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OPERATORS IN $\mathcal{L}(L_a^2(\mathbb{D}))$ AND THE ASSOCIATED SYMBOLS*

Namita Das†

Dedicated to Dr. Dan Tiba on the occasion of his 70th anniversary

Abstract

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{L}(L_a^2(\mathbb{D}))$ be the space of all bounded linear operators from the Bergman space $L_a^2(\mathbb{D})$ into itself. In this paper we shall associate symbols to bounded linear operators in $\mathcal{L}(L_a^2(\mathbb{D}))$ and analyse if a symbol calculus can be obtained.

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1 Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$. Let $dA(z)$ be the area measure on $\mathbb{D}$ normalized so that the area of the disk is 1. Let $L^2(\mathbb{D}, dA)$ be the Hilbert space of Lebesgue measurable functions on $\mathbb{D}$ with the inner product

$$\langle f, g\rangle = \int_{\mathbb{D}} f(z)\overline{g(z)}dA(z), f, g \in L^2(\mathbb{D}).$$

The Bergman space $L_a^2(\mathbb{D})$ is the set of those functions in $L^2(\mathbb{D}, dA)$ that are analytic on $\mathbb{D}$. The norm on $L_a^2(\mathbb{D})$ is also described by $\|f\|^2 = \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1}$,

*Accepted for publication on January 17-th, 2024
†namitadas440@yahoo.co.in P.G. Department of Mathematics, Utkal University, Vanivihar, Bhubaneswar, 751004, Odisha, India

using the power series expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n \in L_a^2(\mathbb{D}), z \in \mathbb{D}$. The Bergman space $L_a^2(\mathbb{D})$ is a closed subspace[25] of $L^2(\mathbb{D}, dA)$, and so there is an orthogonal projection P from $L^2(\mathbb{D}, dA)$ onto $L_a^2(\mathbb{D})$. The map P is called the Bergman projection. Let $K(z, \bar{w})$ be the function on $\mathbb{D} \times \mathbb{D}$ defined by $K(z, \bar{w}) = \overline{K_z(w)} = \frac{1}{(1-z\bar{w})^2}$. The function $K(z, \bar{w})$ is called the Bergman kernel of $\mathbb{D}$ or the reproducing kernel of $L_a^2(\mathbb{D})$ because the formula

$$f(z) = \int_{\mathbb{D}} f(w) K(z, \bar{w}) dA(w)$$

reproduces each f in $L_a^2(\mathbb{D})$. For any $n \geq 0, n \in \mathbb{Z}$, let $e_n(z) = \sqrt{n+1} z^n$, then $\{e_n\}$ forms an orthonormal basis for $L_a^2(\mathbb{D})$ and

$$K(z, \bar{w}) = \sum_{n=0}^{\infty} e_n(z) \overline{e_n(w)} = \frac{1}{(1 - z\bar{w})^2}.$$

Let $k_a(z) = \frac{K(z,\bar{a})}{\sqrt{K(a,\bar{a})}} = \frac{1-|a|^2}{(1-\bar{a}z)^2}, a, z \in \mathbb{D}$. These functions k_a are called the normalized reproducing kernels of $L_a^2(\mathbb{D})$; it is clear that they are unit vectors in $L_a^2(\mathbb{D})$. For any $a \in \mathbb{D}$, let ϕ_a be the analytic mapping on $\mathbb{D}$ defined by $\phi_a(z) = \frac{a-z}{1-\bar{a}z}, z \in \mathbb{D}$. An easy calculation shows that the derivative of ϕ_a at z is equal to $-k_a(z)$. It follows that the real Jacobian determinant of ϕ_a at z is

$$J_{\phi_a}(z) = |k_a(z)|^2 = \frac{(1 - |a|^2)^2}{|1 - \bar{a}z|^4}.$$

Let $L^\infty(\mathbb{D}, dA)$ be the Banach space of all essentially bounded measurable functions f on $\mathbb{D}$ with

$$\|f\|_\infty = \text{ ess sup } \{|f(z)| : z \in \mathbb{D}\} < \infty$$

and $H^\infty(\mathbb{D})$ be the space of bounded analytic functions on $\mathbb{D}$.
For $\phi \in L^\infty(\mathbb{D})$, the Toeplitz operator with symbol ϕ, denoted T_ϕ, is the operator from $L_a^2(\mathbb{D})$ into itself defined by $T_\phi f = P(\phi f)$. We can write T_ϕ as an integral operator as follows :

$$T_\phi f(z) = \int_{\mathbb{D}} \phi(w) K(z, \bar{w}) f(w) dA(w) = \int_{\mathbb{D}} \frac{\phi(w) f(w)}{(1 - z\overline{w})^2} dA(w).$$

Toeplitz operators can also be defined for unbounded symbols ϕ on the open unit disk $\mathbb{D}$. The operator is densely defined, in this case. It is easy to see that

$H^\infty(\mathbb{D})$, which is dense in $L^2_a(\mathbb{D})$, is contained in the domain of the operator T_ϕ. By a harmonic function we mean a complex valued function on $\mathbb{D}$ whose Laplacian is identically 0. Let $h^\infty(\mathbb{D})$ be the space of all bounded harmonic functions on $\mathbb{D}$. Let $\mathcal{L}(L^2_a(\mathbb{D}))$ be the of all bounded linear operators from $L^2_a(\mathbb{D})$ into itself and $\mathcal{LC}(L^2_a(\mathbb{D}))$ be the subspace of $\mathcal{L}(L^2_a(\mathbb{D}))$ consisting of all compact operators from $L^2_a(\mathbb{D})$ into itself. Define the Berezin transform for linear operators $T \in \mathcal{L}(L^2_a(\mathbb{D}))$ by the formula $\rho(T)(z) = \langle Tk_z, k_z\rangle, z \in \mathbb{D}$. Notice that the operator T need not be bounded, and it suffices if its domain contains all $k_z, z \in \mathbb{D}$. Let $V(\mathbb{D}) = \{\phi \in L^\infty(\mathbb{D}) :\ \text{ess lim}_{|z|\to 1}\phi(z) = 0\}$.

Let $\mathcal{T}_\phi$ be a Toeplitz operator on the Hardy space $H^2(\mathbb{T})$. It is easily seen that $\mathcal{T}_z^*\mathcal{T}_\phi\mathcal{T}_z = \mathcal{T}_\phi$ for all $\phi \in L^\infty(\mathbb{T})$. Brown and Halmos[9] showed that the converse also holds: if a bounded linear operator $T : H^2(\mathbb{T}) \to H^2(\mathbb{T})$ satisfies $\mathcal{T}_z^*T T_z = T$, then $T = T_\phi$ for some $\phi \in L^\infty(\mathbb{T})$. This result serves as a starting point for the theory of symbols of operators [23]. Notice that the mapping $\phi \to \mathcal{T}_\phi$ is linear. Englis [10] have shown that Toeplitz operators on $L^2_a(\mathbb{D})$ do not admit the characterization as above. More precisely, if $AT_\phi B = T_\phi$ for all $\phi \in L^\infty(\mathbb{D})$, then $A = cI, B = c^{-1}I$ for some nonzero complex number c. Englis [11] also showed that the set $\{T_\phi : \phi \in L^\infty(\mathbb{D})\}$ is dense in $\mathcal{L}(L^2_a(\mathbb{D}))$ in strong operator topology and the C^*-algebra generated by $\{T_\phi : \phi \in L^\infty(\mathbb{D})\}$ is strictly smaller than $\mathcal{L}(L^2_a(\mathbb{D}))$.

A scalar or matrix-valued function associated with the bounded linear operator and having properties that somehow reflect the properties of the operator is called the symbol of the operator. One assumes that the operators to which a symbol is assigned belong to an algebra and the symbol of an operator also takes values in an algebra. Usually symbols are associated with operators acting on function spaces . For example, Toeplitz, Hankel and composition operators on Hardy and Bergman spaes. The correspondence between symbols and operators is called [7],[8] symbol calculus.

Barria and Halmos [3] and Feintuch [13], [12] introduced the concept of asymptotic Toeplitz operators and asymptotic Hankel operators on the Hardy space. The importance of this notion is that it associates with a class of operators a Toeplitz operator and with a class of operators a Hankel operator where the original operators are not even Toeplitz or Hankel. Thus it is possible to assign a symbol to an operator that is not Toeplitz or Hankel and hence a symbol calculus is obtained. The significance of these results is that it gives distance formulae which can be viewed as operator theoretic analogues of results [13] of Nehari, Hartman and Adamjan, Arov and Krein [15].

In section 2, we investigate if every bounded linear operator in a Hilbert space has a symbol associated with it and whether the norm of the operator is equal to the essential supremum norm of its symbol. We shall use the Cesaro summability method to show that $\mathcal{L}(L^2_a(\mathbb{D}))$ has a certain similarity to $L^\infty(\mathbb{T})$ but there is no natural way to obtain a symbol calculus from it. In section 3, we focus on the characterization of Louhichi and Oloffson [20] and showed that a symbol calculus [7],[8] can be obtained for Toeplitz operators with bounded hamonic symbols. Functions in $L^\infty(\mathbb{T})$ correspond, via the Poisson integral, to bounded harmonic functions on $\mathbb{D}$, so perhaps the restriction to consideration only of Toeplitz and Hankel operators with bounded harmonic symbols is natural. In this section, we also establish that there exists a bounded projection from $\mathcal{L}(L^2_a(\mathbb{D}))$ onto $\{T_\phi : \phi \in h^\infty(\mathbb{D})\}$. In section 4, we focus our attention on Berezin symbols of an operator. We ask the general question of how much information about the bounded linear operators in $\mathcal{L}(L^2_a(\mathbb{D}))$, its Berezin symbol carry and whether a symbol calculus can be obtained. In fact we introduce a class $\mathcal{A} \subset L^\infty(\mathbb{D})$ such that if $\Upsilon \in \mathcal{A}$ and $\Upsilon \geq 0$ then there exist a positive operator $T \in \mathcal{L}(L^2_a(\mathbb{D}))$ such that $\Upsilon(z) = \langle Tk_z, k_z\rangle$ for all $z \in \mathbb{D}$. We also examine what is the Range of the Berezin transform.

2 Symbols of operators in $\mathcal{L}(L^2_a(\mathbb{D}))$

On the Hardy space of the unit circle, the Toeplitz operators and multiplication operators are bounded if their corresponding symbols are [9] essentially bounded. Now we ask the question if every bounded linear operator in a Hilbert space has a symbol associated with it and whether the norm of the operator is equal to the essential supremum norm of its symbol.

In this section we shall use the Cesaro summability method. Let $f \in L^1(\mathbb{T})$ be an integrable function on $\mathbb{T}$. The Nth Cesaro mean $\sigma_N f$ of f is defined by the formula

$$(\sigma_N f)(e^{i\theta}) = \Big(K_N \star f)(e^{i\theta}\Big) = \sum_{k=-N}^{N} \left(1 - \frac{|k|}{N+1}\right) \widehat{f}(k) e^{ik\theta}$$

where $\widehat{f}(k) = \frac{1}{2\pi}\int_{\mathbb{T}} f(e^{i\theta})e^{-ik\theta}d\theta$ is the kth Fourier coefficient of f and $K_N(e^{i\theta}) = \sum_{k=-N}^{N} \left(1 - \frac{|k|}{N+1}\right) e^{ik\theta}$ is the Nth Fejer kernel [17].

Let P_n denote the orthogonal projection of $L^2_a(\mathbb{D})$ onto span $\{e_0, e_1, \cdots e_n\}$. Then the operator norms of $\{P_nTP_n\}$ are uniformly bounded and $\|P_nTP_n\| \leq \|T\|$ for all n and P_nTP_n converges weakly to T. For $k \in \mathbb{Z}$, define

$\Theta_k(T) \in \mathcal{L}(L^2_a(\mathbb{D}))$, by $\langle \Theta_k(T)e_n, e_m\rangle = \langle Te_n, e_m\rangle.\delta_{n-m,k}$ where $\delta_{n-m,k}$ is the Kronecker delta. Define the kth Cesaro mean of $\{\Theta_k(T)\}$ as

$$\sigma_k(T) = \sum_{j=-k}^{k} \frac{k-|j|}{k}\Theta_j(T).$$

In Theorem 1, we shall show that if $T \in \mathcal{L}(L^2_a(\mathbb{D}))$, then $\|T\| = \sup_k \|\sigma_k(T)\|$ where $\sigma_k(T)$ is the kth Cesaro mean of $\Theta_k(T) \in \mathcal{L}(L^2_a(\mathbb{D})$.

Theorem 1. *If $\{\sigma_k(T)\}$ is uniformly bounded then $\|T\| = \sup\limits_k \|\sigma_k(T)\|$.*

Proof. : If $\{\sigma_k(T)\}$ is uniformly bounded then $\sigma_k(T)$ converges weakly to T. It follows therefore that $\|T\| \leq \sup\limits_k \|\sigma_k(T)\|$. Conversely, for $\lambda \in \mathbb{T} = \{\omega \in \mathbb{C} : |\omega| = 1\}$, define the unitary operator W_λ on $L^2_a(\mathbb{D})$ by $W_\lambda e_k = \lambda^k e_k$. For $f, g \in L^2_a(\mathbb{D}), \|f\| = \|g\| = 1$, define $\Phi_{f,g}(\lambda) = \langle TW_\lambda f, W_\lambda g\rangle$. Since $T \in \mathcal{L}(L^2_a(\mathbb{D}))$, hence $\Phi_{f,g} \in L^\infty(\mathbb{T})$. Let $f = \sum\limits_{n=0}^{\infty} a_n e_n, g = \sum\limits_{m=0}^{\infty} b_m e_m, a_n, b_m \in \mathbb{C}$ for all $n, m \in \mathbb{N} \cup \{0\}$. Then $\Phi_{f,g}(\lambda) = \sum\limits_{k=-\infty}^{\infty}\left(\sum\limits_{n-m=k} a_n\bar{b}_m\langle Te_n, e_m\rangle\right).\ \lambda^k$ which is the Fourier series expansion of $\Phi_{f,g}(\lambda)$ as an element of $L^2(\mathbb{T})$. This can be seen as follows : The sequence $P_nTP_n \longrightarrow T$ weakly and therefore

$$\Phi^n_{f,g}(\lambda) = \langle P_nTP_nW_\lambda f, W_\lambda g\rangle \to \Phi_{f,g}(\lambda),$$

for each $\lambda \in \mathbb{T}$ and $\|\Phi^n_{f,g}\|_\infty \leq \|T\|$ for all n. It follows therefore that the sequence $\{\Phi^n_{f,g}\}$ converges to $\Phi_{f,g}$ in L^2 -norm and then that the individual Fourier coefficients converge. An examination of the expansion of $\Phi^n_{f,g}$, which is finite, proves our claim.

Now since $\Phi_{f,g} \in L^\infty(\mathbb{T})$, the Cesaro means of its Fourier expansion are uniformly bounded by $\|\Phi_{f,g}\|_\infty$. Thus

$$\sup_{|\lambda|=1} \|\sigma_k(\Phi_{f,g}(\lambda))\| \leq \|\Phi_{f,g}\|_\infty \leq \|T\|$$

for all k. But

$$\begin{aligned} |\langle\sigma_k\,(T)\,W_\lambda f, W_\lambda g\rangle| &= \left|\textstyle\sum_{j=-k}^{k} \frac{k-|j|}{k} \sum_{n-m=j}\langle Te_n, e_m\rangle a_n\bar{b}_m\lambda^j\right| \\ &= |\sigma_k(\Phi_{f,g}(\lambda))|\,. \end{aligned}$$

Hence for $\|f\| = \|g\| = 1$, $\|\sigma_k(T)\| = \sup\limits_{f,g\in L^2_a} |\langle\sigma_k(T)f, g\rangle| = \sup\limits_{f,g} |(\sigma_k(\Phi_{f,g}(\lambda)))| \leq \|T\|$. The theorem follows. □

Thus we have seen that $\mathcal{L}(L^2_a(\mathbb{D}))$ has a certain similarity to $L^\infty(\mathbb{T})$. In Theorem 1, for $f, g \in L^2_a(\mathbb{D}), \|f\| = \|g\| = 1$, we defined a function $\Phi_{f,g} \in L^\infty(\mathbb{T})$ such that $\|T\| = \sup_{\|f\|=\|g\|=1} \|\Phi_{f,g}\|_\infty = \sup_k \|\sigma_k(T)\|$. But there is no natural way to obtain a symbol correspondence between the operators and the symbols.

3 Toeplitz operators with bounded harmonic symbols

In this section, we focus on the characterization of Louhichi and Oloffson [20] and showed that a symbol calculus can be obtained for Toeplitz operators with bounded hamonic symbols. In [20], the authors have shown that if $T \in \mathcal{L}(L^2_a(\mathbb{D}))$ then T satisfies the operator identity (1) if and only if $T = T_\phi$ is a Toeplitz operator on $L^2_a(\mathbb{D})$ with bounded harmonic symbol ϕ. In this section, we also establish that there exists a bounded projection from $\mathcal{L}(L^2_a(\mathbb{D}))$ onto $\{T_\phi : \phi \in h^\infty(\mathbb{D})\}$.

Consider the following operators on the Bergman space $L^2_a(\mathbb{D})$:

$$(Sf)(z) = zf(z) = \sum_{n=1}^{\infty} a_{n-1} z^n,$$

$$(Rf)(z) = \frac{f(z) - f(0)}{z} = \sum_{n=0}^{\infty} a_{n+1} z^n,$$

$$(Df)(z) = f'(z).$$

It is not difficult to verify that

$$RS = I \text{ and } DSR = D$$

where I is the identity operator and $f = \sum_{n=0}^{\infty} a_n z^n \in L^2_a(\mathbb{D})$. The operator S is called the Bergman shift operator and R is the backward shift operator on $L^2_a(\mathbb{D})$. Notice [2] that $RT_\phi S = T_\phi$ for all $\phi \in H^\infty(\mathbb{D})$. Hence $S^* T_{\bar{\phi}} R^* = T_{\bar{\phi}}$ for all $\phi \in H^\infty(\mathbb{D})$. Let $S' = S(S^*S^{-1})$. The operator S' is a weighted shift operator on $L^2_a(\mathbb{D})$ which act as $(S'f)(z) = \sum_{n=1}^{\infty} \frac{n+1}{n} a_{n-1} z^n, z \in \mathbb{D}, f \in L^2_a(\mathbb{D})$. The Bergman shift operator S satisfies the operator identity

$(S')^*S' = (S^*S)^{-1} = 2I - SS^*$. Thus $(S')^*S' + SS^* = (S^*S)^{-1} + SS^* = 2I$. The adjoint of Bergman shift operator S^* in $\mathcal{L}(L_a^2(\mathbb{D}))$ is given by

$$(S^*f)(z) = \sum_{n=0}^{\infty} \frac{n+1}{n+2} a_{n+1} z^n$$

for $f(z) = \sum_{n=0}^{\infty} a_n z^n \in L_a^2(\mathbb{D}), z \in \mathbb{D}$. The space $\ker S^* = L_a^2(\mathbb{D}) \ominus S(L_a^2(\mathbb{D}))$ consists of the constant functions in $\mathbb{D}$. The operator $R = (S^*S)^{-1}S^* = (S')^*$ in $\mathcal{L}(L_a^2(\mathbb{D}))$ is the left inverse of S. Further $\ker R = \ker S^*$ and $R' = ((S')^*S')^{-1}(S')^* = S^*$ is the left inverse of S' with kernel $\ker R' = \ker S^*$ consisting of the constant functions. Let $T \in \mathcal{L}(L_a^2(\mathbb{D}))$ satisfying the operator identity

$$(S')^*TS' = 2T - STS^* \tag{1}$$

in $\mathcal{L}(L_a^2(\mathbb{D}))$. Louhichi and Olofsson [20]showed that operators in $\mathcal{L}(L_a^2(\mathbb{D}))$ that satisfies the operator identity (1) is a Toeplitz operators on $L_a^2(\mathbb{D})$ with bounded harmonic symbol. Thus we can obtain a symbol calculus for operators satisfying (1).

The work of Louhichi and Olofsson [20] involves the theory of functional calculus and Cesaro summability. Let $T \in \mathcal{L}(L_a^2(\mathbb{D}))$ be a contraction (of norm less than or equal to 1). Let

$$T(k) = \begin{cases} T^k, & \text{for } k \geq 0, \\ T^{*|k|}, & \text{for } k < 0. \end{cases}$$

From the existence of a unitary dilation of T, it follows the existence of a positive $\mathcal{L}(L_a^2(\mathbb{D}))$-valued operator measure dw_T on the unit circle $\mathbb{T}$ such that

$$\int_{\mathbb{T}} e^{ik\theta} dw_T(e^{i\theta}) = T(k), k \in \mathbb{Z}.$$

By an approximate argument the operator measure dw_T is uniquely determined by this action (see [21] and [20]). The following result is established by Louhichi and Olofsson [20].

Theorem 2. *Let H be a Hilbert space and $T \in \mathcal{L}(H)$ be a contraction such that the operator measure dw_T is absolutely continuous with respect to Lebesgue measure on $\mathbb{T}$, and let $f \in L^\infty(\mathbb{T})$. Then*

$$\int_{\mathbb{T}} f(e^{i\theta}) dw_T(e^{i\theta}) = \lim_{N\to\infty} \sum_{k=-N}^{N} \left(1 - \frac{|k|}{N+1}\right) \widehat{f}(k) T(k)$$

with convergence in the strong operator topology in $\mathcal{L}(H)$. Further if $T \in \mathcal{L}(L_a^2(\mathbb{D}))$ then the operator T satisfies the identity (1) *if and only if it has the form of an operator integral*

$$T = \int_{\mathbb{T}} f(e^{i\theta}) dw_S(e^{i\theta})$$

in $\mathcal{L}(L_a^2(\mathbb{D}))$ of a function $f \in L^\infty(\mathbb{T})$. In this case $\|T\| = \|f\|_\infty$.

Using Theorem 2 , Louhichi and Olofsson [20] then characterized Toeplitz operators T_ϕ with harmonic symbol ϕ using the operator identity (1).

Theorem 3. *Let $T \in \mathcal{L}(L_a^2(\mathbb{D}))$. Then T satisfies the operator identity* (1) *if and only if $T = T_\phi$ is a Toeplitz operator on $L_a^2(\mathbb{D})$ with bounded harmonic symbol ϕ.*

For $\psi \in L^\infty(\mathbb{T})$, one can define the Toeplitz operator $\mathcal{T}_\psi$ from $H^2(\mathbb{T})$ into itself as $\mathcal{T}_\psi f = \widetilde{P}(\psi f)$ where $\widetilde{P}$ is the Szegō projection from $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$. It is known [25] that functions in $L^\infty(\mathbb{T})$ correspond via Poisson integral to bounded harmonic functions on $\mathbb{D}$ and the radial limits of functions in $h^\infty(\mathbb{D})$ belong [25] to $L^\infty(\mathbb{T})$.
Hence there is a one-to-one correspondence between $h^\infty(\mathbb{D})$ and $L^\infty(\mathbb{T})$. Let $\mathcal{L}(H^2(\mathbb{T}))$ be the set of all bounded linear operators from $H^2(\mathbb{T})$ into itself. The functions $e_n(z) = \sqrt{n+1}z^n, n = 0, 1, 2, \cdots$ form an orthonormal basis for $L_a^2(\mathbb{D})$ and $\{u_n(t)\}_{n=0}^\infty = \{e^{int}\}_{n=0}^\infty$ form an orthonormal basis for $H^2(\mathbb{T})$. In the following theorem we shall prove the existence of a bounded projection from $\mathcal{L}(L_a^2(\mathbb{D}))$ onto $\{T_\phi : \phi \in h^\infty(\mathbb{D})\}$.

Theorem 4. *There exists a bounded projection from $\mathcal{L}(L_a^2(\mathbb{D}))$ onto $\{T_\phi : \phi \in h^\infty(\mathbb{D})\}$.*

Proof. : The set of functions $e_n(z) = \sqrt{n+1}z^n, n \in \mathbb{Z}_+$(the set of non-negative integers), $z \in \mathbb{D}$ form an orthonormal basis for $L_a^2(\mathbb{D})$. Also every bounded harmonic function f on $\mathbb{D}$ can be written [25] in the form $f = f_1 + \bar{f}_2 = \sum_{n=0}^\infty a_n z^n + \sum_{n=1}^\infty a_{-n}\bar{z}^n, f_1, f_2 \in H^2(\mathbb{D})$.
This implies

$$f(z) = \sum_{n=0}^\infty \frac{a_n}{\sqrt{n+1}} e_n + \sum_{n=1}^\infty \frac{a_{-n}}{\sqrt{n+1}} \bar{e}_n.$$

Then we have

$$\langle T_f e_n, e_m \rangle = \left\langle e_n \sum_{k=0}^\infty \frac{a_k}{\sqrt{k+1}} e_k, e_m \right\rangle + \left\langle e_n \sum_{k=1}^\infty \frac{a_{-k}}{\sqrt{k+1}} \bar{e}_k, e_m \right\rangle$$

$$= \left\langle \sum_{k=0}^{\infty} a_k e_{n+k} \sqrt{\frac{n+1}{n+k+1}}, e_m \right\rangle + \left\langle \sum_{k=1}^{\infty} a_{-k} \bar{e}_{k+m} \sqrt{\frac{m+1}{m+k+1}}, \bar{e}_n \right\rangle$$

$$= \begin{cases} a_0 & \text{if} \quad m = n; \\ a_{m-n}\sqrt{\frac{n+1}{m+1}} & \text{if} \quad m > n; \\ a_{-(n-m)}\sqrt{\frac{m+1}{n+1}} & \text{if} \quad m < n. \end{cases}$$

Thus the matrix of a Toeplitz operator with bounded harmonic symbol f is of the above form where $a_k, k \in \mathbb{Z}$ is the kth Fourier coefficient of $\widetilde{f}(e^{i\theta}) = \lim_{r\to 1^-} f(re^{i\theta})$.

Define W from $H^2(\mathbb{T})$ onto $L^2_a(\mathbb{D})$ as $Wu_n = e_n, n = 0, 1, 2, \cdots$ where $\{u_n\}_{n=0}^{\infty}$ is the standard orthonormal basis for $H^2(\mathbb{T})$ and $\{e_n\}_{n=0}^{\infty}$ is the standard orthonormal basis for $L^2_a(\mathbb{D})$. It is not difficult to show that W is a unitary operator from $H^2(\mathbb{T})$ onto $L^2_a(\mathbb{D})$ and it induces a map σ from $\mathcal{L}(L^2_a(\mathbb{D}))$ into $\mathcal{L}(H^2(\mathbb{T}))$ given by $\sigma(T) = W^*TW$.

In [1], it is shown that there is a positive linear projection Ω from $\mathcal{L}(H^2(\mathbb{T}))$ onto $\{\mathcal{T}_\psi : \psi \in L^\infty(\mathbb{T})\}$ such that $\Omega(\mathcal{T}_\psi) = \mathcal{T}_\psi$ for all $\psi \in L^\infty(\mathbb{T})$. Let $\mathbb{N}$ be the additive semigroup of all positive integers and let Λ be a Banach limit on $\mathbb{N}$. Thus Λ is a state on the commutative C^*-algebra $l^\infty(\mathbb{N})$ (whose value at a bounded sequence $(a_n)_{n\geq 1}$ is denoted by $\Lambda_n a_n$) and which has the additional property that $\Lambda_n a_{n+1} = \Lambda_n a_n, (a_n) \in l^\infty(\mathbb{N})$. Let U denote the bilateral shift defined on the basis $\{u_n\}_{n\in\mathbb{Z}}$ of $L^2(\mathbb{T})$ by $Uu_n = u_{n+1}, n \in \mathbb{Z}$. It is well known [22] that U is a unitary operator and for $x, y \in H^2(\mathbb{T}), A \in \mathcal{L}(H^2(\mathbb{T}))$, we may define the form

$$[x, y] = \Lambda_n \langle U^{*n} A U^n x, y\rangle.$$

A straight forward application of the Schwarz lemma yields a unique operator $\Pi(A) \in \mathcal{L}(H^2(\mathbb{T}))$ such that

$$\langle \Pi(A)x, y\rangle = \Lambda_n \langle U^{*n} A U^n x, y\rangle,$$

$U^*\Pi(A)U = \Pi(A)$ and define $\Omega(A) = \Pi(A)$ which is a Toeplitz operator $\mathcal{T}_\psi$ on the Hardy space for some $\psi \in L^\infty(\mathbb{T})$. As we have seen if $\phi \in h^\infty(\mathbb{D})$, the matrix of the Toeplitz operator on the Bergman space has a special form and it follows easily that if $A = W^*T_\phi W, \phi \in h^\infty(\mathbb{D})$ then $\Omega(W^*T_\phi W) = \Omega(A) = \Pi(A) = \mathcal{T}_{\widetilde{\phi}}$ where $\widetilde{\phi}(e^{i\theta}) = \lim_{r\to 1^-} \phi(re^{i\theta})$ belonging to $L^\infty(\mathbb{T})$. For more details see [1].

It is not difficult to see that there is a one-one map ζ from $\{\mathcal{T}_\psi : \psi \in L^\infty(\mathbb{T})\}$ onto $\{T_\phi : \phi \in h^\infty(\mathbb{D})\}$ such that $\zeta(\mathcal{T}_\psi) = T_{\widehat{\psi}}$ where $\widehat{\psi} \in h^\infty(\mathbb{D})$ is the harmonic extension of ψ. Hence $\zeta \circ \Omega \circ \sigma$ is a map from $\mathcal{L}(L^2_a(\mathbb{D}))$ onto $\{T_\phi : \phi \in h^\infty(\mathbb{D})\}$ and $(\zeta\circ\Omega\circ\sigma)^2 = \zeta\circ\Omega\circ\sigma$. Hence $\{T_\phi : \phi \in h^\infty(\mathbb{D})\}$ can be complemented in $\mathcal{L}(L^2_a(\mathbb{D}))$. It is not difficult to verify that $\{T_\phi : \phi \in h^\infty(\mathbb{D})\}$ is closed in $\mathcal{L}(L^2_a(\mathbb{D}))$. □

Thus the matrix entries of $T_\phi, \phi \in h^\infty(\mathbb{D})$ can be expressed by the Fourier coefficients of ϕ. For any $f \in L^2(\mathbb{D})$, we define a function Bf on $\mathbb{D}$ by

$$Bf(z) = \int_{\mathbb{D}} f(\phi_z(w))dA(w) = \int_{\mathbb{D}} f(w)|k_z(w)|^2 dA(w).$$

The operator B is called the Berezin transform on $\mathbb{D}$.
We shall now prove that if ϕ is a bounded harmonic function on the disk $\mathbb{D}$ then $B\phi = \phi$ and $\|T_\phi\| = \|\phi\|_\infty$.

Theorem 5. *If ϕ is a bounded harmonic function on $\mathbb{D}$, then $B\phi = \phi$ and $\|T_\phi\| = \|\phi\|_\infty$.*

Proof. : Let $\widetilde{\phi}(z) = \langle T_\phi k_z, k_z\rangle$ where k_z is the normalised reproducing kernel for the Hilbert space $L^2_a(\mathbb{D})$. Hence $\widetilde{\phi}(z) = \int_D \phi(w)|k_z(w)|^2 dA(w),\ z \in \mathbb{D}$. For $z \in \mathbb{D}$, let $\phi_z(w) = \frac{z-w}{1-\bar{z}w},\ w \in \mathbb{D}$. Since $\phi_z'(w) = -k_z(w)$ and the real Jacobian determinant of ϕ_z at w is $J_{\phi_z}(w) = |k_z(w)|^2 = \frac{(1-|z|^2)^2}{|1-\bar{z}w|^4}$, by making a change of variable we have $\widetilde{\phi}(z) = \int_{\mathbb{D}} \phi(\phi_z(w))dA(w) = \phi(\phi_z(0)) = \phi(z)$. Thus $|\phi(z)| = |\widetilde{\phi}(z)| = |\langle T_\phi k_z, k_z\rangle| \leq \|T_\phi\|$ for all $z \in \mathbb{D}$. Hence $\|\phi\|_\infty \leq \|T_\phi\|$. Since the Bergman projection has norm 1, we have $\|T_\phi\| = \|\phi\|_\infty$. □

It is to be noted that only for bounded harmonic functions ϕ in $L^\infty(\mathbb{D})$, the area-version of the mean-value property hold and hence $\tilde{\phi}(z) = \phi(z)$ and therefore $\|T_\phi\| = \|\phi\|_\infty$. But there exist functions ϕ in $L^\infty(\mathbb{D})$ that are not harmonic on the disc $\mathbb{D}$ but yet $\|T_\phi\| = \|\phi\|_\infty$ as the following example shows. Again the set of ϕ in $L^\infty(\mathbb{D})$ such that $\|T_\phi\| = \|\phi\|_\infty$ is not linear.

Example 1. *Let ϕ be the characteristic function of the annulus $R \leq |z|^2 < 1$. For $i, j \geq 0$, $\langle T_\phi e_j, e_i\rangle = \langle \phi e_j, e_i\rangle = 0$ if $i \neq j$ and $\{e_n\}_{n\geq 0}$ is the standard orthonormal basis for $L^2_a(\mathbb{D})$. Moreover $\langle T_\phi e_j, e_j\rangle = (j+1)\int_R^1 r^j dr = 1 - R^{j+1}$. Hence the matrix of T_ϕ with respect to the standard orthonormal basis $\{e_n\}_{n\geq 0}$ is diagonal, $\|T_\phi\| = 1$ and $\|\phi\|_\infty = 1$, but ϕ is not harmonic. It will be difficult to describe those ϕ for which $\|T_\phi\| = \|\phi\|_\infty$, because the above example shows that this set is not even linear (i.e., it is not a subspace): the function 1 (constant 1) and the above ϕ both belong to this set, but their difference does not because $T_{1-\phi}$ is a diagonal operator with weights R^{j+1}, so $\|1-\phi\|_\infty = 1 > R = \|T_{1-\phi}\|$.*

4 The Berezin transform of operators in $\mathcal{L}(L^2_a(\mathbb{D}))$

Berezin[4] introduced the notion of covariant and contravariant symbols of an operator. Berger and Coburn [5] [6] are the first to actually use the Berezin symbol (contravariant symbol) of a Toeplitz operator. Further applications of the Berezin symbol can be found in Berger, Coburn and Zhu [8], Stroethoff [24] and Zhu [26]. The Berezin symbol is very effective in many cases in the sense that it contains a lot of information about the operator that induces it. Successful applications of the Berezin transform are so far mainly in the study of Hankel, Toeplitz and composition operators. It is natural to ask the general question of how much information about the operator its Berezin symbol carry and whether a symbol calculus can be obtained.

If $T \in \mathcal{L}(L^2_a(\mathbb{D}))$ then $\rho(T) \in L^\infty(\mathbb{D})$ and $\|\rho(T)\|_\infty \leq \|T\|$ as $|\rho(T)(z)| = |\langle Tk_z, k_z\rangle| \leq \|T\|$ for all $z \in \mathbb{D}$. Further, if $T \in \mathcal{LC}(L^2_a(\mathbb{D}))$, then as $k_z \to 0$ weakly, hence $\rho(T) \in V(\mathbb{D})$.

One may also notice that if $T \in \mathcal{L}(L^2_a(\mathbb{D}))$ is diagonal with respect to the basis $\{e_n\}_{n=0}^\infty$, then $\rho(T)$ is radial. The reason is as follows: Suppose T is diagonal with weights λ_n. Then

$$TK_z(w) = \sum_{n=0}^\infty (n+1)\bar{z}^n \lambda_n w^n.$$

Hence

$$\rho(T)(z) = (1-|z|^2)^2 \sum_{n=0}^\infty (n+1)\lambda_n |z|^{2n},$$

which is a radial function.

Suppose $\phi \in L^\infty(\mathbb{D})$ is a radial function. Then it can be verified that T_ϕ is a diagonal operator with respect to the basis $\{e_n\}_{n=0}^\infty$. Passing to the polar coordinates, we have

$$\langle T_\phi z^n, z^m\rangle = \int_{\mathbb{D}} \phi(z) z^n \bar{z}^m dA(z) = \frac{1}{2\pi}\int_0^1 \int_0^{2\pi} \phi(r) r^{n+m} e^{i(n-m)t} 2r dt dr.$$

If $n \neq m$, this is zero; if $n = m$, it equals $\int_0^1 \psi(r^2) r^{2n} 2r dr = \int_0^1 \psi(t) t^n dt$. Thus $\rho(T_\phi)$ is a radial function.

Example 2. *Let $Tf(z) = f(-z)$. Then T is a diagonal operator with respect to the basis $\{z^k\}$ with eigenvalues $(-1)^k$. After a short computation one sees that $\rho(T)(z) = \langle Tk_z, k_z\rangle = \left(\frac{1-|z|^2}{1+|z|^2}\right)^2$ which is in $V(\mathbb{D})$, yet T is not compact.*

Example 3. *Define $Tf(z) = f(-2z)$. The operator T is a diagonal operator sending z^k to $(-2)^k z^k$ and*

$$\rho(T)(z) = \langle Tk_z, k_z\rangle = \left(\frac{1-|z|^2}{1+2|z|^2}\right)^2$$

which again belongs to $V(\mathbb{D}) \subset L^\infty(\mathbb{D})$ but now T is even unbounded.

The map ρ is is injective[16]. Below we shall give an example of an operator $T \in \mathcal{L}(L^2_a(\mathbb{D}))$ such that $\|\rho(T)\|_\infty \neq \|T\|$ and which also demonstrates that ρ is not bounded below.

Example 4. *Let T be the projection onto the subspace spanned by z^k, i.e., $Tf = (k+1)\langle f, z^k\rangle z^k$. Then $\|T\| = 1$, while a short computation shows that*

$$\rho(T)(z) = (k+1)(1-|z|^2)^2|z|^{2k}.$$

The function $\rho(T)(z)$ attains its maximum for $|z|^2 = \frac{k}{k+2}$, the maximum value being

$$(k+1)\left(\frac{k}{k+2}\right)^k \left(\frac{2}{k+2}\right)^2 \leq \frac{4}{k+2}.$$

Thus $\|T\| = 1$ while $\|\rho(T)\|_\infty \leq \frac{4}{k+2}$.

Thus Rangeρ is not closed [19] in $L^\infty(\mathbb{D})$ as the map ρ is injective and the above example shows that ρ is not bounded below.
Now we shall ask the question what is the range of ρ. That is, we ask given $\Upsilon \in L^\infty(\mathbb{D})$, does there exist $T \in \mathcal{L}(L^2_a(\mathbb{D}))$ such that $\Upsilon(z) = \langle Tk_z, k_z\rangle$ for all $z \in \mathbb{D}$.

Definition 1. *A function $g(x, \bar{y})$ on $\mathbb{D} \times \mathbb{D}$ is called of positive type (or positive definite), written $g \gg 0$, if*

$$\sum_{\mathbf{j},\mathbf{k}=\mathbf{1}}^{\mathbf{n}} \mathbf{c_j \bar{c}_k g(x_j, \bar{x}_k)} \geq \mathbf{0} \tag{2}$$

for any $n-$ tuple of complex numbers $c_1, \cdots, c_n$ and points $x_1, \cdots, x_n \in \mathbb{D}$. We write $g \gg h$ if $g - h \gg 0$.

We shall say $\Upsilon \in \mathcal{A}$ if $\Upsilon \in L^\infty(\mathbb{D})$ and is such that

$$\Upsilon(z) = \Theta(z, \bar{z}) \tag{3}$$

where $\Theta(x, \bar{y})$ is a function on $\mathbb{D} \times \mathbb{D}$ meromorphic in x and conjugate meromorphic in y.

It is a fact that (see [14], [18]) Θ as in (3), if it exists, is uniquely determined by Υ.

Theorem 6. *Let $\Upsilon \in L^\infty(\mathbb{D})$ and $\Upsilon \geq 0$. A necessary and sufficient condition that there exist a positive operator $T \in \mathcal{L}(L_a^2(\mathbb{D}))$ such that $\Upsilon(z) = \langle Tk_z, k_z\rangle$ for all $z \in \mathbb{D}$ is that $\Upsilon \in \mathcal{A}$ and if $\Upsilon(z) = \Theta(z, \bar{z})$ as in (3), then there exists a constant $c > 0$ such that*

$$cK(x, \bar{y}) \gg \Theta(x, \bar{y})K(x, \bar{y}) \gg 0.$$

Further $h^\infty(\mathbb{D}) \subset Range\rho$.

Proof. : Let $T \in \mathcal{L}(L_a^2(\mathbb{D}))$. Let

$$\Theta(x, \bar{y}) = \frac{\langle TK_y, K_x\rangle}{\langle K_y, K_x\rangle} \tag{4}$$

where $K_x = K(., \bar{x})$ is the unnormalized reproducing kernel at x. Then $\Theta(x, \bar{y})$ is a function on $\mathbb{D}\times\mathbb{D}$ meromorphic in x and conjugate meromorphic in y. Let $\Upsilon(z) = \Theta(z, \bar{z})$. Then $\Upsilon(z) = \langle Tk_z, k_z\rangle$ for all $z \in \mathbb{D}$ and $\Upsilon \in L^\infty(\mathbb{D})$ as T is bounded. Thus $\Upsilon \in \mathcal{A}$.

Now let $\Upsilon \in \mathcal{A}$ and $\Upsilon(z) = \Theta(z, \bar{z})$ where $\Theta(x, \bar{y})$ is a function on $\mathbb{D} \times \mathbb{D}$ meromorphic in x and conjugate meromorphic in y. We shall prove the existence of some T (possibly unbounded) such that $\langle Tk_z, k_z\rangle = \Upsilon(z)$. Let

$$Tf(x) = \int_{\mathbb{D}} f(z)\Theta(x, \bar{z})K(x, \bar{z})dA(z). \tag{5}$$

Indeed,

$$\begin{aligned} Tf(x) &= \langle Tf, K_x\rangle \\ &= \langle f, T^*K_x\rangle \\ &= \textstyle\int_{\mathbb{D}} f(z)\overline{\langle T^*K_x, K_z\rangle}dA(z) \\ &= \textstyle\int_{\mathbb{D}} f(z)\langle TK_z, K_x\rangle dA(z) \\ &= \textstyle\int_{\mathbb{D}} f(z)\Theta(x, \bar{z})K(x, \bar{z})dA(z). \end{aligned}$$

Then

$$\begin{aligned} \langle TK_y, K_x\rangle &= \textstyle\int_{\mathbb{D}} K_y(z)\Theta(x, \bar{z})K(x, \bar{z})dA(z) \\ &= \overline{\textstyle\int_{\mathbb{D}} K_y(z)\Theta(x, \bar{z})}\overline{K_x(z)}dA(z) \\ &= \overline{\langle\overline{\Theta(x, \bar{z})}K_x, K_y\rangle} \\ &= \overline{\overline{\Theta(x, \bar{y})}}\langle K_x, K_y\rangle \\ &= \Theta(x, \bar{y})\langle K_y, K_x\rangle. \end{aligned}$$

Hence $\Theta(x, \bar{y}) = \frac{\langle TK_y, K_x\rangle}{\langle K_y, K_x\rangle}$ and $\Upsilon(z) = \Theta(z, \bar{z}) = \langle Tk_z, k_z\rangle$. Notice however that the operator T given by (5) may well be unbounded. We shall now prove a necessary and sufficient condition for T to be bounded and positive is that there exists $c > 0$ such that

$$cK(x,\bar{y}) \gg \Theta(x,\bar{y})K(x,\bar{y}) \gg 0.$$

Suppose there exist $c > 0$ such that for all $x, y \in \mathbb{D}$,

$$cK(x,\bar{y}) \gg \Theta(x,\bar{y})K(x,\bar{y}) \gg 0.$$

We shall show that T is bounded and positive. Let $f = \sum_{j=1}^n c_j K_{x_j}$ where c_j are constants, $x_j \in \mathbb{D}$ for $j = 1, 2\cdots, n$. Then

$$\begin{aligned} \langle Tf, f\rangle &= \langle T(\textstyle\sum_{j=1}^n c_j K_{x_j}), \sum_{j=1}^n c_j K_{x_j}\rangle \\ &= \textstyle\sum_{j,k=1}^n c_j \overline{c_k} \langle TK_{x_j}, K_{x_k}\rangle \\ &= \textstyle\sum_{j,k=1}^n c_j \overline{c_k} \Theta(x_k, \bar{x_j}) K(x_k, \bar{x_j}) \geq 0 \end{aligned}$$

and

$$\begin{aligned} \langle Tf, f\rangle &= \textstyle\sum_{j,k=1}^n c_j \overline{c_k} \langle TK_{x_j}, K_{x_k}\rangle \\ &= \textstyle\sum_{j,k=1}^n c_j \overline{c_k} \Theta(x_k, \bar{x_j}) K(x_k, \bar{x_j}) \\ &\leq c \textstyle\sum_{j,k=1}^n c_j \overline{c_k} K(x_k, \bar{x_j}) = c\|f\|^2. \end{aligned}$$

Since the set of vectors $\{\sum_{j=1}^n c_j K_{x_j}, x_j \in \mathbb{D}, j = 1, 2\cdots, n\}$ is dense in $L^2_a(\mathbb{D})$, hence $0 \leq \langle Tf, f\rangle \leq c\|f\|^2$ for all $f \in L^2_a(\mathbb{D})$ and T is bounded and positive.

Conversely, suppose T is bounded and positive. Then there exists a constant $c > 0$ such that $0 \leq \langle Tf, f\rangle \leq c\|f\|^2$ for all $f \in L^2_a(\mathbb{D})$. That is, if $f = \sum_{j=1}^n c_j K_{x_j}$, then

$$\begin{aligned} 0 &\leq \langle Tf, f\rangle \\ &= \textstyle\sum_{j,k=1}^n c_j \overline{c_k} \langle TK_{x_j}, K_{x_k}\rangle \\ &= \textstyle\sum_{j,k=1}^n c_j \overline{c_k} \Theta(x_k, \bar{x_j}) K(x_k, \bar{x_j}) \\ &\leq c\|f\|^2 \\ &= c \textstyle\sum_{j,k=1}^n c_j \overline{c_k} K(x_k, \bar{x_j}). \end{aligned}$$

Thus $cK(x,\bar{y}) \gg \Theta(x,\bar{y})K(x,\bar{y}) \gg 0$.
We shall now verify that $h^\infty(\mathbb{D}) \subset \text{Range}\rho$. Let $\phi \in h^\infty(\mathbb{D})$. Then by the mean value property we obtain

$$\rho(T_\phi)(z) = \langle T_\phi k_z, k_z\rangle = \int_{\mathbb{D}} \phi(w)|k_z(w)|^2 dA(w)$$

$$= \int_{\mathbb{D}} (\phi \circ \phi_z)(w) dA(w) = (\phi \circ \phi_z)(0) = \phi(z).$$

Hence $\phi \in \text{Range}\rho$. But $\text{Range}\rho \not\subseteq h^\infty(\mathbb{D})$. Example 2 gives an example of a bounded linear operator T such that $\rho(T) \notin h^\infty(\mathbb{D})$. □

Let

$$E = \{T \in \mathcal{L}(L_a^2(\mathbb{D})) : \langle Tk_z, k_z\rangle = \int_{\mathbb{D}} \langle Tk_{\phi_z(w)}, k_{\phi_z(w)}\rangle dA(w) \text{ for all } z \in \mathbb{D}\}.$$

Theorem 7. *The space* $E = \{T_\Phi : \Phi \in h^\infty(\mathbb{D})\}$ *and* $\rho(E) = h^\infty(\mathbb{D})$.

Proof. : Let $T \in E$. Then

$$\langle Tk_z, k_z\rangle = \int_{\mathbb{D}} \langle Tk_{\phi_z(w)}, k_{\phi_z(w)}\rangle dA(w) \text{ for all } z \in \mathbb{D}. \tag{6}$$

Let $\Phi(z) = \rho(T)(z) = \langle Tk_z, k_z\rangle$. Then it follows from (6) that $\Phi(z) = \int_{\mathbb{D}} \Phi(\phi_z(w))dA(w)$ for all $z \in \mathbb{D}$. Hence $\Phi \in h^\infty(\mathbb{D})$ and $\langle Tk_z, k_z\rangle = \rho(T)(z) = \Phi(z) = \langle T_\Phi k_z, k_z\rangle$. Now since the map ρ is injective, we obtain $T = T_\Phi$.

To prove the converse, let $T = T_\Phi, \Phi \in h^\infty(\mathbb{D})$. Then

$$\begin{aligned} \int_{\mathbb{D}} \langle Tk_{\phi_z(w)}, k_{\phi_z(w)}\rangle dA(w) &= \int_{\mathbb{D}} \langle T_\Phi k_{\phi_z(w)}, k_{\phi_z(w)}\rangle dA(w) \\ &= \int_{\mathbb{D}} \Phi(\phi_z(w))dA(w) = \Phi(z) \\ &= \langle T_\Phi k_z, k_z\rangle \\ &= \langle Tk_z, k_z\rangle \\ &= \rho(T)(z) \text{ for all } z \in \mathbb{D}. \end{aligned}$$

Hence $T \in E$.

Let $T = T_\Phi \in E$ and $\Phi \in h^\infty(\mathbb{D})$. Further, $\rho(T) = \Phi \in h^\infty(\mathbb{D})$. Hence $\rho(E) \subseteq h^\infty(\mathbb{D})$. Conversely, let $\Phi \in h^\infty(\mathbb{D})$. Then $T_\Phi \in E$ and $\Phi = \rho(T_\Phi) \in \rho(E)$. Thus $h^\infty(\mathbb{D}) \subseteq \rho(E)$. □

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