Ann. Acad. Rom. Sci.
Ser. Math. Appl.
ISSN 2066-6594 Vol. 16, No. 1/2024

DECAY RATES AND INITIAL VALUES FOR TIME-FRACTIONAL DIFFUSION-WAVE EQUATIONS*

Masahiro Yamamoto [1,2,3] †

Dedicated to Dr. Dan Tiba on the occasion of his 70th anniversary

Abstract

We consider a solution $u(\cdot,t)$ to an initial boundary value problem for time-fractional diffusion-wave equation with the order $\alpha \in (0,2) \setminus \{1\}$ where t is a time variable. We first prove that a suitable norm of $u(\cdot,t)$ is bounded by the rate $t^{-\alpha}$ for $0 < \alpha < 1$ and $t^{1-\alpha}$ for $1 < \alpha < 2$ for all large $t > 0$. Second, we characterize initial values in the cases where the decay rates are faster than the above critical exponents. Differently from the classical diffusion equation $\alpha = 1$, the decay rate can keep some local characterization of initial values. The proof is based on the eigenfunction expansions of solutions and the asymptotic expansions of the Mittag-Leffler functions for large time.

MSC: 35R11, 35B40, 35C20
DOI https://doi.org/10.56082/annalsarscimath.2024.1.77
keywords: fractional diffusion-wave equation, decay rate, initial value

---

*Accepted for publication on November 24-th, 2023

†myama@ms.u-tokyo.ac.jp [1] Graduate School of Mathematical Sciences, The University of Tokyo, Komaba, Meguro, Tokyo 153-8914, Japan [2] Zonguldak Bülent Ecevit University, Turkey [3] Honorary Member of Academy of Romanian Scientists, Ilfov, nr. 3, Bucuresti, Romania [4] Correspondence Member of Accademia Peloritana dei Pericolanti, Palazzo Università, Piazza S. Pugliatti 1 98122 Messina, Italy. Paper written with financial supports of Grant-in-Aid for Scientific Research (A) 20H00117 and Grant-in-Aid for Challenging Research (Pioneering) 21K18142 of Japan Society for the Promotion of Science.

1 Introduction

Let $\Omega \subset R^d$ be a bounded domain with smooth boundary $\partial\Omega$ and let $\nu(x) := (\nu_1(x), ..., \nu_d(x))$ be the unit outward normal vector to $\partial\Omega$ at x. We assume that

$$0 < \alpha < 2, \quad \alpha \neq 1.$$

By ∂_t^α we denote the Caputo derivative:

$$\partial_t^\alpha g(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} \frac{d^n}{ds^n} g(s) ds$$

for $\alpha \notin \mathbf{N}$ satisfying $n-1 < \alpha < n$ with $n \in \mathbf{N}$ (e.g., Podlubny [13]). For $\alpha = 1$, we write $\partial_t g(t) = \frac{dg}{dt}$ and $\partial_t g(x,t) = \frac{\partial g}{\partial t}(x,t)$.

We consider an initial boundary value problem for a time-fractional diffusion-wave equation:

$$\begin{cases} \partial_t^\alpha u(x,t) = -Au(x,t), \quad x \in \Omega, 0 < t < T, \\ u|_{\partial\Omega \times (0,T)} = 0, \\ u(x,0) = a(x), \quad x \in \Omega \quad \text{if } 0 < \alpha \leq 1, \\ u(x,0) = a(x), \quad \partial_t u(x,0) = b(x), \quad x \in \Omega \quad \text{if } 1 < \alpha < 2. \end{cases} \tag{1.1}$$

Throughout this article, we set

$$(-Av)(x) = \sum_{i,j=1}^{d} \partial_i (a_{ij}(x) \partial_j v(x)) + c(x)v(x), \quad x \in \Omega,$$

where $a_{ij} = a_{ji}$, $1 \leq i, j \leq n$ and c are sufficiently smooth on $\overline{\Omega}$, and $c(x) \leq 0$ for $x \in \overline{\Omega}$, and we assume that there exists a constant $\sigma > 0$ such that

$$\sum_{i,j=1}^{d} a_{ij}(x)\zeta_i\zeta_j \geq \sigma \sum_{i=1}^{d} \zeta_i^2 \quad \text{for all } x \in \overline{\Omega} \text{ and } \zeta_1, ..., \zeta_d \in R.$$

For $\alpha \in (0,2) \setminus \{1\}$, the first equation in (1.1) is called a fractional diffusion-wave equation, which models anomalous diffusion in heterogeneous media. As for physical backgrounds, we are restricted to a few references: Metzler and Klafter [11], Roman and Alemany [14], and one can consult Chapter 10 in [13].

The properties such as asymptotic behavior as $t \to \infty$ of solution u to (1.1) are proved to depend on the fractional order α of the derivative. Moreover decay rates can characterize the initial values which is very different from the case $\alpha = 1$. The main purpose of this article is to study these topics.

Throughout this article, $L^2(\Omega)$, $H^\mu(\Omega)$ denote the usual Lebesgue space and Sobolev spaces (e.g., Adams [1]), and by $\|\cdot\|$ and $(\cdot,\cdot)$ we denote the norm and the scalar product in $L^2(\Omega)$ respectively. When we specify the norm in a Hilbert space Y, we write $\|\cdot\|_Y$. All the functions under consideration are assumed to be real-valued.

We define the domain $\mathcal{D}(A)$ of A by $H^2(\Omega)\cap H_0^1(\Omega)$. Then the operator A in $L^2(\Omega)$ has positive eigenvalues with finite multiplicities. We denote the set of all the eigenvalues by

$$0<\lambda_1<\lambda_2\cdots\longrightarrow\infty.$$

We set $\mathrm{Ker}\,(A-\lambda_n):=\{v\in\mathcal{D}(A); Av=\lambda_n v\}$ and $d_n:=\dim\mathrm{Ker}\,(A-\lambda_n)$. We denote an orthonormal basis of Ker $(A-\lambda_n)$ by $\{\varphi_{nk}\}_{1\le k\le d_n}$.

Then we define a fractional power A^γ with $\gamma\in R$ (e.g., Pazy [12]), and we see

$$\mathcal{D}(A^\gamma)=\left\{a\in L^2(\Omega);\ \sum_{n=1}^{\infty}\sum_{k=1}^{d_n}\lambda_n^{2\gamma}(a,\varphi_{nk})^2<\infty\right\}\quad\text{if }\gamma>0$$

and $\mathcal{D}(A^\gamma)\supset L^2(\Omega)$ if $\gamma\le 0$,

$$\begin{cases} A^\gamma a=\sum_{n=1}^{\infty}\lambda_n^\gamma\sum_{k=1}^{d_n}(a,\varphi_{nk})\varphi_{nk}, \\ \|A^\gamma a\|=\left(\sum_{n=1}^{\infty}\sum_{k=1}^{d_n}\lambda_n^{2\gamma}(a,\varphi_{nk})^2\right)^{\frac12}, \quad a\in\mathcal{D}(A^\gamma). \end{cases} \tag{1.2}$$

In particular,

$$A^{-1}a=\sum_{n=1}^{\infty}\sum_{k=1}^{d_n}\frac{1}{\lambda_n}(a,\varphi_{nk})\varphi_{nk},\quad \|A^{-1}a\|=\left(\sum_{n=1}^{\infty}\sum_{k=1}^{d_n}\frac{1}{\lambda_n^2}(a,\varphi_{nk})^2\right)^{\frac12}. \tag{1.3}$$

Moreover it is known that

$$\mathcal{D}(A^\gamma)\subset H^{2\gamma}(\Omega)\quad\text{for }\gamma\ge 0.$$

The well-posedness for (1.1) is studied for example in Gorenflo, Luchko and Yamamoto [8], Kubica, Ryszewska and Yamamoto [10], Sakamoto and Yamamoto [15]. As for the asymptotic behavior, we know

$$\|u(\cdot,t)\|\le\frac{C}{t^\alpha}\|a\|,\quad t>0 \tag{1.4}$$

(e.g., [10], [15], Vergara and Zacher [16]). The article [16] first established (1.4) for t-dependent symmetric operator A. Moreover by the eigenfunction expansion of $u(x,t)$ (e.g., [15]), one can prove

$$\|u(\cdot,t)\|\le\frac{C}{t^\alpha}\|a\|+\frac{C}{t^{\alpha-1}}\|b\|,\quad t>0 \tag{1.5}$$

for $1<\alpha<2$.

First we improve (1.4) and (1.5) with stronger norm of u.

Theorem 1. *Let $t_0 > 0$ be arbitrarily fixed. There exists a constant $C > 0$ depending on t_0 such that*

$$\|u(\cdot,t)\|_{H^2(\Omega)} \le \begin{cases} \frac{C}{t^\alpha}\|a\| & \text{if } 0 < \alpha < 1, \\ \frac{C}{t^\alpha}\|a\| + \frac{C}{t^{\alpha-1}}\|b\| & \text{if } 1 < \alpha < 2 \end{cases}$$

for $t \ge t_0$.

This theorem means that the Sobolev regularity of initial values is improved by 2 after any time $t > 0$ passes.

The fractional diffusion-wave equation (1.1) models slow diffusion, which the decay estimates (1.4) and (1.5) describe. For $\alpha = 1$, by the eigenfunction expansion of u, we can readily prove that $\|u(\cdot,t)\| \le e^{-\lambda_1 t}\|a\|$. Needless to say, Theorem 1 does not reject the exponential decay $e^{-\lambda_1 t}$, but as this article shows, the decay rates in the theorem are the best possible in a sense.

For further statements, we introduce a bounded linear operator $F : \mathcal{D}(A^\gamma) \longrightarrow Y$, where $\gamma > 0$ and Y is a Hilbert space with the norm $\|\cdot\|_Y$. We interpret that F is an observation mapping, and we consider the following four kinds of F.

Case 1. Let $\omega \subset \Omega$ be a subdomain. Let

$$F_1(v) = v|_\omega, \quad \mathcal{D}(F_1) = L^2(\Omega), \quad Y = L^2(\omega). \tag{1.6}$$

Then $F_1 : L^2(\Omega) \longrightarrow L^2(\omega)$ is bounded.

Case 2. Let $\Gamma \subset \partial\Omega$ be a subboundary. Let

$$F_2(v) = \partial_{\nu_A} v|_\Gamma, \quad \mathcal{D}(F_2) = H^2(\Omega), \quad Y = L^2(\Gamma). \tag{1.7}$$

Here we set

$$\partial_{\nu_A} v := \sum_{i,j=1}^{d} a_{ij}(x)(\partial_i v)(x)\nu_j(x).$$

The trace theorem (e.g., Adams [1]) implies that $F_2 : H^2(\Omega) \longrightarrow L^2(\Gamma)$ is bounded.

Case 3. Let $x^1, ..., x^M \in \Omega$ be fixed and let $\gamma > \frac{d}{4}$, where d is the spatial dimensions. We consider

$$F_3(v) = (v(x^1), ..., v(x^M)), \quad \mathcal{D}(F_3) = \mathcal{D}(A^\gamma), \quad Y = R^M. \tag{1.8}$$

Then the Sobolev embedding implies that $\mathcal{D}(F_3) \subset C(\overline{\Omega})$, and so $F_3 : \mathcal{D}(A^\gamma) \longrightarrow R^M$ is bounded. We interpret that F_3 are pointwise data.

Case 4. Let $\rho_1, ..., \rho_M \in L^2(\Omega)$ be given and let $Y = R^M$. Let

$$F_4(v) = \left(\int_\Omega \rho_k(x) v(x) dx \right)_{1 \le k \le M}, \quad \mathcal{D}(F_4) = L^2(\Omega), \quad Y = R^M. \tag{1.9}$$

Then $F_4 : L^2(\Omega) \longrightarrow R^M$ is bounded and corresponds to distributed data with weight functions ρ_k whose supports concentrate around some points in Ω.

Now we state

Theorem 2. *In (1.1) we assume that $a, b \in L^2(\Omega)$ for F_1, F_2, F_4 and $a, b \in \mathcal{D}(A^{\gamma_0})$ with $\gamma_0 = 0$ if $\frac{d}{4} < 1$ and $\gamma_0 > \frac{d}{4} - 1$ if $\frac{d}{4} \ge 1$ for F_3. Let $u = u(x,t)$ satisfy (1.1). For $j = 3, 4$, let F_j satisfy $F_j|_{Ker(\lambda_n - A)}$ is injective for all $n \in \mathbf{N}$.*

Furthermore we assume that for $j = 1, 2, 3, 4$, there exist sequences τ_n, $n \in \mathbf{N}$ and $C_n > 0$, $n \in \mathbf{N}$ which may depend on u, such that

$$\tau_n > 0, \quad \lim_{n \to \infty} \tau_n = \infty \tag{1.10}$$

and

$$\|F_j(u(\cdot, t))\|_Y \le \frac{C_n}{t^{\tau_n}} \quad \text{as } t \to \infty \text{ for all } n \in \mathbf{N}. \tag{1.11}$$

Then $u = 0$ in $\Omega \times (0, \infty)$.

For $0 < \alpha < 1$, a similar result is proved as Theorem 4.3 in [15], and Theorem 2 is an improvement.

Example 1 (Example of F_3 such that $F_3|_{\mathrm{Ker}(\lambda_n - A)}$ is injective)**.** *Let*

$$A = -\Delta, \quad d = 2, \quad \Omega = \{(x_1, x_2); 0 < x_1 < L_1, 0 < x_2 < L_2\}.$$

Then dim Ker $(A - \lambda_n) = 1$ for each $n \in \mathbf{N}$ if $\frac{L_1}{L_2} \notin \mathbf{Q}$. Indeed, the eigenvalues are given by

$$\lambda_{mn} := \left(\frac{m^2}{L_1^2} + \frac{n^2}{L_2^2} \right) \pi^2, \qquad m, n \in \mathbf{N}$$

and the corresponding eigenfunction $\varphi_{mn}(x)$ is given by $\sin \frac{m\pi}{L_1} x_1 \sin \frac{n\pi}{L_2} x_2$. Therefore, by $\frac{L_1}{L_2} \notin \mathbf{Q}$ we see that if $\lambda_{mn} = \lambda_{m'n'}$ with $m, n, m', n' \in \mathbf{N}$, then $m = m'$ and $n = n'$.

Let $x^1 = (x_1^1, x_2^1) \in \Omega$ satisfy $\frac{x_1^1}{L_1}, \frac{x_2^1}{L_2} \notin \mathbf{Q}$. We set $F_3(v) := v(x^1)$ and $M = 1$. Then we can readily verify that $F_3|_{Ker(\lambda_n - A)}$ is injective for all $n \in \mathbf{N}$.

The corresponding result to Theorem 2 can be proved for the classical diffusion equation $\alpha = 1$: if there exist sequences τ_n, $n \in \mathbf{N}$ and $C_n > 0$, $n \in \mathbf{N}$ which can depend on u such that $\tau_n > 0$, $\lim_{n \to \infty} \tau_n = \infty$ and

$$\|u(\cdot, t)\|_{L^2(\omega)} \le C_n e^{-\tau_n t} \quad \text{as } t \to \infty,$$

then $u = 0$ in $\Omega \times (0, \infty)$.

Next we consider characterizations of initial values yielding local faster decay than $\frac{1}{t^\alpha}$ and/or $\frac{1}{t^{\alpha-1}}$.

Theorem 3. **(i)** *Let F_1 be defined by (1.6).*
Case I: $0 < \alpha < 1$.
If

$$\|u(\cdot,t)\|_{L^2(\omega)} = o\left(\frac{1}{t^\alpha}\right) \quad as\ t \to \infty, \tag{1.12}$$

then

$$A^{-1}a = a = 0 \quad in\ \omega. \tag{1.13}$$

Moreover, assuming further that either $a \ge 0$ in Ω or $a \le 0$ in Ω, then (1.12) yields $a = 0$ in Ω.
Case II: $1 < \alpha < 2$.
If

$$\|u(\cdot,t)\|_{L^2(\omega)} = o\left(\frac{1}{t^{\alpha-1}}\right) \quad as\ t \to \infty, \tag{1.14}$$

then

$$A^{-1}b = b = 0 \quad in\ \omega. \tag{1.15}$$

If (1.12) holds, then we have $u(x_0,0) = \partial_t u(x_0,0) = 0$. Moreover, assuming further that either $b \ge 0$ in Ω or $b \le 0$ in Ω, then (1.14) yields $b = 0$ in Ω, and the same conclusion holds for a.
(ii) *Let F_3 be defined by (1.8) with $M = 1$ and $a, b \in \mathcal{D}(A^\gamma)$ with $\gamma > \frac{d}{4}$.*
Case I: $0 < \alpha < 1$.

$$|Au(x_0,t)| = |\partial_t^\alpha u(x_0,t)| = o\left(\frac{1}{t^\alpha}\right) \quad as\ t \to \infty \tag{1.16}$$

if and only if $u(x_0,0) = 0$.
Case II: $1 < \alpha < 2$.

$$|Au(x_0,t)| = |\partial_t^\alpha u(x_0,t)| = o\left(\frac{1}{t^{\alpha-1}}\right) \quad as\ t \to \infty \tag{1.17}$$

if and only if $\partial_t u(x_0,0) = 0$.

Moreover (1.16) holds if and only if $u(x_0,0) = \partial_t u(x_0,0) = 0$.

Theorem 3 asserts that the faster decay than $\frac{1}{t^\alpha}$ or $\frac{1}{t^{\alpha-1}}$ provides information that initial values vanishes at some point or in a subdomain.

In a special case, we prove

Proposition 1. *Let $x_0 \in \Omega$ be arbitrarily given. Let $a, b \in \mathcal{D}(A^\gamma)$ with $\gamma > \frac{d}{4}$, and*

$$\begin{cases} a \ge 0 \quad in\ \Omega \quad or \quad a \le 0 \quad in\ \Omega, \\ b \ge 0 \quad in\ \Omega \quad or \quad b \le 0 \quad in\ \Omega. \end{cases} \tag{1.18}$$

Case I: $0 < \alpha < 1$**.**

$$|u(x_0,t)| = o\left(\frac{1}{t^\alpha}\right) \quad as\ t \to \infty, \tag{1.19}$$

if and only if $u(x,0) = 0$ for $x \in \Omega$.
Case II: $1 < \alpha < 2$**.**

$$|u(x_0,t)| = o\left(\frac{1}{t^{\alpha-1}}\right) \quad as\ t \to \infty \tag{1.20}$$

if and only if $\partial_t u(x,0) = 0$ for $x \in \Omega$.

We cannot expect similar results to Theorem 2 for the classical diffusion equation, i.e., $\alpha = 1$.

Example 2 (Example of the classical diffusion equation)**.**

$$\begin{cases} \partial_t u(x,t) = \partial_x^2 u(x,t), \quad 0 < x < 1, t > 0, \\ u(0,t) = u(1,t) = 0, \quad t > 0, \\ u(x,0) = a(x), \quad 0 < x < 1. \end{cases}$$

Then it is well-known that for arbitrary $x_0 \in \Omega$ and $a \in L^2(0,1)$,

$$|u(x_0,t)| = o(e^{-\pi^2 t}) \quad as\ t \to \infty$$

if and only if

$$\sin \pi x_0 \int_0^1 a(x) \sin \pi x dx = 0. \tag{1.21}$$

In other words, Theorem 2 means that for $\alpha \in (0,2) \setminus \{1\}$, the faster decay at a point x_0 or in a subdomain ω still keeps some information of the initial value $a(x)$ at x_0 or in ω. On the other hand, in the case of $\alpha = 1$, the decay rate is influenced only by averaged information (1.21) of the initial value. However under extra assumption that the initial value a does not change the signs, by (1.21) we can conclude that $a = 0$ in Ω by $\sin \pi x \ge 0$ for $0 < x < 1$ if $\sin \pi x_0 \ne 0$. This is true for general dimensions, because one can prove that the eigenfunction for λ_1 does not change the signs.

This article is composed of five sections. In Section 2, we show lemmata which we use for the proofs of Theorems 1 - 3 and Proposition 1. Sections 3 and 4 are devoted to the proofs of Theorems 1-2 and Theorem 3 and Proposition 1, respectively. In Section 5, we give concluding remarks.

2 Preliminaries

For $\alpha > 0$, we define the Mittag-Leffler functions by

$$E_{\alpha,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}, \quad E_{\alpha,2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+2)}, \quad z \in \mathbf{C}$$

and it is know that $E_{\alpha,1}(z)$ and $E_{\alpha,2}(z)$ are entire functions in $z \in \mathbf{C}$ (e.g. Gorenflo, Kilbas, Mainardi and Rogosin [7], Podlubny [13]).

First we show

Lemma 1. *Let* $\beta = 1,2$ *and* $\alpha \in (0,2) \setminus \{1\}$.
(i) For $p \in \mathbf{N}$ *we have*

$$E_{\alpha,\beta}(-\eta) = \sum_{\ell=1}^{p} \frac{(-1)^{\ell+1}}{\Gamma(\beta-\alpha\ell)} \frac{1}{\eta^{\ell}} + O\left(\frac{1}{\eta^{p+1}}\right) \quad as\ \eta > 0, \to \infty. \tag{2.1}$$

(ii)

$$|E_{\alpha,\beta}(\eta)| \le \frac{C}{1+\eta} \quad for\ all\ \eta > 0. \tag{2.2}$$

Proof. As for (2.1), see Proposition 3.6 (pp.25-26) in [7] or Theorem 1.4 (pp.33-34) in [13]. The estimate (2.2) is seen by Theorem 1.6 (p.35) in [13] for example. Thus the proof of Lemma 1 is complete. □

Moreover, by the eigenfunction expansion of the solution u to (1.1) (e.g., Theorems 2.1 and 2.3 in [15]), we have

Lemma 2.

$$\begin{cases} u(x,t) = \sum_{n=1}^{\infty} E_{\alpha,1}(-\lambda_n t^{\alpha}) \sum_{k=1}^{d_n} (a,\varphi_{nk})\varphi_{nk}(x) \quad if\ 0<\alpha<1, \\ u(x,t) = \sum_{n=1}^{\infty} \Big[E_{\alpha,1}(-\lambda_n t^{\alpha}) \sum_{k=1}^{d_n} (a,\varphi_{nk})\varphi_{nk}(x) \\ + \quad tE_{\alpha,2}(-\lambda_n t^{\alpha}) \sum_{k=1}^{d_n} (b,\varphi_{nk})\varphi_{nk}(x) \Big] \quad if\ 1<\alpha<2 \end{cases} \tag{2.4}$$

in $C([0,T];L^2(\Omega)) \cap C((0,T];H^2(\Omega)\cap H_0^1(\Omega))$.

By Lemma 1, we can prove

Lemma 3. *(i) Let* $a,b \in \mathcal{D}(A^{\gamma_0})$ *where* $\gamma_0 = 0$ *if* $\frac{d}{4} < 1$ *and* $\gamma_0 > \frac{d}{4} - 1$ *if* $\frac{d}{4} \ge 1$. *We fix* $t_0 \in (0,T)$ *arbitrarily. Then the series in (2.4) are convergents in* $C(\overline{\Omega} \times [t_0,T])$.

(ii) Let $a, b \in L^2(\Omega)$. Then

$$\partial_{\nu_A} u(x,t) = \sum_{n=1}^{\infty} E_{\alpha,1}(-\lambda_n t^\alpha) \sum_{k=1}^{d_n} (a, \varphi_{nk}) \partial_{\nu_A} \varphi_{nk}(x) \quad \text{if } 0 < \alpha < 1,$$

$$\partial_{\nu_A} u(x,t) = \sum_{n=1}^{\infty} \Bigg[E_{\alpha,1}(-\lambda_n t^\alpha) \sum_{k=1}^{d_n} (a, \varphi_{nk}) \partial_{\nu_A} \varphi_{nk}(x)$$
$$+ \quad t E_{\alpha,2}(-\lambda_n t^\alpha) \sum_{k=1}^{d_n} (b, \varphi_{nk}) \partial_{\nu_A} \varphi_{nk}(x) \Bigg] \quad \text{if } 1 < \alpha < 2$$

in $C([t_0, T]; L^2(\partial\Omega))$.

For the proof of Lemma 3, we show

Lemma 4. *Let $\gamma \in R$, and let $t_0 \in (0, T)$ be given arbitrarily. We assume that $a, b \in \mathcal{D}(A^\gamma)$. Then there exists a constant $C = C(t_0, \gamma) > 0$ such that*

$$\|A^{\gamma+1} u(\cdot, t)\| \le \begin{cases} C t^{-\alpha} \|A^\gamma a\| & \text{if } 0 < \alpha < 1, \\ C(t^{-\alpha} \|A^\gamma a\| + t^{-\alpha+1} \|A^\gamma b\|) & \text{if } 1 < \alpha < 2 \end{cases}$$

for all $t \ge t_0$.

Proof of Lemma 4. For $\gamma \in R$, by each $u_0 \in \mathcal{D}(A^\gamma)$, applying (1.2), we see

$$A^{\gamma+1}(u_0, \varphi_{nk}) \varphi_{nk} = (u_0, \varphi_{nk}) \lambda_n^{\gamma+1} \varphi_{nk} = \lambda_n (u_0, \lambda_n^\gamma \varphi_{nk}) \varphi_{nk}$$
$$= \quad \lambda_n (u_0, A^\gamma \varphi_{nk}) \varphi_{nk} = \lambda_n (A^\gamma u_0, \varphi_{nk}) \varphi_{nk}.$$

Here we used $(u_0, A^\gamma \varphi_{nk}) = (A^\gamma u_0, \varphi_{nk})$ by (1.2). Therefore, in view of (2.4), we have

$$A^{\gamma+1} u(x,t) = \sum_{n=1}^{\infty} \lambda_n E_{\alpha,1}(-\lambda_n t^\alpha) \sum_{k=1}^{d_n} (A^\gamma a, \varphi_{nk}) \varphi_{nk}(x)$$
$$+ \quad t \sum_{n=1}^{\infty} \lambda_n E_{\alpha,2}(-\lambda_n t^\alpha) \sum_{k=1}^{d_n} (A^\gamma b, \varphi_{nk}) \varphi_{nk}(x)$$

in $C([0,T]; L^2(\Omega))$. We fix $t_0 > 0$ arbitrarily. Let $1 < \alpha < 2$. By (2.2) we see

$$\|A^{\gamma+1} u(\cdot, t)\|^2 \le \sum_{n=1}^{\infty} \lambda_n^2 |E_{\alpha,1}(-\lambda_n t^\alpha)|^2 \sum_{k=1}^{d_n} |(A^\gamma a, \varphi_{nk})|^2$$
$$+ \quad t^2 \sum_{n=1}^{\infty} \lambda_n^2 |E_{\alpha,2}(-\lambda_n t^\alpha)|^2 \sum_{k=1}^{d_n} |(A^\gamma b, \varphi_{nk})|^2$$
$$\le \quad C(t_0) \left(\frac{1}{t^{2\alpha}} \sum_{n=1}^{\infty} \lambda_n^2 \sum_{k=1}^{d_n} |(A^\gamma a, \varphi_{nk})|^2 \frac{1}{\lambda_n^2} + \frac{1}{t^{2\alpha-2}} \sum_{n=1}^{\infty} \lambda_n^2 \sum_{k=1}^{d_n} |(A^\gamma b, \varphi_{nk})|^2 \frac{1}{\lambda_n^2} \right)$$

for $t \geq t_0$. The proof for $0 < \alpha < 1$ is similar. Thus we complete the proof of Lemma 4. □

Now we proceed to

Proof of Lemma 3. By the condition on γ, we apply the Sobolev embedding to have

$$\|u(\cdot,t)\|_{C(\overline{\Omega})} \leq C\|A^{\gamma+1}u(\cdot,t)\|_{L^2(\Omega)}.$$

Therefore, Lemma 4 yields that the series in (2.4) converge in $C(\overline{\Omega}\times[t_0,T])$. Part (ii) is seen by the trace theorem:

$$\|\partial_{\nu_A}u(\cdot,t)\|_{L^2(\partial\Omega)} \leq C\|Au(\cdot,t)\|_{L^2(\Omega)}.$$

Thus the proof of Lemma 3 is complete. □

We conclude this section with

Lemma 5. *We assume that $p_n \in R$, $\{\ell_m\}_{m\in\mathbf{N}} \subset \mathbf{N}$ satisfying $\lim_{m\to\infty}\ell_m = \infty$, and there exist constants $C > 0$ and $\theta_0 \geq 0$ such that*

$$\sup_{n\in\mathbf{N}}|p_n| \leq C\lambda_n^{\theta_0}. \tag{2.5}$$

If

$$\sum_{n=1}^{\infty}\frac{p_n}{\lambda_n^{\ell_m}} = 0 \quad \textit{for all } m\in\mathbf{N},$$

then $p_n = 0$ for all $n\in\mathbf{N}$.

Proof. By μ_n, $n\in\mathbf{N}$, we renumber the eigenvalues λ_n of A according to the multiplicities:

$$\mu_k = \lambda_1 \text{ for } 1 \leq k \leq d_1, \quad \mu_k = \lambda_2 \text{ for } d_1+1 \leq k \leq d_1+d_2, \cdots.$$

Then $\mu_n \leq \lambda_n$ for $n\in\mathbf{N}$.

On the other hand, there exists a constant $c_1 > 0$ such that

$$\mu_n = c_1 n^{\frac{2}{d}} + o(1) \quad \text{as } n\to\infty$$

(e.g., Agmon [2], Theorem 15.1). Here we recall that d is the spatial dimensions. Therefore, we can find a constant $c_2 > 0$ such that $\lambda_n \geq c_2 n^{\frac{2}{d}}$ as $n\to\infty$. Hence, we can choose a large constant $\theta_1 > 0$, for example $\theta_1 > \frac{d}{2}$, such that

$$\sum_{n=1}^{\infty}\frac{1}{\lambda_n^{\theta_1}} < \infty.$$

We set

$$r_n := \frac{p_n}{\lambda_n^{\theta_0+\theta_1}}, \quad n \in \mathbf{N}.$$

Then (2.5) implies

$$\sum_{n=1}^{\infty} |r_n| \le \sum_{n=1}^{\infty} \left|\frac{p_n}{\lambda_n^{\theta_0}}\right| \frac{1}{\lambda_n^{\theta_1}} \le C \sum_{n=1}^{\infty} \frac{1}{\lambda_n^{\theta_1}} < \infty.$$

Since $\sum_{n=1}^{\infty} \frac{p_n}{\lambda_n^{\ell_m}} = 0$, we obtain $\sum_{n=1}^{\infty} \frac{r_n}{\lambda_n^{\kappa_m}} = 0$ for all $m \in \mathbf{N}$, where $\kappa_m = \ell_m - \theta_0 - \theta_1$, so that

$$\frac{r_1}{\lambda_1^{\kappa_m}} + \sum_{n=2}^{\infty} \frac{r_n}{\lambda_n^{\kappa_m}} = 0, \quad \text{that is,} \quad r_1 + \sum_{n=2}^{\infty} r_n \left(\frac{\lambda_1}{\lambda_n}\right)^{\kappa_m} = 0.$$

Hence

$$|r_1| = \left| - \sum_{n=2}^{\infty} r_n \left(\frac{\lambda_1}{\lambda_n}\right)^{\kappa_m} \right| \le \left(\sum_{n=2}^{\infty} |r_n|\right) \left(\frac{\lambda_1}{\lambda_2}\right)^{\kappa_m}.$$

By $0 < \lambda_1 < \lambda_2 < ....$, we see that $\left|\frac{\lambda_1}{\lambda_2}\right| < 1$. Letting $m \to \infty$, we see that $\kappa_m \to \infty$, and so $r_1 = 0$, that is, $p_1 = 0$. Therefore,

$$\sum_{n=2}^{\infty} \frac{r_n}{\lambda_n^{\kappa_m}} = 0.$$

Repeating the above argument, we have $p_2 = p_3 = \cdots = 0$. Thus the proof of Lemma 5 is complete. □

3 Proofs of Theorems 1 and 2

3.1 Proof of Theorem 1.

Now, by noting that $\|u(\cdot,t)\|_{H^2(\Omega)} \le C\|Au(\cdot,t)\|$ by $u(\cdot,t) \in \mathcal{D}(A)$, Theorem 1 follows directly from Lemma 4 with $\gamma = 0$ in Section 2.

3.2 Proof of Theorem 2.

First Step. It suffices to prove in the case $1 < \alpha < 2$, because the case $0 < \alpha < 1$ is similar and even simpler. In view of Lemma 3, for a and b satisfying the conditions in the theorem, we have

$$F_j(u(\cdot,t)) = \sum_{n=1}^{\infty} E_{\alpha,1}(-\lambda_n t^{\alpha}) F_j\left(\sum_{k=1}^{d_n} (a,\varphi_{nk})\varphi_{nk}\right)$$
$$+ \quad t\sum_{n=1}^{\infty} \lambda_n E_{\alpha,2}(-\lambda_n t^{\alpha}) F_j\left(\sum_{k=1}^{d_n} (b,\varphi_{nk})\varphi_{nk}\right), \quad j = 1,2,3,4$$

in $C([t_0,T];Y)$, where

$$Y = \begin{cases} L^2(\omega) & \text{for } F_1, \\ L^2(\partial\Omega) & \text{for } F_2, \\ R^M & \text{for } F_3 \text{ and } F_4. \end{cases}$$

Applying (2.1) in Lemma 1, we obtain

$$\begin{aligned} F_j(u(\cdot,t)) = &\sum_{\ell=1}^{p} \frac{(-1)^{\ell+1}}{\Gamma(1-\alpha\ell)t^{\alpha\ell}} \sum_{n=1}^{\infty} \frac{1}{\lambda_n^\ell} F_j\left(\sum_{k=1}^{d_n}(a,\varphi_{nk})\varphi_{nk}\right) \\ &+O\left(\frac{1}{t^{\alpha p+\alpha}}\right)\sum_{n=1}^{\infty} F_j\left(\sum_{k=1}^{d_n}(a,\varphi_{nk})\varphi_{nk}\right) \\ &+\sum_{\ell=1}^{p} \frac{(-1)^{\ell+1}}{\Gamma(2-\alpha\ell)t^{\alpha\ell-1}} \sum_{n=1}^{\infty} \frac{1}{\lambda_n^\ell} F_j\left(\sum_{k=1}^{d_n}(b,\varphi_{nk})\varphi_{nk}\right) \\ &+O\left(\frac{1}{t^{\alpha p+\alpha-1}}\right)\sum_{n=1}^{\infty} F_j\left(\sum_{k=1}^{d_n}(b,\varphi_{nk})\varphi_{nk}\right). \end{aligned} \tag{3.1}$$

Therefore, (3.1) yields

$$\begin{aligned} F_j(u(\cdot,t)) = &\sum_{\ell=1}^{p} \frac{(-1)^{\ell+1}}{\Gamma(1-\alpha\ell)t^{\alpha\ell}} \sum_{n=1}^{\infty} \frac{p_n}{\lambda_n^\ell} \\ &+\sum_{\ell=1}^{p} \frac{(-1)^{\ell+1}}{\Gamma(2-\alpha\ell)t^{\alpha\ell-1}} \sum_{n=1}^{\infty} \frac{q_n}{\lambda_n^\ell} + O\left(\frac{1}{t^{\alpha p+\alpha-1}}\right) \quad \text{as } t\to\infty. \end{aligned} \tag{3.2}$$

Here we set

$$p_n = F_j\left(\sum_{k=1}^{d_n}(a,\varphi_{nk})\varphi_{nk}\right), \quad q_n = F_j\left(\sum_{k=1}^{d_n}(b,\varphi_{nk})\varphi_{nk}\right)$$

for $j=1,2,3,4$.

In the above series, we exclude $\ell \in \mathbf{N}$ such that $1-\alpha\ell, 2-\alpha\ell \in \{0,-1,-2,...\}$, that is, these terms do not appear if $\alpha\ell \in \mathbf{N}$.

Second Step. We see that

$$\{\ell \in \mathbf{N};\, \alpha\ell \notin \mathbf{N}\} \text{ is an infinite set if } \alpha \notin \mathbf{N}. \tag{3.3}$$

Indeed, if not, then $\mathbf{N} = \{\ell \in \mathbf{N};\, \alpha\ell \notin \mathbf{N}\} \cup \{\ell \in \mathbf{N};\, \alpha\ell \in \mathbf{N}\}$ implies that there exists $N_0 \in \mathbf{N}$ such that $\{\ell \in \mathbf{N};\, \alpha\ell \in \mathbf{N}\} \supset \{N_0, N_0+1, ...\}$. Therefore $\alpha N_0,\ \alpha(N_0+1) \in$

$\mathbf{N}$, which yields $\alpha = \alpha(N_0+1) - \alpha N_0 \in \mathbf{N}$. By $\alpha \not\in \mathbf{N}$, this is impossible. Therefore (3.3) holds.

We number the infinite set $\{\ell \in \mathbf{N};\, \alpha\ell \not\in \mathbf{N}\}$ by $\ell_1, \ell_2, \ell_3, ...$ and for each $N \in \mathbf{N}$, we can rewrite (3.2) as

$$F_j(u(\cdot,t)) = \sum_{m=1}^{N} \frac{(-1)^{\ell_m+1}}{\Gamma(1-\alpha\ell_m)t^{\alpha\ell_m}} \sum_{n=1}^{\infty} \frac{p_n}{\lambda_n^{\ell_m}}$$

$$+ \sum_{m=1}^{N} \frac{(-1)^{\ell_m+1}}{\Gamma(2-\alpha\ell_m)t^{\alpha\ell_m-1}} \sum_{n=1}^{\infty} \frac{q_n}{\lambda_n^{\ell_m}} + O\left(\frac{1}{t^{\alpha\ell_{N+1}-1}}\right) \quad \text{as } t \to \infty. \tag{3.4}$$

Moreover

$$\{\alpha n\}_{n\in\mathbf{N}} \cap \{\alpha n - 1\}_{n\in\mathbf{N}} = \emptyset \quad \text{for } 1 < \alpha < 2. \tag{3.5}$$

Indeed let $\alpha n' = \alpha n'' - 1$ with some $n', n'' \in \mathbf{N}$. Then $\alpha\ell_0 = 1$ with $\ell_0 := n'' - n'$, which means $\alpha \le 1$ and this is a contradiction by $1 < \alpha < 2$.

By (3.5), we number $\{\alpha\ell_m\}_{m\in\mathbf{N}} \cup \{\alpha\ell_m - 1\}_{m\in\mathbf{N}}$ by $\alpha\ell_1 - 1 =: s_1 < s_2 < \cdots < s_{2N} := \alpha\ell_N$ and then

$$F_j(u(\cdot,t)) = \sum_{m=1}^{2N} \frac{Q_m}{t^{s_m}} + O\left(\frac{1}{t^{\alpha\ell_{N+1}-1}}\right) \quad \text{in } C([t_0,T];Y) \text{ as } t \to \infty, \tag{3.6}$$

where

$$Q_m = \frac{(-1)^{\ell_m+1}}{\Gamma(1-\alpha\ell_m)} \sum_{n=1}^{\infty} \frac{p_n}{\lambda_n^{\ell_m}} \quad \text{or} \quad Q_m = \frac{(-1)^{\ell_m+1}}{\Gamma(2-\alpha\ell_m)} \sum_{n=1}^{\infty} \frac{q_n}{\lambda_n^{\ell_m}}.$$

Third Step. We fix $N \in \mathbf{N}$ arbitrarily. In terms of (1.11), by (3.6) we see that for each $n \in \mathbf{N}$ there exists a constant $C_n > 0$ such that

$$\frac{\|Q_1\|_Y}{t^{s_1}} - \sum_{m=2}^{2N} \frac{\|Q_m\|_Y}{t^{s_m}} - \frac{C}{t^{\alpha\ell_{N+1}-1}} \le \frac{C_n}{t^{\tau_n}}.$$

Then

$$\|Q_1\|_Y \le \sum_{m=2}^{2N} \frac{\|Q_m\|_Y}{t^{s_m-s_1}} + \frac{C}{t^{\alpha\ell_{N+1}-1-s_1}} + \frac{C_n}{t^{\tau_n-s_1}}.$$

We note that $\alpha\ell_N < \alpha\ell_{N+1} - 1$ by $\alpha > 1$ and $\ell_n, \ell_{N+1} \in \mathbf{N}$, so that $s_{2N} < \alpha\ell_{N+1} - 1$.

Since $\lim_{n\to\infty}\tau_n = \infty$, we can choose $n \in \mathbf{N}$ such that $\tau_n > s_1$. Hence, letting $t \to \infty$, we have $Q_1 = 0$ in Y. Continuing this argument, we reach $Q_m = 0$ for $1 \le m \le 2N$. Since $N \in \mathbf{N}$ is arbitrary, we obtain $Q_m = 0$ for all $m \in \mathbf{N}$, that is,

$$\sum_{n=1}^{\infty} \frac{p_n}{\lambda_n^{\ell_m}} = \sum_{n=1}^{\infty} \frac{q_n}{\lambda_n^{\ell_m}} = 0 \quad \text{for all } m \in \mathbf{N}.$$

In order to apply Lemma 5, we have to verify (2.5). It suffices to consider for p_n, because the verification for q_n is the same.

Case: $F_1(u(\cdot,t))$**.** By the Sobolev embedding (e.g., [1]), fixing $\mu_0 > 0$ with $2\mu_0 > d$, we have

$$\begin{aligned} \|p_n\|_{C(\overline{\Omega})} \le C\|p_n\|_{H^{\mu_0}(\Omega)} &\le C\left\| A^{\frac{\mu_0}{2}}\left(\sum_{k=1}^{d_n}(a,\varphi_{nk})\varphi_{nk}\right)\right\|_{L^2(\Omega)} \\ = \quad C\lambda_n^{\frac{\mu_0}{2}}\left\|\sum_{k=1}^{d_n}(a,\varphi_{nk})\varphi_{nk}\right\|_{L^2(\Omega)} &\le C\lambda_n^{\frac{\mu_0}{2}}\|a\|_{L^2(\Omega)}. \end{aligned}$$

For the second inequality, we need sufficient smoothness of the coefficients a_{ij} and c of the elliptic operator A (e.g., Gilbarg and Trudinger [6]). Therefore

$$\|p_n\|_{C(\overline{\Omega})} \le C\lambda_n^{\frac{\mu_0}{2}}, \quad n \in \mathbf{N}.$$

Therefore, we see (2.5) for F_1, F_3 and F_4 with $\theta_0 = \frac{\mu_0}{2}$.

Case: $F_2(u(\cdot,t))$**.** We fix $\mu_0 > 0$ such that $2\mu_0 > d$. Then by the Sobolev embedding, we obtain

$$\begin{aligned} \left\|\partial_{\nu_A}\left(\sum_{k=1}^{d_n}(a,\varphi_{nk})\varphi_{nk}\right)\right\|_{C(\partial\Omega)} &\le C\left\|\left(\sum_{k=1}^{d_n}(a,\varphi_{nk})\varphi_{nk}\right)\right\|_{C^1(\overline{\Omega})} \\ \le \quad C\left\|\sum_{k=1}^{d_n}(a,\varphi_{nk})\varphi_{nk}\right\|_{H^{\mu_0+1}(\Omega)} &\le C\left\|A^{\frac{\mu_0}{2}+\frac{1}{2}}\sum_{k=1}^{d_n}(a,\varphi_{nk})\varphi_{nk}\right\|_{L^2(\Omega)} \\ = \quad C\lambda_n^{\frac{\mu_0}{2}+\frac{1}{2}}\left\|\sum_{k=1}^{d_n}(a,\varphi_{nk})\varphi_{nk}\right\|_{L^2(\Omega)} &\le C\lambda_n^{\frac{\mu_0}{2}+\frac{1}{2}}\|a\|_{L^2(\Omega)}. \end{aligned}$$

Hence (2.5) is satisfied with $\theta_0 = \frac{\mu_0}{2} + \frac{1}{2}$.

Therefore, Lemma 5 yields $p_n = q_n = 0$ for all $n \in \mathbf{N}$, that is,

$$F_j\left(\sum_{k=1}^{d_n}(a, \varphi_{nk})\varphi_{nk}\right)$$

$$= F_j\left(\sum_{k=1}^{d_n}(b, \varphi_{nk})\varphi_{nk}\right) = 0, \quad j = 1,2,3,4, \quad n \in \mathbf{N}. \tag{3.7}$$

Fourth Step. It suffices to verify that $p_n = 0$ for all $n \in \mathbf{N}$ imply $a = 0$ in Ω. For F_3 and F_4, the assumption in Theorem 2 yields

$$\sum_{k=1}^{d_n}(a, \varphi_{nk})\varphi_{nk} = \sum_{k=1}^{d_n}(b, \varphi_{nk})\varphi_{nk} = 0 \quad \text{in } \Omega$$

for all $n \in \mathbf{N}$. Therefore, $a = b = 0$ in Ω, that is, $u = 0$ in $\Omega \times (0, \infty)$. Thus the proof of Theorem 2 is complete for F_3 and F_4.

Case: F_1**.** By (3.7), we have

$$p_n(x) = \sum_{k=1}^{d_n}(a, \varphi_{nk})\varphi_{nk}(x) = 0, \quad n \in \mathbf{N}, x \in \omega.$$

Since $(A - \lambda_n)p_n = 0$ in Ω, we apply the unique continuation for the elliptic operator $A - \lambda_n$ (e.g., Choulli [3], Hörmander [9]) to see that $p_n = 0$ in Ω for $n \in \mathbf{N}$. Since $a = \sum_{n=1}^{\infty} p_n$ in $L^2(\Omega)$, we reach $a = 0$ in Ω.

Case: F_2**.** We set $u_n(x) = \sum_{k=1}^{d_n}(a, \varphi_{nk})\varphi_{nk}(x)$ for $x \in \Omega$. By $u_n \in \mathcal{D}(A)$, we have $u_n = 0$ on Γ and so

$$\partial_{\nu_A} u_n(x) = u_n(x) = 0, \quad n \in \mathbf{N}, x \in \Gamma.$$

Therefore, since $(A - \lambda_n)u_n = 0$ in Ω, the unique continuation (e.g., [3], [9]) yields $\sum_{k=1}^{d_n}(a, \varphi_{nk})\varphi_{nk}(x) = 0$ for all $n \in \mathbf{N}$ and $x \in \Omega$. Hence, we can see $a = 0$ in Ω. Thus the proof of Theorem 2 is complete.

4 Proofs of Theorem 3 and Proposition 1

4.1 Proof of Theorem 3

Case: F_1**.** It is sufficient to prove the case $1 < \alpha < 2$. Let (1.14) hold. By (3.2) with $p = 1$, noting that $\Gamma(1-\alpha)$ and $\Gamma(2-\alpha)$ are finite, we see

$$\left\| \frac{1}{\Gamma(1-\alpha)} \frac{1}{t^{\alpha}} \sum_{n=1}^{\infty} \sum_{k=1}^{d_n} \frac{(a, \varphi_{nk})\varphi_{nk}}{\lambda_n} + \frac{1}{\Gamma(2-\alpha)} \frac{1}{t^{\alpha-1}} \sum_{n=1}^{\infty} \sum_{k=1}^{d_n} \frac{(b, \varphi_{nk})\varphi_{nk}}{\lambda_n} \right\|_{L^2(\omega)}$$

$$= o\left(\frac{1}{t^{\alpha-1}}\right) \quad \text{as } t \to \infty. \tag{4.1}$$

Therefore, in terms of (1.3), we obtain

$$\frac{1}{\Gamma(2-\alpha)} \frac{1}{t^{\alpha-1}} \|A^{-1}b\|_{L^2(\omega)} - \frac{1}{\Gamma(1-\alpha)} \frac{1}{t^{\alpha}} \|A^{-1}a\|_{L^2(\omega)} = o\left(\frac{1}{t^{\alpha-1}}\right)$$

as $t \to \infty$. Multiplying with $t^{\alpha-1}$ and letting $t \to \infty$, we obtain $A^{-1}b = 0$ in ω.

Next let (1.12) hold. Then, by $o\left(\frac{1}{t^{\alpha}}\right) \le o\left(\frac{1}{t^{\alpha-1}}\right)$, we have also (1.14), so that we have already proved $A^{-1}b = 0$ in ω. Therefore, since

$$\frac{(-1)^2}{\Gamma(2-\alpha)} \frac{1}{t^{\alpha-1}} \sum_{n=1}^{\infty} \sum_{k=1}^{d_n} \frac{(b, \varphi_{nk})\varphi_{nk}}{\lambda_n} = \frac{1}{\Gamma(2-\alpha)} \frac{1}{t^{\alpha-1}} A^{-1}b = 0 \quad \text{in } \omega,$$

equality (3.2) with $p = 1$ and (1.12) yield

$$\frac{1}{\Gamma(1-\alpha)} \frac{1}{t^{\alpha}} \|A^{-1}a\|_{L^2(\omega)} + o\left(\frac{1}{t^{2\alpha}}\right) + o\left(\frac{1}{t^{2\alpha-1}}\right) = o\left(\frac{1}{t^{\alpha}}\right).$$

Multiplying with t^{α} and letting $t \to \infty$, by $\alpha - 1 > 0$, we see that $A^{-1}a = 0$ in ω.

Moreover $A^{-1}a = 0$ in ω implies $a = 0$ in ω. Indeed, setting $g := A^{-1}a$ in Ω, we have $g = 0$ in ω and $Ag = a$ in Ω. Therefore, $a = A0 = 0$ in ω. Similarly $A^{-1}b = 0$ in ω yields $b = 0$ in ω.

Finally we have to prove that the extra condition

$$a \ge 0 \text{ in } \Omega \text{ or } a \le 0 \text{ in } \Omega, \tag{4.2}$$

implies $a = 0$ in Ω.

Let $a \ge 0$ in Ω. Then $g := A^{-1}a$ satisfies

$$\sum_{i,j=1}^{d} \partial_i(a_{ij}(x)\partial_j g(x)) + c(x)g(x) \ge 0 \quad \text{in } \Omega.$$

By $c \le 0$ in Ω and $g = 0$ on $\partial\Omega$, the weak maximum principle (e.g., Theorem 3.1 (p.32) in Gilbarg and Trudinger [6]) implies that $g \le 0$ on $\overline{\Omega}$. Since $g(x) = 0$ for $x \in \omega$, we see that g achieves the maximum 0 at an interior point $x_0 \in \Omega$. Again by $c \le 0$ in Ω, the strong maximum principle (e.g., Theorem 3.5 (p.35) in [6]) yields that g is a constant function, that is, $g(x) = 0$ for all $x \in \Omega$. Hence, $a = Ag = 0$ in Ω. Thus the proof in the case F_1 is complete.

Case: F_3**.** It suffices to prove only in the case $1 < \alpha < 2$. By Lemma 2, for arbitrarily chosen $t_0 \in (0,T)$, we see

$$Au(x,t) = \sum_{n=1}^{\infty} E_{\alpha,1}(-\lambda_n t^{\alpha}) \sum_{k=1}^{d_n} (a, \varphi_{nk})\lambda_n\varphi_{nk}$$
$$+ \quad \sum_{n=1}^{\infty} tE_{\alpha,2}(-\lambda_n t^{\alpha}) \sum_{k=1}^{d_n} (b, \varphi_{nk})\lambda_n\varphi_{nk} \quad \text{in } C([t_0,T];L^2(\Omega)).$$

Using $a,b \in \mathcal{D}(A^{\gamma})$ with $\gamma > \frac{d}{4}$ and noting

$$A^{\gamma}(a, \varphi_{nk})\lambda_n\varphi_{nk} = \lambda_n^{1+\gamma}(a, \varphi_{nk})\varphi_{nk} = \lambda_n(a, A^{\gamma}\varphi_{nk})\varphi_{nk} = \lambda_n(A^{\gamma}a, \varphi_{nk})\varphi_{nk},$$

we obtain

$$A^{1+\gamma}u(x,t) = \sum_{n=1}^{\infty} E_{\alpha,1}(-\lambda_n t^{\alpha}) \sum_{k=1}^{d_n} (A^{\gamma}a, \varphi_{nk})\lambda_n\varphi_{nk}$$
$$+ \quad \sum_{n=1}^{\infty} tE_{\alpha,2}(-\lambda_n t^{\alpha}) \sum_{k=1}^{d_n} (A^{\gamma}b, \varphi_{nk})\lambda_n\varphi_{nk}.$$

Consequently, by Lemma 1, we can prove

$$\|A^{1+\gamma}u\|_{L^{\infty}(t_0,T;L^2(\Omega))} < \infty,$$

and so the above series is convergent in $L^{\infty}(t_0,T;L^2(\Omega))$. Since the Sobolev embedding implies $\mathcal{D}(A^{\gamma}) \subset C(\overline{\Omega})$ with $\gamma > \frac{d}{4}$, we obtain

$$Au(x_0,t) = \sum_{n=1}^{\infty} E_{\alpha,1}(-\lambda_n t^{\alpha}) \sum_{k=1}^{d_n} (a, \varphi_{nk})\lambda_n\varphi_{nk}(x_0)$$
$$+ \quad \sum_{n=1}^{\infty} tE_{\alpha,2}(-\lambda_n t^{\alpha}) \sum_{k=1}^{d_n} (b, \varphi_{nk})\lambda_n\varphi_{nk}(x_0), \quad t_0 < t < T \quad \text{in } C[t_0,T].$$

Substituting (2.1) with $p=1$ and $\beta = 1,2$, we have

$$Au(x_0,t) = \frac{1}{\Gamma(1-\alpha)}\sum_{n=1}^{\infty}\sum_{k=1}^{d_n}(a,\,\varphi_{nk})\varphi_{nk}(x_0)\frac{1}{t^{\alpha}}$$
$$+\quad \frac{1}{\Gamma(2-\alpha)}\sum_{n=1}^{\infty}\sum_{k=1}^{d_n}(b,\,\varphi_{nk})\varphi_{nk}(x_0)\frac{1}{t^{\alpha-1}}+O\left(\frac{1}{t^{2\alpha-1}}\right)\quad \text{as } t\to\infty.$$

By $a,b\in\mathcal{D}(A^{\gamma})\subset C(\overline{\Omega})$, we find

$$Au(x_0,t) = \frac{1}{\Gamma(1-\alpha)t^{\alpha}}a(x_0)+\frac{1}{\Gamma(2-\alpha)t^{\alpha-1}}b(x_0)+O\left(\frac{1}{t^{2\alpha-1}}\right)\quad \text{as } t\to\infty. \tag{4.3}$$

By an argument similar to Case F_1 in Theorem 3, we see that (1.16) and (1.17) imply $a(x_0)=0$ and $b(x_0)=0$ respectively. The converse assertion in the theorem directly follows from (4.3).

4.2 Proof of Proposition 1

It is sufficient to prove in the case $1<\alpha<2$. By $a,b\in\mathcal{D}(A^{\gamma})\subset C(\overline{\Omega})$ with $\gamma>\frac{d}{4}$, similarly to (4.1), we obtain

$$\begin{aligned} & u(x_0,t)\\ =\ & \frac{1}{\Gamma(1-\alpha)}\frac{1}{t^{\alpha}}\sum_{n=1}^{\infty}\sum_{k=1}^{d_n}\frac{(a,\,\varphi_{nk})}{\lambda_n}\varphi_{nk}(x_0)\\ +\ & \frac{1}{\Gamma(2-\alpha)}\frac{1}{t^{\alpha-1}}\sum_{n=1}^{\infty}\sum_{k=1}^{d_n}\frac{(b,\,\varphi_{nk})}{\lambda_n}\varphi_{nk}(x_0)+O\left(\frac{1}{t^{2\alpha-1}}\right)\\ =\ & \frac{1}{\Gamma(1-\alpha)}\frac{1}{t^{\alpha}}(A^{-1}a)(x_0)+\frac{1}{\Gamma(2-\alpha)}\frac{1}{t^{\alpha-1}}(A^{-1}b)(x_0)+O\left(\frac{1}{t^{2\alpha-1}}\right)\end{aligned}$$

as $t\to\infty$. Similarly to the case F_1 in the proof of Theorem 3, we can prove that (1.20) implies $(A^{-1}b)(x_0)=0$. Under the assumption that b does not change the signs in Ω, in view of the weak and the strong maximum principles, we can argue similarly to the final part of the proof of Theorem 3 in the case of F_1, so that we can reach $b=0$ in Ω. Therefore, we prove that (1.20) implies $b(x)=0$ for $x\in\Omega$. The converse statement of the proposition is readily seen. Thus the proof of Proposition 1 is complete.

5 Concluding remarks

5.1. Time-fractional diffusion-wave equations with order $\alpha \in (0,2)\setminus\{1\}$ describe slow diffusion and are known not to have strong smoothing property as the classical diffusion equation. Such a weak smoothing property is characterized by the norm equivalence between $\|u(\cdot,t)\|_{H^2(\Omega)}$ and $\|u(\cdot,0)\|_{L^2(\Omega)}$ for any $t>0$ in the case of $0<\alpha<1$. The weak smoothing property allows that the backward problem in time is well-posed for $\alpha \in (0,2)\setminus\{1\}$ (Floridia, Li and Yamamoto [4], Floridia and Yamamoto [5], Sakamoto and Yamamoto [15]), which is a remarkable difference from the case $\alpha = 1$.

The present article establishes that local properties of initial values affect the decay rate of solution as $t\to\infty$, which indicates that a time-fractional equation can keep some profile of the initial value even for very large $t>0$. This property can be understood related to the backward well-posedness in time and is essentially different from the case $\alpha = 1$.

The essence of the argument relies on that the behavior of a solution u for large $t>0$ admits an asymptotic expansion with respect to $\left(\frac{1}{t}\right)^{\alpha\ell}$ and $\left(\frac{1}{t}\right)^{\alpha\ell-1}$ with $\ell\in\mathbf{N}$.

5.2. We can generalize Theorem 3 (ii). For simplicity, we consider only the case $0<\alpha<1$.

Proposition 2. *Let $a\in\mathcal{D}(A^\gamma)$ with $\gamma>\frac{d}{4}$ and $0<\alpha,\beta<1$. Then*

$$|\partial_t^\beta u(x_0,t)| \le \frac{C}{t^\beta}\|a\|.$$

If

$$|\partial_t^\beta u(x_0,t)| = o\left(\frac{1}{t^\beta}\right) \quad \text{as } t\to\infty,$$

then $u(x_0,0)=0$.

The proof relies on

$$\partial_t^\beta u(x,t) = -t^{\alpha-\beta}\sum_{n=1}^{\infty}\lambda_n E_{\alpha,\alpha+1-\beta}(-\lambda_n t^\alpha)\sum_{k=1}^{d_n}(a,\varphi_{nk})\varphi_{nk}(x)$$
$$\text{in } C((0,T];L^2(\Omega)) \tag{5.1}$$

and then we can argue similarly to Theorem 3 (ii) by (2.1). The equation (5.1) can be verified as follows:

$$\partial_t^\beta(t^{\alpha k}) = \frac{\Gamma(\alpha k+1)}{\Gamma(\alpha k+1-\beta)}t^{\alpha k-\beta}, \quad k\in\mathbf{N},$$

and so the termwise differentiation yields

$$\partial_t^\beta E_{\alpha,1}(-\lambda_n t^\alpha) = -\lambda_n t^{\alpha-\beta} E_{\alpha,\alpha+1-\beta}(-\lambda_n t^\alpha), \quad t > 0.$$

Then (2.4) yields (5.1).

We omit the details of the proof of Proposition 2.

References

[1] R.A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[2] S. Agmon, *Lectures on Elliptic Boundary Value Problems*, D. van Nostrand, Princeton, 1965.

[3] M. Choulli, *Applications of Elliptic Carleman Inequalities to Cauchy and Inverse Problems*, Springer-Verlag, Berlin, 2016.

[4] G. Floridia, Z. Li, and M. Yamamoto, Well-posedness for the backward problems in time for general time-fractional diffusion equation, *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.* 31:593-610, 2020.

[5] G. Floridia and M. Yamamoto, Backward problems in time for fractional diffusion-wave equation, *Inverse Problems* 36: 125016, 14 pp., 2020

[6] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin, 2001.

[7] R. Gorenflo, A.A. Kilbas, F. Mainardi and S. V. Rogosin, *Mittag-Leffler Functions, Related Topics and Applications*, Springer-Verlag, Berlin, 2014.

[8] R. Gorenflo, Y. Luchko and M. Yamamoto, Time-fractional diffusion equation in the fractional Sobolev spaces, *Fract. Calc. Appl. Anal.* 18:799-820, 2015.

[9] L. Hörmander, *Linear Partial Differential Operators*, Springer-Verlag, Berlin, 1976.

[10] A. Kubica, K. Ryszewska, and M. Yamamoto, *Theory of Time-fractional Differential Equations An Introduction*, Springer Japan, Tokyo, 2020.

[11] R. Metzler and J. Klafter, The random walk's guide to anomalous diffusion: a fractional dynamics approach, *Phyics Reports* 339: 1–77, 2000.

[12] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, Berlin, 1983

[13] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[14] H.E. Roman and P.A. Alemany, Continuous-time random walks and the fractional diffusion equation, *J. Phys. A:Math. Gen.* 27: 3407–3410, 1994.

[15] K. Sakamoto and M. Yamamoto, Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems, *J. Math. Anal. Appl.* 382: 426-447, 2011.

[16] V. Vergara and R. Zacher, Optimal decay estimates for time-fractional and other nonlocal subdiffusion equation via energy methods, *SIAM J. Math. Anal.* 47: 210-239, 2015.