (X, d) be a metric space and $t : X \to X$ be a single-valued operator. Then $x \in X$ is called a fixed point for t if $x = t(x)$. The symbol

$$Fix(t) := \{x \in X : \ x = t(x)\}$$

denotes the fixed point set of t.

Remark 1 *A sequence $(x_n)_{n\in\mathbb{N}}$ from X satisfying*

$$x_0 \in X, \ x_{n+1} = t(x_n), \ \textit{for each } n \in \mathbb{N},$$

is called an iterative sequence of Picard type starting from x_0 for the single-valued operator $t : X \to X$.

If $x, y \in \mathbb{R}^m$, $x = (x_1, ..., x_m)$ and $y = (y_1, ..., y_m)$, then, by definition

$$x \preceq y \text{ if and only if } x_i \leq y_i, \text{ for each } i \in \{1, 2, \cdots, m\}.$$

Through this section, we will make an identification between row and column vectors in $\mathbb{R}^m$.

We can now recall the notion of vector-valued metric space, see e.g., [4]. By definition, (X, d) is a vector-valued metric space if X is a nonempty set and $d : X \times X \to \mathbb{R}^m_+$ satisfies all the axioms of the usual metric, where the inequalities from the axioms of the metric are given with respect to $\preceq$.

We may suppose that

$$d(x, y) := \begin{pmatrix} d_1(x, y) \\ \cdots \\ d_m(x, y) \end{pmatrix}, \text{ for } x, y \in X.$$

We denote by $M_{m,m}(\mathbb{R}_+)$ the set of all $m \times m$ matrices with positive elements, by $I_{m,m}$ the identity $m \times m$ matrix and by $O_{m,m}$ the null $m \times m$ matrix. Also, the symbol O_m denotes the null vector of $\mathbb{R}^m$.

By definition, $K \in M_{m,m}(\mathbb{R}_+)$ is said to be convergent to zero if $K^n \to O_{m,m}$ as $n \to \infty$. The following result will be important for our next considerations, see e.g. [17].

Theorem 1 *Let $K \in M_{m,m}(\mathbb{R}_+)$. The following assertions are equivalent:*

(i) $K^n \to O_{m,m}$ as $n \to \infty$;

(ii) The spectral radius $\rho(K)$ of K is strictly less than 1, i.e., the eigenvalues of K are in the open unit disc;

(iii) The matrix $(I_{m,m} - K)$ is nonsingular and

$$(I_{m,m} - K)^{-1} = I_{m,m} + K + ... + K^n + ...; \tag{1}$$

(iv) The matrix $(I_{m,m} - K)$ is nonsingular and $(I_{m,m} - K)^{-1}$ has nonnegative elements.

Using the above properties, Perov [4] proved the following result.

Theorem 2 (Perov) *Let (X, d) be a complete vector-valued metric space and let $t : X \to X$ be an K-contraction, i.e., $K \in M_{m,m}(\mathbb{R}_+)$ converges to zero and*

$$d(t(x), t(y)) \preceq K d(x, y), \text{ for all } x, y \in X.$$

Then:

(1) *$Fix(t) = \{x^*\}$;*

(2) *the sequence $(x_n)_{n\in\mathbb{N}}$, $x_n := t^n(x_0)$ of Picard iterates for t starting from any $x_0 \in X$ is convergent to x^*;*

(3) *the following estimation holds*

$$d(x_n, x^*) \preceq K^n (I_{m,m} - K)^{-1} d(x_0, x_1), \text{ for every } n \in \mathbb{N}; \tag{2}$$

The purpose of this paper is to give some data dependence and stability results for a fixed point equation in complete vector-valued metric spaces, using the admissible perturbation technique. Our approach generalizes some recent results in metric fixed point theory, see [1], [7], [8], [12], [13], [15].

2 Main results

Let us recall first the notion of admissible perturbation and its relation with fixed point theory. For related notions and results see [11], [13], [16].

Let X be a nonempty set and $A : X \times X \to X$ be a mapping satisfying the following two conditions:

(AP1) $A(x, x) = x$, for each $x \in X$;

(AP2) if $x, y \in X$ satisfy $A(x, y) = x$, then $y = x$.

The concept of admissible perturbation for the single-valued case was proposed by I.A. Rus in [11]. For the multi-valued case see [6].

Definition 1 *Let X be a nonempty set and $A : X \times X \to X$ be a mapping having the properties* (AP1) *and* (AP2)*. Let $t : X \to X$ be a single-valued operator. Then, the single-valued operator $t_A : X \to X$ given by $t_A(x) := A(x, t(x))$ is called the admissible perturbation of t corresponding to A.*

The following important properties hold.

Lemma 1 *(see [11]) If X is a nonempty set and $t : X \to X$ is a single-valued operator which admits an admissible perturbation t_A, then $Fix(t) = Fix(t_A)$.*

Some examples of mappings A which generate admissible perturbations are given now.

Example 1 *(see [11]) Let $\mathcal{E}$ be a linear space, $\lambda \in \mathbb{R}\setminus\{0\}$ and $A : \mathcal{E}\times\mathcal{E} \to \mathcal{E}$ be given by*

$$A(x, y) := \lambda x + (1 - \lambda)y.$$

If $t : X \to X$ is a single-valued operator, then $t_A : X \to X$ given by

$$t_A(x) := \lambda x + (1 - \lambda)t(x)$$

is the admissible perturbation of t corresponding to A.

Example 2 *([2], [3]) A pair (X, F) is called a convex prestructure in the sense of Gudder if X is a nonempty set and $F : [0, 1] \times X \times X \to X$ is a given mapping. Suppose that the convex prestructure (X, F) satisfies the following axioms:*

(AX1) $F(\lambda, x, y) = F(1 - \lambda, y, x)$, for every $\lambda \in [0, 1]$ and each $x, y \in X$;

(AX2) $F(\lambda, x, F(\mu, y, z)) = F(\lambda + (1 - \lambda)\mu, F(\frac{\lambda}{\lambda+(1-\lambda)\mu}, x, y), z)$, for every $\lambda, \mu \in [0, 1]$ with $\lambda + (1 - \lambda)\mu \neq 0$ and for each $x, y, z \in X$;

(AX3) $F(\lambda, x, x) = x$, for every $\lambda \in [0, 1]$ and each $x \in X$;

(AX4) If for some $\lambda \in [0, 1)$ and $x \in X$ we have that $F(\lambda, x, y) = F(\lambda, x, z)$, then $y = z$;

(AX5) $F(0, x, y) = y$, for every $x, y \in X$.

Then the pair (X, F) is called a convex structure in the sense of Gudder.

If (X, F) is a convex structure, then a set $Y \in P(X)$ is called convex if for every $\lambda \in [0, 1]$ and every $x, y \in X$ we have that $F(\lambda, x, y) \in Y$.

Let (X, F) be a convex structure and $\lambda \in (0, 1)$. We define now the operator $A : X \times X \to X$ by

$$A(x, y) := F(\lambda, x, y).$$

If $t : X \to X$ *is a single-valued operator, then* $t_A : X \to X$ *given by*

$$t_A(x) := F(\lambda, x, t(x))$$

is the admissible perturbation of t *corresponding to* A.

Example 3 *([18], [19]) Let* (X, d) *be a vector-valued metric space and* $W : X \times X \times [0,1] \to X$ *be an operator satisfying the following relation:*

$$d(u, W(x, y, \lambda)) \preceq \lambda d(u, x) + (1-\lambda) d(u, y), \text{ for every } u, x, y \in X \text{ and } \lambda \in (0,1).$$

The pair (X, W) *is called a convex vector-valued metric space in the sense of Takahashi if* (X, d) *is a vector-valued metric space and* $W : X \times X \times [0,1] \to X$ *is an operator with the above property. A set* $Y \in P(X)$ *is called convex in the sense of Takahashi if* $W(x, y, \lambda) \in Y$, *for every* $\lambda \in [0,1]$ *and every* $x, y \in X$.

Let (X, W) *be a convex vector-valued metric space in the sense of Takahashi, such that the following implication holds*

$$\lambda \in (0,1), x, y \in X \text{ with } W(x, y, \lambda) = x \text{ imply that } y = x. \tag{3}$$

Then, for $\lambda \in (0,1)$ *let us define the operator* $A : X \times X \to X$ *by*

$$A(x, y) := W(x, y, \lambda).$$

If $t : X \to X$ *is a single-valued operator, then* $t_A : X \to X$ *given by*

$$t_A(x) := W(x, t(x), \lambda)$$

is the admissible perturbation of t *corresponding to* A.

Let us consider now some concepts related to some stability theorems for the fixed points of single-valued operators in vector-valued metric spaces.

Definition 2 *Let* (X, d) *be a vector-valued metric space,* $t : X \to X$ *be a single-valued operator such that* $Fix(t) \neq \emptyset$ *and there exists* $r : X \to Fix(t)$ *a set retraction. Then*

$$X = \bigcup_{x^* \in Fix(t)} r^{-1}(x^*)$$

is the fixed point partition of X *corresponding to* r.

Definition 3 *Let (X, d) be a vector-valued metric space and $t : X \to X$ be a single-valued operator with $Fix(t) \neq \emptyset$. Let $\Psi : \mathbb{R}^m_+ \to \mathbb{R}^m_+$ be an increasing (with respect to the componentwise partial order) function such that $\Psi(O_m) = O_m$ and ψ is continuous at O_m. If there exists a set retraction $r : X \to Fix(t)$ such that*

$$d(x, r(x)) \preceq \Psi(d(x, t(x))), \text{ for each } x \in X,$$

then we say that the retraction-displacement condition on t corresponding to r holds.

We will introduce now some data dependence and stability property for the fixed points of single-valued operators in vector-valued metric spaces. For related notions and results see [5], [9], [10].

Definition 4 *Let (X, d) be a vector-valued metric space. If $t : X \to X$ is a single-valued operator with at least one fixed point, then we say that the data dependence phenomenom for the fixed point set of t holds if for an operator $s : X \to X$ with $Fix(s) \neq \emptyset$ and for which there exists $\eta := (\eta_1, \cdots, \eta_m)$ (with $\eta_i > 0$ for each $i \in \{1, 2, \cdots, m\}$), such that*

$$d(t(x), s(x)) \preceq \eta, \text{ for each } x \in X,$$

there exists an increasing (with respect to the componentwise partial order) function $\chi : \mathbb{R}^m_+ \to \mathbb{R}^m_+$ which is continuous in O_m and satisfies the relation $\chi(O_m) = O_m$, such that

$$H(Fix(t), Fix(s)) \preceq \chi(\eta).$$

Definition 5 *Let (X, d) be a vector-valued metric space and $t : X \to X$ be a single-valued operator. Then, we say that the generalized Ulam-Hyers stability property for the fixed point equation $x = t(x), x \in X$ holds if there exists a function $\mu : \mathbb{R}^m_+ \to \mathbb{R}^m_+$ which is increasing (with respect to the componentwise partial order), continuous in O_m, with $\mu(O_m) = O_m$, such that for every $\epsilon := (\epsilon_1, \cdots, \epsilon_m)$ (with $\epsilon_i > 0$ for each $i \in \{1, 2, \cdots, m\}$) and any $z \in X$ with $d(z, t(z)) \preceq \epsilon$, there exists $x^* \in Fix(t)$ satisfying the relation*

$$d(z, x^*) \preceq \mu(\epsilon).$$

Definition 6 *Let (X, d) be a vector-valued metric space and $t : X \to X$ be a single-valued operator, such that $Fix(t) \neq \emptyset$ and there exists $r : X \to Fix(t)$*

a set retraction. Then, we say that the well-posedness property (in the sense of Reich and Zaslavski) of the fixed point equation $x = t(x)$ with respect to the fixed point partition X corresponding to r holds if for each $x^* \in Fix(t)$ and for each sequence $\{u_n\}_{n\in\mathbb{N}} \subset r^{-1}(x^*)$ with $d(u_n, t(u_n)) \to 0$, we have that $u_n \to x^*$ as $n \to \infty$.

Definition 7 *Let (X, d) be a vector-valued metric space and $t : X \to X$ be a single-valued operator such that $Fix(t) \neq \emptyset$. Let $r : X \to Fix(t)$ be a set retraction. The fixed point equation $x = t(x)$ has the Ostrowski stability property with respect to the fixed point partition of X corresponding to r if for each $x^* \in Fix(t)$ and for each sequence $\{w_n\}_{n\in\mathbb{N}} \subset r^{-1}(x^*)$ with $d(w_{n+1}, t(w_n)) \to 0$ as $n \to \infty$, we have that $w_n \to x^*$ as $n \to \infty$.*

The following general class of mappings will be considered for our next main results.

Definition 8 *Let (X, d) be a vector-valued metric space and $t : X \to X$ be a single-valued operator. Then, t is called a weakly Picard operator if for each $u_0 \in X$ the sequence $(u_n)_{n\in\mathbb{N}}$ of Picard iterates for t starting from u_0 (i.e., $u_n := t^n(u_0)$ or equivalently $u_{n+1} = t(u_n), n \in \mathbb{N}$) converges to a fixed point of t.*

Definition 9 *Let (X, d) be a vector-valued metric space and $t : X \to X$ be a weakly Picard operator. We define the operator $t^\infty : X \to Fix(t)$, given by $t^\infty(u) := \lim\limits_{n\to\infty} t^n(u)$.*

Definition 10 *Let (X, d) be a vector-valued metric space and $t : X \to X$ be a weakly Picard operator. Then, t is called a Γ-weakly Picard operator if $\Gamma : \mathbb{R}^m_+ \to \mathbb{R}^m_+$ is increasing (with respect to the componentwise partial order), continuous in O_m with $\Gamma(O_m) = O_m$ and*

$$d(x, t^\infty(x)) \leq \Gamma(d(x, t(x))), \text{ for all } x \in X. \tag{4}$$

In particular, if $t : X \to X$ is a weakly Picard operator for which there exists $K \in M_{m,m}(\mathbb{R}_+) \setminus \{O_{m,m}\}$ such that

$$d(x, t^\infty(x)) \leq K d(x, t(x)), \text{ for all } x \in X, \tag{5}$$

then t is called a K-weakly Picard operator.

Example 4 *Let (X,d) be a complete vector-valued metric space and $t : X \to X$ be a single-valued operator with closed graph. If there exists a matrix $\kappa \in M_{m,m}(\mathbb{R}_+)$ such that*

$$d(t(x), t^2(x)) \leq \kappa d(x, t(x)), \text{ for every } x \in X, \tag{6}$$

then t is a $(I_{m,m} - \kappa)^{-1}$-weakly Picard operator. Notice that an operator satisfying the condition (6) is called a graph κ-contraction.

The following are the main results of the paper.

Theorem 3 *Let (X,d) be a vector-valued metric space and let $t, s : X \to X$ be two single-valued operators. Let $A : X \times X \to X$ be an operator satisfying the conditions* (AP1) *and* (AP2). *Suppose:*
(a) the admissible perturbation $t_A : X \to X, t_A(x) := A(x, t(x))$ is a Γ-weakly Picard operator;
(b) the admissible perturbation $s_A : X \to X, s_A(x) := A(x, s(x))$ is a Υ-weakly Picard operator;
(c) there exists $L \in M_{m,m}(\mathbb{R}_+)\backslash\{O_{m,m}\}$ such that $d(x, t_A(x)) \preceq Ld(x, t(x))$, for each $x \in X$;
(d) there exists $Q \in M_{m,m}(\mathbb{R}_+)\backslash\{O_{m,m}\}$ such that $d(x, s_A(x)) \preceq Qd(x, s(x))$, for each $x \in X$;
(e) there exists $R \in \mathbb{R}^m_+$ (with $R_i > 0$ for each $i \in \{1, 2, \cdots, m\}$) such that $d(t(x), s(x)) \leq R$, for each $x \in X$.
Then

$$H(Fix(t), Fix(s)) \leq \Theta(R) := \begin{pmatrix} \max\{\Gamma_1(QR), \Upsilon_1(LR)\} \\ \cdots \\ \max\{\Gamma_m(QR), \Upsilon_m(LR)\} \end{pmatrix}.$$

Proof. We will show that for every $x \in Fix(t)$ there exists $y \in Fix(s)$ such that $d(x,y) \leq \Theta(R)$ and the analogue relation that for every $y \in Fix(s)$ there exists $x \in Fix(t)$ such that $d(x,y) \leq \Theta(R)$.
Let $x \in Fix(t)$. Since s_A is a Υ-weakly Picard operator, we have

$$d(x_0, s_A^\infty(x_0)) \preceq \Upsilon(d(x_0, s_A(x_0))), \text{ for all } x_0 \in X.$$

Taking $x_0 := x$ we get that $s_A^\infty(x) \in Fix(s)$ and

$$d(x, s_A^\infty(x)) \preceq \Upsilon(d(x, s_A(x))) \preceq \Upsilon(Qd(x, s(x))) = \Upsilon(Qd(t(x), s(x))) \preceq \Upsilon(QR).$$

Similarly, for $y \in Fix(s)$ we have that $t_A^\infty(y) \in Fix(t)$ and the following relations hold

$$d(y, t_A^\infty(y)) \preceq \Gamma(d(y, t_A(y))) \preceq \Gamma(Ld(y, t(y))) = \Gamma(Ld(s(y), t(y))) \leq \Gamma(LR).$$

By the above relations we get $H(Fix(t), Fix(s)) \preceq \Theta(R)$, which completes the proof. □

Our next result proves two important properties of the fixed point equation.

Theorem 4 *Let (X, d) be a vector-valued metric space and let $t : X \to P(X)$ be a multi-valued operator. Let $A : X \times X \to X$ be an operator satisfying the conditions* (AP1) *and* (AP2). *Suppose:*
(a) the admissible perturbation $t_A : X \to X, t_A(x) := A(x, t(x))$ is a Γ-weakly Picard operator;
(b) there exists $L \in M_{m,m}(\mathbb{R}_+)\setminus\{O_{m,m}\}$ such that $d(x, t_A(x)) \preceq Ld(x, t(x))$, for each $x \in X$.

Then, the fixed point equation $x = t(x), x \in X$ is well-posed and satisfies the Ulam-Hyers stability property.

Proof. Since t_A is a Γ-weakly Picard operator, we have that $Fix(t) \neq \emptyset$ and for the operator t_A^∞ we have

$$d(x_0, t_A^\infty(x_0)) \preceq \Gamma(d(x_0, t_A(x_0))), \text{ for all } x_0 \in X.$$

A. (Well-posedness) Consider the fixed point partition of X corresponding to t_A^∞, i.e.,

$$X = \bigcup_{x^* \in Fix(t)} (t_A^\infty)^{-1}(x^*).$$

Let $u^* \in Fix(t)$. Then, for any sequence $\{u_n\}_{n\in\mathbb{N}} \subset (t_A^\infty)^{-1}(x^*)$ with $d(u_n, t(u_n)) \to 0$ we have

$$d(u_n, x^*) = d(u_n, t_A^\infty(u_n)) \preceq \Gamma(d(u_n, t_A(u_n))) \preceq \Gamma(Ld(u_n, t(u_n))) \to 0$$

as $n \to \infty$.

B. (Ulam-Hyers stability) Take any $\epsilon := (\epsilon_1, \cdots, \epsilon_m)$ (with $\epsilon_i > 0$ for each $i \in \{1, 2, \cdots, m\}$) and any $z \in X$ with the property $d(z, t(z)) \preceq \epsilon$. Denote $x^* := t_A^\infty(z) \in Fix(t)$. Then, we have

$$d(z, x^*) \preceq d(z, t_A^\infty(z)) \preceq \Gamma(d(z, t_A(z))) \preceq \Gamma(Ld(z, t(z))) \leq \Gamma(L\epsilon).$$

The proof is now complete. □

We discuss now the Ostrowski stability property for the fixed point equation $x = t(x), x \in X$ with a weakly Picard operator in a vector-valued metric space.

Theorem 5 *Let (X, d) be a vector-valued metric space and let $t : X \to X$ be a multi-valued operator. Let $A : X \times X \to X$ be an operator satisfying the conditions* (AP1) *and* (AP2). *Suppose:*
(a) the admissible perturbation $t_A : X \to X, t_A(x) := A(x, t(x))$ is a weakly Picard operator;
(b) t is a K-quasi contraction with respect to the fixed point partition corresponding to t_A^∞, i.e., there exists a matrix $K \in M_{m,m}(\mathbb{R}_+)$ which converges to zero such that

$$d(t(x), t_A^\infty(x)) \preceq Kd(x, t_A^\infty(x)), \text{ for every } x \in X.$$

(b) there exists $L \in M_{m,m}(\mathbb{R}_+)\setminus\{O_{m,m}\}$ such that $d(x, t_A(x)) \preceq Ld(x, t(x))$, for each $x \in X$.

Then, the fixed point equation $x = t(x), x \in X$ has the Ostrowski stability property.

Proof. Since t_A is a weakly Picard operator, the fixed point set $Fix(t)$ is nonempty. Let $x^* \in Fix(t)$ and let $\{w_n\}_{n\in\mathbb{N}} \subset (t_A^\infty)^{-1}(x^*)$ such that $d(w_{n+1}, t(w_n)) \to 0$ as $n \to \infty$.
Then, we have

$$d(w_{n+1}, x^*) = d(w_{n+1}, t_A^\infty(w_n)) \preceq d(w_{n+1}, t_A(w_n)) + d(t_A(w_n), t_A^\infty(w_n)) \preceq$$
$$Ld(w_{n+1}, t(w_n)) + Kd(w_n, t_A^\infty(w_n)) = Ld(w_{n+1}, t(w_n)) + Kd(w_n, x^*) \preceq$$
$$Ld(w_{n+1}, t(w_n)) + K[Ld(w_n, t(w_{n-1})) + Kd(w_{n-1}, x^*)] \preceq \cdots \preceq$$
$$Ld(w_{n+1}, t(w_n)) + KLd(w_n, t(y_{w-1})) + \cdots + K^n Ld(w_1, t(w_0)) + K^{n+1}d(w_0, x^*).$$

By the vectorial version of the Cauchy-Toeplitz Lemma (see e.g. [7], [14]) we get the conclusion. □

As consequences, for each example of operator A we can get corresponding results concerning the stability properties of the fixed point equation.

For example, in the case of the convex structure of Gudder we obtain the following result for a single-valued graph contraction in a complete vector-valued metric space.

Theorem 6 *Let (X, d) be a complete metric space endowed with a convex structure in the sense of Gudder $F : [0, 1] \times X \times X \to X$. Let $t : X \to X$ be a single-valued operator with closed graph. Suppose that:*

(a) there exists a matrix $\kappa \in M_{m,m}(\mathbb{R}_+)$ which converges to zero such that

$$d(F(\lambda, x, t(x)), F(\lambda, F(\lambda, x, t(x)), t(F(\lambda, x, t(x))))) \preceq$$

$\kappa d(x, F(\lambda, x, t(x)))$, *for every* $x \in X$ *and* $\lambda \in (0, 1)$.

(b) there exists $Q \in M_{m,m}(\mathbb{R}_+) \setminus \{O_{m,m}\}$ *such that* $d(x, F(\lambda, x, t(x)) \leq Qd(x, t(x))$*, for each* $x \in X$ *and* $\lambda \in (0, 1)$*;*

(c) if $\{x_n\}_{n\in\mathbb{N}}, \{v_n\}_{n\in\mathbb{N}}$ *are two sequences in* X *such that* $x_n \to x$ *and if the sequence* $u_n := F(\lambda, x_n, v_n), n \in \mathbb{N}$ *is convergent in* X *to* u*, then there exists* $v \in X$ *such that* $v_n \to v$ *as* $n \to \infty$ *and* $u = F(\lambda, x, v)$.

Then, the following conclusions hold:

(i) t_A *is a* K*-weakly Picard operator, with* $K := (I_{m.m} - \kappa)^{-1}$*;*

(ii) the fixed point equation $x = t(x), x \in X$ *is well-posed and has the Ulam-Hyers stability property.*

Proof. By (a) we get that the admissible perturbation t_A of t is a graph κ-contraction. By (c) we obtain that t_A has closed graph. By Exemple 4 we obtain the conclusion (i). The conclusion (ii) follows by Theorem 4. □

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