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# DIFFERENCE DOUBLE SEQUENCES OF BI-COMPLEX NUMBERS*

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**Abstract**

In this article, we have introduced the notion of convergence of difference double sequences in Pringsheim's sense, difference null in Pringsheim's sense, bounded difference, bounded convergence difference, bounded null difference, regular convergence difference and regular null difference double sequences of bi-complex numbers. We have proved that these are linear spaces. With the help of the Euclidean norm defined on bi-complex numbers, we have established their different algebraic and topological properties, as well as some of their geometric properties. Suitable examples have been discussed to support the introduction of these classes of sequences and during the investigation of their properties for failure cases.

**Keywords:** double sequence, bi-complex numbers, Orlicz function, solid, symmetric.

**MSC:** 40A05, 40C05, 46A45, 46B45.

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## 1 Introduction

In 1892, Segre [15] introduced the bi-complex numbers. The most comprehensive study of bi-complex numbers is done by Price [12]. Later on, Wagh

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[21], Değirmen and Sağir [4], Bera and Tripathy [1, 2], Sager and Sağir [16] and many researchers have studied some algebraic, topological and geometrical properties of bi-complex sequence spaces.
In this section, we procure detailed preliminaries on bi-complex numbers.
Throughout $\mathbb{C}_0$, $\mathbb{C}_1$ and $\mathbb{C}_2$ denote the set of real, complex and bi-complex numbers and $\theta$, the zero elements of $\mathbb{C}_2$.
Segre [15] defined the bi-complex numbers as follows:

$$\begin{aligned}\gamma &= u_1 + i_2 u_2 \\ &= w + i_1 x + i_2 y + i_1 i_2 z,\end{aligned}$$

where $u_1, u_2 \in \mathbb{C}_1; w, x, y, z \in \mathbb{C}_0$, and $i_1, i_2$ are two distinct imaginary unit whose square is $-1$, and $i_1 i_2$ is hyperbolic unit whose square is 1.
The set of bi-complex numbers is denoted by $\mathbb{C}_2$ and defined by

$$\mathbb{C}_2 = \{w + i_1 x + i_2 y + i_1 i_2 z : w, x, y, z \in \mathbb{C}_0\}.$$

There are three types of conjugations of bi-complex numbers defined by Rochon and Shapiro [14] as follows:
(i) $i_1$-conjugation of bi-complex number $\gamma$ is $\gamma^* = \overline{u_1} + i_2 \overline{u_2}$, for all $u_1, u_2 \in \mathbb{C}_1$ and $\overline{u_1}, \overline{u_2}$ are complex conjugates of $u_1$, $u_2$ respectively.
(ii) $i_2$-conjugation of bi-complex number $\gamma$ is $\tilde{\gamma} = u_1 - i_2 u_2$, for all $u_1, u_2 \in \mathbb{C}_1$.
(iii) $i_1 i_2$-conjugation of bi-complex number $\gamma$ is $\gamma^{'} = \overline{u_1} - i_2 \overline{u_2}$, for all $u_1, u_2 \in \mathbb{C}_1$ and $\overline{u_1}, \overline{u_2}$ are complex conjugates of $u_1$, $u_2$ respectively.
A bi-complex number $\gamma = u_1 + i_2 u_2$ is called hyperbolic if $\gamma' = \gamma$, the $i_1 i_2$-conjugation of $\gamma$.
The set of all hyperbolic elements is denoted by $\mathcal{H}$ and defined by

$$\mathcal{H} = \{w + i_1 i_2 z : w, z \in \mathbb{C}_0\}.$$

A bi-complex number $\gamma = u_1 + i_1 u_2$ is called singular if $|u_1^2 + u_2^2| = 0$ and otherwise it is called non-singular.
In $\mathbb{C}_2$, there are exactly two non-trivial idempotent elements $e_1$ and $e_2$, where

$$e_1 = \frac{1 + i_1 i_2}{2} \text{ and } e_2 = \frac{1 - i_1 i_2}{2}.$$

Obviously, $e_1 + e_2 = 1$ and $e_1 \cdot e_2 = e_2 \cdot e_1 = 0$.
Every bi-complex number $\gamma = u_1 + i_2 u_2$ has a unique idempotent representation as
$\gamma = \mu_1 e_1 + \mu_2 e_2$, where $\mu_1 = u_1 - i_1 u_2$ and $\mu_2 = u_1 + i_1 u_2$ are called the

idempotent components of $\gamma$.
Norm (Euclidean Norm) on $\mathbb{C}_2$ is defined by

$$\|\gamma\|_{\mathbb{C}_2} = \sqrt{w^2 + x^2 + y^2 + z^2} = \sqrt{|u_1|^2 + |u_2|^2} = \sqrt{\frac{|\mu_1|^2 + |\mu_2|^2}{2}}.$$

$\mathbb{C}_2$ becomes a modified Banach algebra concerning this norm in the sense that

$$\|\gamma \cdot t\|_{\mathbb{C}_2} \leq \sqrt{2}\|\gamma\|_{\mathbb{C}_2} \cdot \|t\|_{\mathbb{C}_2}.$$

The set $\mathbb{C}_2$ is a Banach space w.r.t. the Euclidean norm.
The notion of the double sequence was introduced by Pringsheim [13]. It is also found in Bromwich [3]. Later on, Hardy [6] introduced the notion of regular convergence of double sequences and Tripathy and Sarma [18, 19, 20], Móricz [11], and many researchers have studied their different properties. A double sequence $x = (x_{lm})$ is said to be bounded if $\|x\| = \sup_{l,m\geq 0} |x_{lm}| < \infty$. The symbol ${}_2\ell_\infty$ stands for the set of all bounded double sequences.
A double sequence $(x_{lm})$ of real or complex terms is said to converge in Pringsheim's sense to $K$, if for each $\varepsilon > 0$, $\exists$ a natural number $n_0$ such that $|x_{lm} - K| < \varepsilon$, for all $l, m \geq n_0$, which is written as

$$P - \lim_{\substack{l\to\infty \\ m\to\infty}} x_{lm} = K, \text{ where } l, m \to \infty, \text{ independent to one another.}$$

A double sequence $x = (x_{lm})$ is said to converge regularly (introduced by Hardy [6]) if it converges in Pringsheim's sense and the following limits exist,

$$\lim_{l\to\infty} x_{lm} = P_m, \text{ exists for each } m \in \mathbb{N}, \& \lim_{m\to\infty} x_{lm} = Q_l, \text{ exists for each } l \in \mathbb{N}.$$

Kizmaz introduced the notion of difference for single sequence spaces [8] as follows:

$$Z(\Delta) = \{(x_l) \in \omega : (\Delta x_l) \in Z\},$$

for $Z = c, c_0$ and $\ell_\infty$, where $\Delta x_l = x_l - x_{l+1}$ for all $l \in \mathbb{N}$. The above spaces are Banach spaces normed by

$$\|(x_l)\|_\Delta = |x_1| + \sup_{l\geq 1} |\Delta x_l|.$$

Later on, the notion was further investigated by Tripathy [5] and many others. Tripathy and Sarma [17, 18] introduced the notion of difference double sequence spaces defined over the seminormed space $(X, q)$.

$$Z(\Delta, q) = \{(\gamma_{lm}) \in {}_2\omega(q) : (\Delta\gamma_{lm}) \in Z(q)\},$$

where $Z = {}_2\ell_\infty, {}_2c, {}_2c_0, {}_2c^R, {}_2c_0^R, {}_2c^B, {}_2c_0^B$ and $\Delta\gamma_{lm} = \gamma_{lm} - \gamma_{l+1,m} - \gamma_{l,m+1} + \gamma_{l+1,m+1}$, for all $l, m \in \mathbb{N}$.

## 2 Definitions and preliminaries

In this section, we procure the existing definitions that will be used and introduce the meaning of this article.

**Definition 1** *[10] A double sequence space $E$ of bi-complex numbers is said to be solid if $(\alpha_{lm}\gamma_{lm}) \in E$, whenever $(\gamma_{lm}) \in E$, for all double sequences $(\alpha_{lm})$ of scalars with $|\alpha_{lm}| \leq 1$, for all $l, m \in \mathbb{N}$.*

**Definition 2** *[10] A double sequence space $E$ of bi-complex numbers is said to be symmetric if $(\gamma_{lm}) \in E \Longrightarrow (\gamma_{\pi(l,m)}) \in E$, where $\pi$ is the permutation of $\mathbb{N} \times \mathbb{N}$.*

**Definition 3** *[10] Let $K = \{(l_i, m_j) : i, j \in \mathbb{N}; l_1 < l_2 < .... \text{ and } m_1 < m_2 < ....\} \subseteq \mathbb{N} \times \mathbb{N}$ and $E$ be a double sequence space. A $K$-step space of $E$ is a sequence space*

$$\lambda_K^E = \{(\gamma_{l_i m_j}) \in {}_2\omega(\mathbb{C}_2) : (\gamma_{lm}) \in E\}.$$

*A canonical pre-image of a sequence $(\gamma_{lm}) \in E$ is defined as follows:*

$$t_{lm} = \begin{cases} \gamma_{lm}, & \text{if } (l,m) \in K; \\ \theta, & \text{otherwise.} \end{cases}$$

*A canonical pre-image of a step space $\lambda_K^E$ is a set of canonical pre-images of all elements in $\lambda_K^E$.*

**Definition 4** *[10] A double sequence space $E$ is monotone if it contains the canonical pre-images of all its step spaces.*

**Remark 1** *[7] "If a sequence space $E$ is solid, then sequence space $E$ is monotone."*

**Definition 5** *[10] A double sequence space $E$ is said to be convergence free if $(t_{lm}) \in E$, whenever $(\gamma_{lm}) \in E$ and $t_{lm} = \theta$, whenever $\gamma_{lm} = \theta$, where $\theta$ is the zero element of $\mathbb{C}_2$.*

**Definition 6** *[10] A double sequence space $E$ is said to be a sequence algebra if $(\gamma_{lm}) \star (t_{lm}) = (\gamma_{lm} t_{lm}) \in E$, whenever $(\gamma_{lm}), (t_{lm}) \in E$.*

In this article, we consider the termwise product of the sequences.

**Definition 7** *[9] Let $E$ be a subset of the linear space $X$. Then $E$ is said to be convex if $(1-\lambda)(\gamma_{lm}) + \lambda(t_{lm}) \in E$ for all $(\gamma_{lm}), (t_{lm}) \in E$ and scalar $\lambda \in [0,1]$.*

**Definition 8** *[9] A Banach space $X$ is said to be strictly convex if $(\gamma_{lm}), (t_{lm}) \in S_X$ with $(\gamma_{lm}) \neq (t_{lm})$ implies that $\|\lambda(\gamma_{lm}) + (1-\lambda)(t_{lm})\|_X < 1$, for all $\lambda \in (0,1)$, where $S_X$ is unit sphere.*

We introduce the difference double sequence of bi-complex numbers as follows:

$$U[\Delta, \|\cdot\|_{\mathbb{C}_2}] = \{(\gamma_{lm}) \in {}_2\omega(\mathbb{C}_2) : (\Delta\gamma_{lm}) \in U\},$$

where $U = {}_2\ell_\infty(\mathbb{C}_2), {}_2c(\mathbb{C}_2), {}_2c_0(\mathbb{C}_2), {}_2c^R(\mathbb{C}_2), {}_2c_0^R(\mathbb{C}_2), {}_2c^B(\mathbb{C}_2), {}_2c_0^B(\mathbb{C}_2)$ and $\Delta\gamma_{lm} = \gamma_{lm} - \gamma_{l+1,m} - \gamma_{l,m+1} + \gamma_{l+1,m+1}$, for all $l, m \in \mathbb{N}$.

**Lemma 1** *[10] Every regular convergent double sequence of bi-complex numbers is always bounded.*

## 3 Main results

In this section, we establish the results of this article.

**Theorem 1** *The classes $U[\Delta, \|\cdot\|_{\mathbb{C}_2}]$, where $U = {}_2\ell_\infty(\mathbb{C}_2), {}_2c(\mathbb{C}_2), {}_2c_0(\mathbb{C}_2)$, ${}_2c^R(\mathbb{C}_2)$, ${}_2c_0^R(\mathbb{C}_2), {}_2c^B(\mathbb{C}_2), {}_2c_0^B(\mathbb{C}_2)$ are linear spaces.*

**Proof.** We established the theorem for the case ${}_2\ell_\infty(\mathbb{C}_2)[\Delta, \|\cdot\|_{\mathbb{C}_2}]$.
Let $(\gamma_{lm}), (t_{lm})$ be two arbitrary double sequence of bi-complex numbers in ${}_2\ell_\infty(\mathbb{C}_2)[\Delta, \|\cdot\|_{\mathbb{C}_2}]$. Then we have

$$\sup_{l,m} \|(\Delta\gamma_{lm})\|_{\mathbb{C}_2} < \infty \text{ and } \sup_{l,m} \|(\Delta t_{lm})\|_{\mathbb{C}_2} < \infty.$$

Let $a, b \in \mathbb{C}_0$, then

$$\begin{aligned}
\sup_{l,m} \|(\Delta\{a\gamma_{lm} + bt_{lm}\})\|_{\mathbb{C}_2} &< \sup_{l,m} \|(\Delta\{a\gamma_{lm}\} + \Delta\{bt_{lm}\})\|_{\mathbb{C}_2} \\
&\leq \sup_{l,m} \|(\Delta\{a\gamma_{lm}\})\|_{\mathbb{C}_2} + \sup_{l,m} \|(\Delta\{bt_{lm}\})\|_{\mathbb{C}_2} \\
&\leq |a| \sup_{l,m} \|(\Delta\gamma_{lm})\|_{\mathbb{C}_2} + |b| \sup_{l,m} \|(\Delta t_{lm})\|_{\mathbb{C}_2} \\
&< \infty.
\end{aligned}$$

Therefore, the double sequence $(a\gamma_{lm} + bt_{lm}) \in {}_2\ell_\infty(\mathbb{C}_2)[\Delta, \|\cdot\|_{\mathbb{C}_2}]$. Hence, ${}_2\ell_\infty(\mathbb{C}_2)[\Delta, \|\cdot\|_{\mathbb{C}_2}]$ is a linear space. Similarly, the other cases can be established.

**Theorem 2** *The classes* $U[\Delta, \|\cdot\|_{\mathbb{C}_2}]$, *where* $U = {}_2\ell_\infty(\mathbb{C}_2), {}_2c^R(\mathbb{C}_2), {}_2c_0^R(\mathbb{C}_2), {}_2c^B(\mathbb{C}_2), {}_2c_0^B(\mathbb{C}_2)$ *are Banach spaces under the norm* $\|\cdot\|_\Delta$ *defined by*

$$\|(\gamma_{lm})\|_\Delta = \sup_l \|\gamma_{l,1}\|_{\mathbb{C}_2} + \sup_m \|\gamma_{1,m}\|_{\mathbb{C}_2} + \sup_{l,m} \|\Delta\gamma_{lm}\|_{\mathbb{C}_2} < \infty.$$

**Proof.** We establish the result for the space ${}_2\ell_\infty(\mathbb{C}_2)[\Delta, \|\cdot\|_{\mathbb{C}_2}]$.
Let $(A^i)$ be a Cauchy sequence in ${}_2\ell_\infty(\mathbb{C}_2)[\Delta, \|\cdot\|_{\mathbb{C}_2}]$, where $A^i = (\gamma_{lm}^i)$ for each $i \in \mathbb{N}$. Thus for a given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$. such that

$$\|A^i - A^j\|_\Delta < \varepsilon, \text{ for all } i, j \geq n_0.$$

Then

$$\sup_l \|\gamma_{l,1}^i - \gamma_{l,1}^j\|_{\mathbb{C}_2} + \sup_m \|\gamma_{1,m}^i - \gamma_{1,m}^j\|_{\mathbb{C}_2} + \sup_{l,m} \|\Delta\gamma_{lm}^i - \Delta\gamma_{lm}^j\|_{\mathbb{C}_2} < \varepsilon, \forall\, i, j \geq n_0.$$

Therefore, $\|\gamma_{lm}^i - \gamma_{lm}^j\|_{\mathbb{C}_2} < \varepsilon,$ for all $i, j \geq n_0,$ for each $l, m \in \mathbb{N}$.
Hence, $(\gamma_{lm}^i)$ is a Cauchy sequence in $\mathbb{C}_2$, for each $l, m \in \mathbb{N}$.
Thus, the double sequence $(\gamma_{lm}^i)$ converges $\gamma_{lm} \in \mathbb{C}_2$ (say), for each $l, m \in \mathbb{N}$.
i.e., there exists

$$\lim_{i\to\infty} \gamma_{lm}^i = \gamma_{lm}, \text{ for each } l, m \in \mathbb{N}.$$

Further for each $\varepsilon > 0$, there exists $n_0 = n_0(\varepsilon)$, such that for all $i, j \geq n_0$ and for all $l, m \in \mathbb{N}$.
$\|\gamma_{l,1}^i - \gamma_{l,1}^j\|_{\mathbb{C}_2} < \varepsilon, \quad \|\gamma_{1,m}^i - \gamma_{1,m}^j\|_{\mathbb{C}_2} < \varepsilon, \quad \|\Delta\gamma_{lm}^i - \Delta\gamma_{lm}^j\|_{\mathbb{C}_2} < \varepsilon$, and

$$\lim_{j\to\infty} \|\gamma_{l,1}^i - \gamma_{l,1}^j\|_{\mathbb{C}_2} = \|\gamma_{l,1}^i - \gamma_{l,1}\|_{\mathbb{C}_2} < \varepsilon, \text{ for all } i > n_0, \text{ for each } l \in \mathbb{N}.$$

$$\lim_{j\to\infty} \|\gamma_{1,m}^i - \gamma_{1,m}^j\|_{\mathbb{C}_2} = \|\gamma_{1,m}^i - \gamma_{1,m}\|_{\mathbb{C}_2} < \varepsilon, \text{ for all } i > n_0, \text{ for each } m \in \mathbb{N}.$$

$$\text{Now,} \quad \lim_{j\to\infty} \|\Delta\gamma_{lm}^i - \Delta\gamma_{lm}^j\|_{\mathbb{C}_2} = \|\Delta\gamma_{lm}^i - \Delta\gamma_{lm}\|_{\mathbb{C}_2} < \varepsilon,$$

for all $i > n_0,$ for each $l, m \in \mathbb{N}$. Since $\varepsilon$ does not depend on $l, m$;

$$\sup_{l,m} \|\Delta\gamma_{lm}^i - \Delta\gamma_{lm}\|_{\mathbb{C}_2} < \varepsilon, \text{ for all } i > n_0.$$

Consequently we have, $\|\gamma_{lm}^i - \gamma_{lm}\|_{\mathbb{C}_2} < 3\varepsilon,$ for all $i > n_0$.
Hence, we obtain $\gamma_{lm}^i \to \gamma_{lm}$ as $i \to \infty$ in ${}_2\ell_\infty(\mathbb{C}_2)[\Delta, \|\cdot\|_{\mathbb{C}_2}]$. Now we have to show that $(\gamma_{lm}) \in {}_2\ell_\infty(\mathbb{C}_2)[\Delta, \|\cdot\|_{\mathbb{C}_2}]$,

$$\begin{aligned}
&\|\gamma_{lm} - \gamma_{l+1,m+1}\|_{\mathbb{C}_2} \\
&= \|\gamma_{lm} - \gamma_{lm}^k + \gamma_{lm}^k - \gamma_{l+1,m+1}^k + \gamma_{l+1,m+1}^k - \gamma_{l+1,m+1}\|_{\mathbb{C}_2} \\
&\leq \|\gamma_{lm} - \gamma_{lm}^k\|_{\mathbb{C}_2} + \|\gamma_{lm}^k - \gamma_{l+1,m+1}^k\|_{\mathbb{C}_2} + \|\gamma_{l+1,m+1}^k - \gamma_{l+1,m+1}\|_{\mathbb{C}_2} \\
&= O(1).
\end{aligned}$$

This implies, $(\gamma_{lm}) \in {}_2\ell_\infty(\mathbb{C}_2)[\Delta, \|\cdot||_{\mathbb{C}_2}]$.
Since, ${}_2\ell_\infty(\mathbb{C}_2)[\Delta, \|\cdot||_{\mathbb{C}_2}]$ is linear space. Hence, ${}_2\ell_\infty(\mathbb{C}_2)[\Delta, \|\cdot||_{\mathbb{C}_2}]$ is complete. Similarly, the others.

**Corollary 1** *The spaces* $U[\Delta, \|\cdot||_{\mathbb{C}_2}]$, *where* $U = {}_2\ell_\infty(\mathbb{C}_2), {}_2c(\mathbb{C}_2), {}_2c_0(\mathbb{C}_2), {}_2c^R(\mathbb{C}_2), {}_2c_0^R(\mathbb{C}_2), {}_2c^B(\mathbb{C}_2), {}_2c_0^B(\mathbb{C}_2)$ *are not monotone.*

**Proof.** The spaces are not monotone, as shown in the following Example.

**Example 1** *Consider the double sequence* $(\gamma_{lm})$ *of bi-complex numbers defined by*

$$\gamma_{lm} = e_1 + e_2, \text{ for all } l, m \in \mathbb{N}.$$

*Consider the sequence* $(t_{lm})$ *in the pre-image space defined by*

$$t_{lm} = \begin{cases} \gamma_{lm}, & \text{for } l = m = i^2, i \in \mathbb{N}; \\ \theta, & \text{otherwise} . \end{cases}$$

*Then* $(\gamma_{lm}) \in U[\Delta, \|\cdot||_{\mathbb{C}_2}]$ *but* $(t_{lm}) \notin U[\Delta, \|\cdot||_{\mathbb{C}_2}]$ *for* $U = {}_2c(\mathbb{C}_2), {}_2c_0(\mathbb{C}_2), {}_2c^R(\mathbb{C}_2), {}_2c_0^R(\mathbb{C}_2), {}_2c^B(\mathbb{C}_2), {}_2c_0^B(\mathbb{C}_2)$. *Hence* $U[\Delta, \|\cdot||_{\mathbb{C}_2}]$ *for* $U = {}_2c(\mathbb{C}_2), {}_2c_0(\mathbb{C}_2), {}_2c^R(\mathbb{C}_2), {}_2c_0^R(\mathbb{C}_2), {}_2c^B(\mathbb{C}_2), {}_2c_0^B(\mathbb{C}_2)$ *are not monotone.*

**Example 2** *For the space* ${}_2\ell_\infty(\mathbb{C}_2)[\Delta, \|\cdot||_{\mathbb{C}_2}]$. *Consider the double sequence* $(\gamma_{lm})$ *of bi-complex numbers defined by*

$$\gamma_{lm} = (l + m)\{e_1 + e_2\}, \text{ for all } l, m \in \mathbb{N}.$$

*Consider the sequence* $(t_{lm})$ *in the pre-image space defined by*

$$t_{lm} = \begin{cases} \gamma_{lm}, & \text{for } l = m = i^2, i \in \mathbb{N}; \\ \theta, & \text{otherwise} . \end{cases}$$

*Then,* $(\gamma_{lm}) \in {}_2\ell_\infty(\mathbb{C}_2)[\Delta, \|\cdot||_{\mathbb{C}_2}]$, *but* $(t_{lm}) \notin {}_2\ell_\infty(\mathbb{C}_2)[\Delta, \|\cdot||_{\mathbb{C}_2}]$. *Hence,* ${}_2\ell_\infty(\mathbb{C}_2)[\Delta, \|\cdot||_{\mathbb{C}_2}]$ *is not monotone.*

**Corollary 2** *The spaces* $U[\Delta, \|\cdot||_{\mathbb{C}_2}]$, *where* $U = {}_2\ell_\infty(\mathbb{C}_2), {}_2c(\mathbb{C}_2), {}_2c_0(\mathbb{C}_2), {}_2c^R(\mathbb{C}_2), {}_2c_0^R(\mathbb{C}_2), {}_2c^B(\mathbb{C}_2), {}_2c_0^B(\mathbb{C}_2)$ *are not solid.*

**Proof.** The proof follows from the Remark 1.

**Corollary 3** *The spaces* $U[\Delta, \|\cdot||_{\mathbb{C}_2}]$, *where* $U = {}_2\ell_\infty(\mathbb{C}_2), {}_2c(\mathbb{C}_2), {}_2c_0(\mathbb{C}_2), {}_2c^R(\mathbb{C}_2), {}_2c_0^R(\mathbb{C}_2), {}_2c^B(\mathbb{C}_2), {}_2c_0^B(\mathbb{C}_2)$ *are not symmetric.*

**Proof.** The result follows from the following Example.

**Example 3** *Consider the double sequence $(\gamma_{lm})$ of bi-complex numbers defined by*

$$\gamma_{lm} = \begin{cases} e_1 + e_2, & \text{for all } l = 1 \text{ and } m \in \mathbb{N}; \\ e_1 e_2, & \text{otherwise .} \end{cases}$$

*Then, $(\gamma_{lm}) \in U[\Delta, \|\cdot||_{\mathbb{C}_2}]$ for $U = {}_2c(\mathbb{C}_2), {}_2c_0(\mathbb{C}_2), {}_2c^R(\mathbb{C}_2),\ {}_2c_0^R(\mathbb{C}_2), {}_2c^B(\mathbb{C}_2), {}_2c_0^B(\mathbb{C}_2)$.*
*Consider the rearranged double sequence $(t_{lm})$ of bi-complex numbers of $(\gamma_{lm})$ defined by*

$$t_{lm} = \begin{cases} e_1 + e_2, & \text{for } l = m, \text{ and } l, m \in \mathbb{N}; \\ \theta, & \text{otherwise .} \end{cases}$$

$$\Delta t_{lm} = \begin{cases} 2(e_1 + e_2), & \text{for } l = m, \text{ for all } l, m \in \mathbb{N}; \\ -(e_1 + e_2), & \text{for } l = m + 1 \text{ and } m = l + 1, \text{ and } l, m \in \mathbb{N}; \\ \theta, & \text{otherwise .} \end{cases}$$

*Then, $(t_{lm}) \notin U[\Delta, \|\cdot\|_{\mathbb{C}_2}]$. Hence the spaces $U[\Delta, \|\cdot\|_{\mathbb{C}_2}]$ for $U = {}_2c(\mathbb{C}_2), {}_2c_0(\mathbb{C}_2), {}_2c^R(\mathbb{C}_2),\ {}_2c_0^R(\mathbb{C}_2), {}_2c^B(\mathbb{C}_2), {}_2c_0^B(\mathbb{C}_2)$ are not symmetric.*

**Example 4** *Consider the double sequence $(\gamma_{lm})$ of bi-complex numbers defined by*

$$\gamma_{lm} = \begin{cases} (e_1 + e_2)l, & \text{for } m = 1 \text{ and all } l \in \mathbb{N}; \\ e_1 e_2, & \text{otherwise .} \end{cases}$$

*Then, $(\gamma_{lm}) \in {}_2\ell_\infty(\mathbb{C}_2)[\Delta, \|\cdot||_{\mathbb{C}_2}]$. Let $(t_{lm})$ be rearangement of $(\gamma_{lm})$ defined by*

$$t_{lm} = \begin{cases} i\{e_1 + e_2\}, & \text{for } l = m \text{ and } l = i^2, i \in \mathbb{N}; \\ \theta, & \text{otherwise .} \end{cases}$$

$$\Delta t_{lm} = \begin{cases} (e_1 + e_2)i, & \text{for } l = m, \text{ and } l = i^2, i \in \mathbb{N}; \\ -(e_1 + e_2)i, & \text{for } l = m - 1 \text{ and } m = l - 1, \text{ and } l = i^2, l, m \in \mathbb{N}; \\ \theta, & \text{otherwise .} \end{cases}$$

*Then, $(t_{lm}) \notin {}_2\ell_\infty(\mathbb{C}_2)[\Delta, \|\cdot||_{\mathbb{C}_2}]$. Hence the space ${}_2\ell_\infty(\mathbb{C}_2)[\Delta, \|\cdot||_{\mathbb{C}_2}]$ is not symmetric.*

**Corollary 4** *The spaces $U[\Delta, \|\cdot||_{\mathbb{C}_2}]$ where $U = {}_2\ell_\infty(\mathbb{C}_2), {}_2c(\mathbb{C}_2), {}_2c_0(\mathbb{C}_2), {}_2c^R(\mathbb{C}_2), {}_2c_0^R(\mathbb{C}_2),\ {}_2c^B(\mathbb{C}_2), {}_2c_0^B(\mathbb{C}_2)$ are not convergence free.*

**Example 5** *Consider the double sequence $(\gamma_{lm})$ of bi-complex numbers defined by*

$$\gamma_{lm} = \begin{cases} (e_1 + e_2), & \text{if } m = 1 \text{ and for all } l \in \mathbb{N}; \\ (e_1 e_2), & \text{otherwise .} \end{cases}$$

*Then the sequence* $(\gamma_{lm}) \in {}_2c(\mathbb{C}_2)[\Delta, \|\cdot\|_{\mathbb{C}_2}]$.
*Construct the double sequence* $(t_{lm})$ *of bi-complex numbers by*

$$t_{lm} = \begin{cases} (e_1 + e_2)l, & \text{if } m = l; \\ (e_1 e_2), & \text{otherwise }. \end{cases}$$

*Clearly,* $(t_{lm}) \notin {}_2c(\mathbb{C}_2)[\Delta, \|\cdot\|_{\mathbb{C}_2}]$. *Hence,* ${}_2c(\mathbb{C}_2)[\Delta, \|\cdot\|_{\mathbb{C}_2}]$ *is not convergence free.*

**Theorem 3** *The spaces* $U[\Delta, \|\cdot\|_{\mathbb{C}_2}]$, *where* $U = {}_2\ell_\infty(\mathbb{C}_2), {}_2c(\mathbb{C}_2), {}_2c_0(\mathbb{C}_2)$, ${}_2c^R(\mathbb{C}_2)$, ${}_2c_0^R(\mathbb{C}_2), {}_2c^B(\mathbb{C}_2), {}_2c_0^B(\mathbb{C}_2)$ *are sequence algebras.*

**Proof.** The conclusion comes from the inequality given below.
Let the double sequences $(\gamma_{lm}), (t_{lm}) \in {}_2c_0^R(\mathbb{C}_2)[\Delta, \|\cdot\|_{\mathbb{C}_2}]$ of bi-complex numbers. Then for given $\varepsilon > 0$, there exists $k_1, k_2, k_3 \in \mathbb{N}$ such that

$$\begin{aligned} \|\gamma_{lm}\|_{\mathbb{C}_2} &< \varepsilon, \text{ for all } l > k_1 \text{ and all } m \in \mathbb{N}; \\ \|\gamma_{lm}\|_{\mathbb{C}_2} &< \varepsilon, \text{ for all } m > k_2 \text{ and all } l \in \mathbb{N}; \\ \text{and } \|\gamma_{lm}\|_{\mathbb{C}_2} &< \varepsilon, \text{ for all } l > k_3 \text{ and } m > k_3, \end{aligned}$$

and

$$\begin{aligned} \|t_{lm}\|_{\mathbb{C}_2} &< \varepsilon, \text{ for all } l > k_1 \text{ and all } m \in \mathbb{N}; \\ \|t_{lm}\|_{\mathbb{C}_2} &< \varepsilon, \text{ for all } m > k_2 \text{ and all } l \in \mathbb{N}; \\ \text{and } \|t_{lm}\|_{\mathbb{C}_2} &< \varepsilon, \text{ for all } l > k_3 \text{ and } m > k_3. \end{aligned}$$

Thus, we have for the product (termwise) of the two sequences

$$\begin{aligned} \|\gamma_{lm}t_{lm}\|_{\mathbb{C}_2} &< \varepsilon, \text{ for all } l > k_1 \text{ and all } m \in \mathbb{N}; \\ \|\gamma_{lm}t_{lm}\|_{\mathbb{C}_2} &< \varepsilon, \text{ for all } m > k_2 \text{ and all } l \in \mathbb{N}; \\ \text{and } \|\gamma_{lm}t_{lm}\|_{\mathbb{C}_2} &< \varepsilon, \text{ for all } l > k_3 \text{ and } m > k_3. \end{aligned}$$

Clearly, $(\gamma_{lm}) \star (t_{lm}) \in {}_2c_0^R(\mathbb{C}_2)[\Delta, \|\cdot\|_{\mathbb{C}_2}]$. Hence, the double sequence space ${}_2c_0^R(\mathbb{C}_2)[\Delta, \|\cdot\|_{\mathbb{C}_2}]$ is a sequence algebra.
Similarly, the other cases can be established.

**Theorem 4** *The inclusion relation* $U \subset U_0[\Delta, \|\cdot\|_{\mathbb{C}_2}]$, *for* $U = {}_2c(\mathbb{C}_2), {}_2c^R(\mathbb{C}_2)$, ${}_2c^B(\mathbb{C}_2)$ *holds and is strict.*

**Proof.** Let $(\gamma_{lm}) \in {}_2c(\mathbb{C}_2)$. Then for a given $\varepsilon > 0$, there exists $n_0, k_0 \in \mathbb{N}$ such that

$$\|\gamma_{lm} - L\|_{\mathbb{C}_2} < \frac{\varepsilon}{4}, \text{ for all } l \geq n_0, m \geq k_0. \tag{1}$$

Hence, for all $l \geq n_0, m \geq k_0$,

$$\begin{aligned}
&\|(\Delta\gamma_{lm})\|_{\mathbb{C}_2} \\
&\leq \|\gamma_{lm} - L\|_{\mathbb{C}_2} + \|\gamma_{l,m+1} - L\|_{\mathbb{C}_2} + \|\gamma_{l+1,m} - L\|_{\mathbb{C}_2} + \|\gamma_{l+1,m+1} - L\|_{\mathbb{C}_2} \\
&< \varepsilon, \text{ by Equation (1)} . \qquad (2)
\end{aligned}$$

Thus, $(\Delta\gamma_{lm}) \in {}_2c_0(\mathbb{C}_2)$. Hence the sequence $(\gamma_{lm}) \in {}_2c_0(\mathbb{C}_2)[\Delta, \|\cdot\|_{\mathbb{C}_2}]$. Similarly, the other cases can be established.

**Example 6** *Consider the double sequence $(\gamma_{lm})$ of bi-complex numbers defined by*

$$\gamma_{lm} = (l + m - 1)(e_1 + e_2), \text{ for all } l, m \in \mathbb{N}.$$

*Then $\Delta\gamma_{lm} = \theta$, for all $l, m \in \mathbb{N}$.*
*Hence, $(\gamma_{lm}) \in {}_2c_0^R(\Delta, \|\cdot\|_{\mathbb{C}_2}) \subset {}_2c_0(\Delta, \|\cdot\|_{\mathbb{C}_2})$, but $(\gamma_{lm}) \notin {}_2c(\|\cdot\|_{\mathbb{C}_2})$.*

**Theorem 5** *The inclusion $U \subset U[\Delta, \|\cdot\|_{\mathbb{C}_2}]$, for $U = {}_2\ell_\infty(\mathbb{C}_2), {}_2c(\mathbb{C}_2), {}_2c_0(\mathbb{C}_2)$, ${}_2c^R(\mathbb{C}_2)$, ${}_2c_0^R(\mathbb{C}_2), {}_2c^B(\mathbb{C}_2), {}_2c_0^B(\mathbb{C}_2)$ holds and is strict.*

**Proof.** The Pringsheim's sense convergence part for the inclusion follows from the Equation (2). Row-wise and column-wise convergence can be established. The inclusion is strict follows from the following Examples.

**Example 7** *Consider the double sequence $(\gamma_{lm})$ of bi-complex numbers defined by*

$$\gamma_{lm} = \begin{cases} (e_1 + e_2)l, & \text{if } l \text{ is odd and all } m \in \mathbb{N}; \\ (e_1 + e_2)m, & \text{otherwise} . \end{cases}$$

*Then*

$$\Delta\gamma_{lm} = \begin{cases} (e_1 + e_2), & \text{if } l \text{ is odd and all } m \in \mathbb{N}; \\ -(e_1 + e_2), & \text{otherwise} . \end{cases}$$

*Thus, $(\Delta\gamma_{lm}) \in {}_2\ell_\infty(\mathbb{C}_2)$. Hence the sequence $(\gamma_{lm}) \in {}_2\ell_\infty(\mathbb{C}_2)[\Delta, \|\cdot\|_{\mathbb{C}_2}]$, but $(\gamma_{lm}) \notin {}_2\ell_\infty(\mathbb{C}_2)$.*

**Example 8** *Consider the double sequence $(\gamma_{lm})$ of bi-complex numbers defined by*

$$\gamma_{lm} = (e_1 + e_2), \text{ for all } l, m \in \mathbb{N}.$$

*Then*

$$\Delta\gamma_{lm} = \theta, \text{ for all } l, m \in \mathbb{N}.$$

*Thus, $(\Delta\gamma_{lm}) \in U$, for $U = {}_2c_0^R(\mathbb{C}_2), {}_2c_0^B(\mathbb{C}_2)$. Hence the sequence $(\gamma_{lm}) \in U[\Delta, \|\cdot\|_{\mathbb{C}_2}]$, for $U = {}_2c_0^R(\mathbb{C}_2), {}_2c_0^B(\mathbb{C}_2)$, but $(\gamma_{lm}) \notin U$, where $U = {}_2c_0^R(\mathbb{C}_2), {}_2c_0^B(\mathbb{C}_2)$.*

**Example 9** *Consider the double sequence $(\gamma_{lm})$ of bi-complex numbers defined by*

$$\gamma_{lm} = (e_1 + e_2)lm, \text{ for all } l, m \in \mathbb{N}.$$

*Then*

$$\Delta\gamma_{lm} = \begin{cases} \theta, & \text{if } l = 1 \text{ and } m = 1; \\ e_1 + e_2, & \text{otherwise .} \end{cases}$$

*Thus, $(\Delta\gamma_{lm}) \in U$, for $U = {}_2c(\mathbb{C}_2), {}_2c^R(\mathbb{C}_2), {}_2c^B(\mathbb{C}_2)$. Hence the sequence $(\gamma_{lm}) \in U[\Delta, \|\cdot||_{\mathbb{C}_2}]$, but $(\gamma_{lm}) \notin U$, where $U = {}_2c(\mathbb{C}_2), {}_2c^R(\mathbb{C}_2), {}_2c^B(\mathbb{C}_2)$.*

**Example 10** *Consider the double sequence $(\gamma_{lm})$ of bi-complex numbers defined by*

$$\gamma_{lm} = \begin{cases} e_1e_2, & \text{if } m = 1 \text{ for all } l \in \mathbb{N}; \\ (e_1 + e_2)(l + m), & \text{otherwise .} \end{cases}$$

*Then*

$$\Delta\gamma_{lm} = \begin{cases} (e_1 + e_2), & \text{if } m = 1 \text{ for all } l \in \mathbb{N}; \\ \theta, & \text{otherwise .} \end{cases}$$

*Thus, $(\Delta\gamma_{lm}) \in {}_2c_0(\mathbb{C}_2)$. Hence the sequence $(\gamma_{lm}) \in {}_2c_0(\mathbb{C}_2)[\Delta, \|\cdot||_{\mathbb{C}_2}]$, but $(\gamma_{lm}) \notin {}_2c_0(\mathbb{C}_2)$.*

**Theorem 6** *The inclusion relation $U[\Delta, \|\cdot||_{\mathbb{C}_2}] \subset {}_2\ell_\infty(\mathbb{C}_2)[\Delta, \|\cdot||_{\mathbb{C}_2}]$, for $U = {}_2c^R(\mathbb{C}_2), {}_2c_0^R(\mathbb{C}_2), {}_2c^B(\mathbb{C}_2), {}_2c_0^B(\mathbb{C}_2)$ holds and is strict.*

**Proof.** The proof is trivial by using Lemma 1.
The inclusion is strict follows from the following Examples.

**Example 11** *Consider the double sequence $(\gamma_{lm})$ of bi-complex numbers defined by*

$$\gamma_{lm} = \begin{cases} (e_1 + e_2)(l + m), & \text{if } l \text{ is odd and all } m \in \mathbb{N}; \\ (e_1 + e_2)l, & \text{otherwise .} \end{cases}$$

*Then*

$$\Delta\gamma_{lm} = \begin{cases} -(e_1 + e_2), & \text{if } l \text{ is odd and all } m \in \mathbb{N}; \\ (e_1 + e_2), & \text{otherwise .} \end{cases}$$

*Then $(\gamma_{lm}) \in {}_2\ell_\infty(\mathbb{C}_2)[\Delta, \|\cdot||_{\mathbb{C}_2}]$, but $(\gamma_{lm}) \notin U[\Delta, \|\cdot||_{\mathbb{C}_2}]$ for $U = {}_2c^R(\mathbb{C}_2)$, ${}_2c_0^R(\mathbb{C}_2), {}_2c^B(\mathbb{C}_2), {}_2c_0^B(\mathbb{C}_2)$.*

**Theorem 7** *The classes of double sequences ${}_2\ell_\infty(\mathbb{C}_2)[\Delta, \|\cdot||_{\mathbb{C}_2}]$ and ${}_2\omega(\mathbb{C}_2)[\Delta, \|\cdot||_{\mathbb{C}_2}]$ are convex.*

**Proof.** Let $(\gamma_{lm}), (t_{lm}) \in {}_2\ell_\infty(\mathbb{C}_2)[\Delta, \|\cdot\|_{\mathbb{C}_2}]$ and $\lambda \in \mathbb{C}_0$ satisfying $0 \leq \lambda \leq 1$. Then, $\sup_{l,m\in\mathbb{N}} \|\Delta\gamma_{lm}\|_{\mathbb{C}_2}$ and $\sup_{l,m\in\mathbb{N}} \|\Delta t_{lm}\|_{\mathbb{C}_2}$ are finite. Then we have

$$\begin{aligned}
&\sup_{l,m\in\mathbb{N}} \|\Delta\{\lambda\gamma_{lm} + (1-\lambda)t_{lm}\}\|_{\mathbb{C}_2} \\
&\leq \sup_{l,m\in\mathbb{N}} \{\|\Delta\lambda\gamma_{lm}\|_{\mathbb{C}_2} + \|(1-\lambda)\Delta t_{lm}\|_{\mathbb{C}_2}\} \\
&= \lambda \sup_{l,m\in\mathbb{N}} \{\|\Delta\gamma_{lm}\|_{\mathbb{C}_2}\} + (1-\lambda) \sup_{l,m\in\mathbb{N}} \{\|\Delta t_{lm}\|_{\mathbb{C}_2}\},
\end{aligned}$$

which implies that $\{\lambda(\gamma_{lm}) + (1-\lambda)(t_{lm})\} \in {}_2\ell_\infty(\mathbb{C}_2)[\Delta, \|\cdot\|_{\mathbb{C}_2}]$.

**Corollary 5** *The class of double sequence* ${}_2\ell_\infty(\mathbb{C}_2)[\Delta, \|\cdot\|_{\mathbb{C}_2}]$ *is not strictly convex.*

**Example 12** *Let*

$$\gamma_{lm} = \begin{pmatrix} 1 & -i_1 & \theta & \theta & \dots \\ i_1 & \theta & \theta & \theta & \dots \\ \theta & \theta & \theta & \theta & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, t_{lm} = \begin{pmatrix} -1 & -i_1 & \theta & \theta & \dots \\ i_1 & \theta & \theta & \theta & \dots \\ \theta & \theta & \theta & \theta & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

*Then,*

$$\Delta\gamma_{lm} = \begin{pmatrix} 1 & -i_1 & \theta & \theta & \dots \\ i_1 & \theta & \theta & \theta & \dots \\ \theta & \theta & \theta & \theta & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \Delta t_{lm} = \begin{pmatrix} -1 & -i_1 & \theta & \theta & \dots \\ i_1 & \theta & \theta & \theta & \dots \\ \theta & \theta & \theta & \theta & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

*Then we have* $\|\Delta\gamma_{lm}\|_{{}_2\ell_\infty(\mathbb{C}_2)[\Delta,\|\cdot\|_{\mathbb{C}_2}]} = 1, \|\gamma_{lm}\|_{{}_2\ell_\infty(\mathbb{C}_2)[\Delta,\|\cdot\|_{\mathbb{C}_2}]} = 1$ *and*

$$
\begin{aligned}
&\|\Delta(\lambda\gamma_{lm} + (1-\lambda)t_{lm})\|_{{}_2\ell_\infty(\mathbb{C}_2)[\Delta,\|\cdot\|_{\mathbb{C}_2}]} \\
&= \sup_{l\in\mathbb{N}}\{\|\lambda\gamma_{l1} + (1-\lambda)t_{l1}\|_{\mathbb{C}_2}\} + \sup_{m\in\mathbb{N}}\{\|\lambda\gamma_{1m} + (1-\lambda)t_{1m}\|_{\mathbb{C}_2}\} \\
&+ \sup_{l,m\in\mathbb{N}}\{\|\Delta\lambda\gamma_{lm} + (1-\lambda)\Delta t_{lm}\|_{\mathbb{C}_2}\} \\
&= \sup_{l\in\mathbb{N}}\left\|\lambda\begin{pmatrix} 1 & -i_1 & \theta & \theta & \dots \\ i_1 & \theta & \theta & \theta & \dots \\ \theta & \theta & \theta & \theta & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} + (1-\lambda)\begin{pmatrix} -1 & -i_1 & \theta & \theta & \dots \\ i_1 & \theta & \theta & \theta & \dots \\ \theta & \theta & \theta & \theta & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}\right\|_{\mathbb{C}_2} \\
&+ \sup_{m\in\mathbb{N}}\left\|\lambda\begin{pmatrix} 1 & -i_1 & \theta & \theta & \dots \\ i_1 & \theta & \theta & \theta & \dots \\ \theta & \theta & \theta & \theta & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} + (1-\lambda)\begin{pmatrix} -1 & -i_1 & \theta & \theta & \dots \\ i_1 & \theta & \theta & \theta & \dots \\ \theta & \theta & \theta & \theta & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}\right\|_{\mathbb{C}_2} \\
&+ \sup_{l,m\in\mathbb{N}}\left\|\lambda\begin{pmatrix} 1 & -i_1 & \theta & \theta & \dots \\ i_1 & \theta & \theta & \theta & \dots \\ \theta & \theta & \theta & \theta & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} + (1-\lambda)\begin{pmatrix} -1 & -i_1 & \theta & \theta & \dots \\ i_1 & \theta & \theta & \theta & \dots \\ \theta & \theta & \theta & \theta & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}\right\|_{\mathbb{C}_2} \\
&= \sup_{l\in\mathbb{N}}\left\|\begin{pmatrix} (2\lambda-1) & -i_1 & \theta & \theta & \dots \\ i_1 & \theta & \theta & \theta & \dots \\ \theta & \theta & \theta & \theta & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}\right\|_{\mathbb{C}_2} \\
&+ \sup_{m\in\mathbb{N}}\left\|\begin{pmatrix} (2\lambda-1) & -i_1 & \theta & \theta & \dots \\ i_1 & \theta & \theta & \theta & \dots \\ \theta & \theta & \theta & \theta & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}\right\|_{\mathbb{C}_2} \\
&+ \sup_{l,m\in\mathbb{N}}\left\|\begin{pmatrix} (2\lambda-1) & -i_1 & \theta & \theta & \dots \\ i_1 & \theta & \theta & \theta & \dots \\ \theta & \theta & \theta & \theta & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}\right\|_{\mathbb{C}_2} \\
&= \sup_{l\in\mathbb{N}}\{\|(2\lambda-1)\|_{\mathbb{C}_2}, \|-i_1\|_{\mathbb{C}_2}, \theta, \theta, ..., \|i_1\|_{\mathbb{C}_2}, \theta, \theta, \theta, ..., .....\} \\
&+ \sup_{m\in\mathbb{N}}\{\|(2\lambda-1)\|_{\mathbb{C}_2}, \|-i_1\|_{\mathbb{C}_2}, \theta, \theta, ..., \|i_1\|_{\mathbb{C}_2}, \theta, \theta, \theta, ..., .....\} \\
&+ \sup_{l,m\in\mathbb{N}}\{\|(2\lambda-1)\|_{\mathbb{C}_2}, \|-i_1\|_{\mathbb{C}_2}, \theta, \theta, ..., \|i_1\|_{\mathbb{C}_2}, \theta, \theta, \theta, ..., .....\} \\
&= 1, \ \textit{for all } \lambda \in (0,1).
\end{aligned}
$$

*That is to say that* ${}_2\ell_\infty(\mathbb{C}_2)[\Delta, \|\cdot\||_{\mathbb{C}_2}]$ *is not strictly convex.*

# 4 Conclusion

In this article, we have studied the convergence of difference double sequences in Pringsheim's sense, difference null in Pringsheim's sense, bounded difference, bounded convergence difference, bounded null difference, regular convergence difference and regular null difference double sequence of bi-complex numbers. We have examined its various algebraic and topological properties and discussed some examples.

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