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COUNTING PATHS OF GRAPHS VIA INCIDENCE MATRICES*

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Abstract

Operating only by means of the incidence matrix of a connected graph G, a new algebraic combinatorial method for determining the paths of length $(q-1)$ of G together with the generators of the corresponding generalized graph ideal $I_q(G)$ is discussed and developed. The stated formulae are obtained and shown even by changing techniques appropriately when the difficulties of calculation increased.

MSC: 05B20, 05C38, 05C50

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Introduction

Algebraically speaking, determining some paths of length $(q-1)$, q positive integer, of a connected undirected graph G, means to find generators of a monomial ideal to which G can be associated, the generalized graph ideal $I_q(G)$ (see [3, 4, 5]).
The problem of computing, using only the incidence matrix of G, the number and structure of paths of fixed length in G, and the generators of the relative generalized graph ideal, presents aspects useful in various scientific and statistical research areas.

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In more detail, the number of paths of length at least 2 is obtained in terms of multiplicity of pairs of rows in the incidence matrix of G. Their structure is determined by joining alternatively on the rows and the columns of the incidence matrix specific entries 1 related to the vertices of the paths beginning from the inside, that is either from a combination of pairs of 1's on the rows or from the two 1's on the columns of the matrix, respectively. Increasing the length, by considering appropriate and sometimes innovative strategies in the calculations, the problem is solved up to $q = 6$.
The extension method is not at all simple, due especially to the presence of cycle subgraphs in the graph G. Along the script, selected explanatory examples are given.

1 Preliminary notions

Throughout the paper, following [1], let G be a graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G) = \{e_1, \ldots, e_m\}$.
A *walk* of G is a sequence of alternating vertices (or points) and edges $v_1, e_1, v_2, \ldots, e_{q-1}, v_q$, where each $e_i = \{v_i, v_{i+1}\}$. A walk is *closed* if endpoints coincide. The *length* of a walk is the number of its edges.
A graph G is *connected* if all the pairs of vertices in G are joined by a *path*, namely a walk with no repeated vertices and edges.
A *tree* is a connected graph with no *cycles*, that is with no closed paths.
A *cycle graph* or a *circular graph* consists of a single cycle. A cycle graph with n vertices is denoted by C_n.
A graph is *complete* if every pair of its distinct vertices is connected by an edge. A complete graph with n vertices is denoted by K_n.

Definition 1 Let G be a connected graph having vertices $v_1, \ldots, v_n$. The *generalized graph ideal* $I_q(G)$, $\mathbb{N} \ni q \leq n$, is the ideal of the polynomial ring $K[x_1, \ldots, x_n]$, where K is a field and each variable x_i corresponds to v_i, generated by all the square-free monomials $x_{i_1} \cdots x_{i_q}$ of degree q such that the vertex v_{i_j} is adjacent to $v_{i_{j+1}}$, for all $1 \leq j \leq (q-1)$.

Remark 1 $I_2(G)$ is the generalized graph ideal generated by the edges of G, the so-called edge ideal. More generally, the generators of $I_q(G)$ are paths of G of length $q-1$, simply called $(q-1)$-paths.

Definition 2 A monomial ideal $L_q \subset K[x_1, \ldots, x_m; y_1, \ldots, y_n]$ is called an *ideal of mixed products* if it is writable as $L_q = I_p J_r + I_s J_t$, where

$$q = p + r = s + t, \quad \text{for } p, r, s, t \text{ non-negative integers;}$$

I_p or I_s (resp. J_r or J_t) are ideals of $K[x_1,\ldots,x_m;\, y_1,\ldots,y_n]$ generated by square-free monomials of degree p or s (resp. r or t) in the variables $x_1,\ldots,x_m$ (resp. $y_1,\ldots,y_n$).

Remark 2 Setting $I_0 = J_0 = R$, there are essentially only the following two cases for these ideals:

a)	$L_q = I_p\, J_r + I_s\, J_t$	$0 \le p < s$ or $0 \le t < r$,
b)	$L_q = I_p\, J_r$	$p \ge 1$ or $r \ge 1$.

Proposition 1 [7] *All the powers of the ideals I_p, J_r are integrally closed (or complete). Therefore I_p, J_r are normal ideals.*

Some interesting properties and results about normality of L_q can be found in [6], where a complete characterization of normal ideals of mixed products is given.

Theorem 1 *Let $R = K[x_1,\ldots,x_m; y_1,\ldots,y_n]$ be a polynomial ring over a field K. Let $L_q = I_p\, J_r + I_s\, J_t \neq R$ be an ideal of mixed products with $q = p + r = s + t$. Then L_q is normal if and only if it can be written (up to permutation of p, s and r, t) in one of the following forms:*

(a) $L_q = I_p\, J_r + I_{p+1}\, J_{r-1}$ $\qquad$ $p \ge 0$ *and* $r \ge 1$;

(b) $L_q = I_p\, J_r$ $\qquad$ $p \ge 1$ *or* $r \ge 1$;

(c) $L_q = I_p\, J_r + I_s\, J_t$ $\qquad$ $0 = p < s = m$, *or* $0 = t < r = n$, *or* $p = t = 0$, $s = 1$.

Proof. See [6, Theorem 2.9]. □

Definition 3 A graph G is called *bipartite* if $V(G)$ can be partitioned into two subsets $V_1 = \{x_1,\ldots,x_m\}$ and $V_2 = \{y_1,\ldots,y_n\}$ such that every edge of G joins a vertex in V_1 to a vertex in V_2.
A bipartite graph G is *complete* if it contains all the edges that can join V_1 and V_2. It is denoted by $K_{m,n}$.

For a complete bipartite graph G, the associated generalized graph ideal $L_q(G)$ is a particular ideal of mixed products.
To this purpose, the following results hold.

Proposition 2 *Let G be a complete bipartite graph having vertex set $V(G) = \{x_1,\ldots,x_m;\, y_1,\ldots,y_n\}$. Then, for $2 \le q < (m+n)$, the generalized graph ideal $L_q(G)$ is of the form*

$$L_q(G) = \begin{cases} I_h\, J_{h+1} + I_{h+1}\, J_h & \text{if} \quad q = 2h+1 \\ I_h\, J_h & \text{if} \quad q = 2h \end{cases} .$$

Such an ideal is normal for all $q \geq 2$.

Proof. Since G is complete bipartite, every vertex x_i has degree n and every vertex y_j has degree m . The edges of G are pairs of the form $\{ x_i, y_j \}$ and their number is $m\,n$.
Evidently $L_q(G)$, for all $q \geq 2$, is an ideal of one of the forms as in the enunciate, and its normality is a consequence of Theorem 1. □

Proposition 3 *Let* $L_q \subset K[\,x_1,\ldots,x_m;\, y_1,\ldots,y_n\,]$, $2 \leq q < (m+n)$ *, be an ideal of mixed products of the form*

a) $I_h\, J_{h+1}$, *or* $I_{h+1}\, J_h$, *or* $I_h\, J_{h+1} + I_{h+1}\, J_h$ *for* $h = \dfrac{q-1}{2}$;

b) $I_h\, J_h$ *for* $h = \dfrac{q}{2}$.

Then $L_q = L_q(G)$*, where* G *is a complete bipartite graph with* $m+n$ *vertices.*

Proof. It is enough apply Proposition 2. □

Definition 4 Let G be an undirected graph with n vertices and m edges. The *incidence matrix* M_G of G is a $(n \times m)$-matrix whose entries a_{ij} are equal to 1 if the i-th vertex of G belongs to the j-th edge, 0 otherwise.

Remark 3 Each row of M_G has as many entries 1 as the degree of the corresponding vertex of G. Each column of M_G has two entries 1 and the remaining are 0.

Definition 5 Let G be a connected graph and M_G be the incidence matrix of G. When two rows in M_G correspond to a pair of vertices in G, having degrees α and β respectively, we call *multiplicity* of such pair of rows the product $(\alpha-1)\,(\beta-1)$, and denote it by $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$.

Remark 4 Let G and M_G be as in Definition 5. The multiplicity of a pair of rows in M_G, that correspond to a pair of vertices of G joined by a $(q-1)$-path, $q \geq 2$, and that have the degrees $\alpha \geq 2$ and $\beta \geq 2$, gives the number of walks of length $q+1$ in G containing the $(q-1)$-path inside.

2 Generators of generalized graph ideals

In this section we analyze the problem of computing up to $q=6$ the number and the structure of the $(q-1)$-paths of a connected graph G and the generators of the corresponding generalized graph ideal $I_q(G)$ using only the incidence matrix of G. We count in lexicographical order.

– **2-paths of G and generators of $I_3(G)$**

Theorem 2 *A connected graph G having $n \geq 3$ vertices and m edges has $\sum_{i=1}^{n} \binom{\lambda_i}{2}$ 2-paths, where λ_i denotes the number of entries 1 in the i-th row of the incidence matrix M_G.*
On each row of M_G let's consider every pair $a_{i_2\,h}$, $a_{i_2\,k}$ of entries 1, and let $a_{i_1\,h}$, $a_{i_3\,k}$ be the other entries 1 of the corresponding columns.
Then all the 2-paths of G and the generators of $I_3(G)$ are of the type $x_{i_1}\,x_{i_2}\,x_{i_3}$, $1 \leq i_1 \neq i_2 \neq i_3 \leq n$.

Proof. Observe that the number of non-zero entries on any row of the incidence matrix of a graph G represents the number of edges incident to the vertex of the graph related to such a row.
All the possible combinations of pairs of these edges determine 2-paths for G having that vertex as inner vertex.
With an analogous reasoning on every vertex of G, the number of the 2-paths follows.
Moreover, taking in pairs the λ non-zero entries of the row R_{i_2} of M_G, we give place to $\binom{\lambda}{2}$ distinct 2-paths for G, each having inner vertex x_{i_2}.
The ends x_{i_1}, x_{i_3} of such 2-paths are related to the rows R_{i_1}, R_{i_3} of M_G on which the remaining non-zero entry of each of the two columns determined by every single pair stays.
From the definition of generalized graph ideal, the generators of $I_3(G)$ are all the 2-paths of G obtained as above for every choice of R_{i_2}, up to permutation of indices. □

– **3-paths of G and generators of $I_4(G)$**

Theorem 3 *A connected graph G having $n \geq 4$ vertices, m edges, and s cycle subgraphs C_3, has $\sum_{j=1}^{m} \begin{bmatrix} \alpha_j \\ \beta_j \end{bmatrix} - 3\,s$ 3-paths, where $\begin{bmatrix} \alpha_j \\ \beta_j \end{bmatrix}$, $\alpha_j \geq \beta_j \geq 2$, denotes the multiplicity of the rows of the incidence matrix M_G on which the entries 1 of its j-th column lie.*

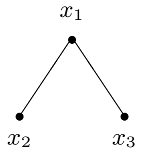
If $a_{i_2\,j}\,,\, a_{i_3\,j}$ are such entries, let's combine the pairs of 1's on the row R_{i_2} that contain $a_{i_2\,j}$ together with the pairs of 1's on R_{i_3} that contain $a_{i_3\,j}$, and let $a_{i_2\,h}\,,\, a_{i_3\,k}$ be the other entries 1 in the pairs for each of these combinations, $a_{i_1\,h}\,,\, a_{i_4\,k}$ be the remaining entries 1 of the relative columns. Then all the 3-paths of G are of the type $x_{i_1}\,x_{i_2}\,x_{i_3}\,x_{i_4}\,,\; 1 \le i_1 \ne i_2 \ne i_3 \ne i_4 \le n\,,$ and the generators of $I_4(G)$ are the 3-paths of G different from one another for at least an index.

Proof. A 3-path of the graph G can be thought as a pair of 2-paths having a common edge and the other two edges of it without a common vertex.
Observe that the non-zero entries on any column of the incidence matrix of G represent an edge of G with ends the vertices of G related to the rows on which such entries stay.
If ℓ is the multiplicity of such rows, all the possible pairs of 2-paths having a common edge related to that column determine ℓ 3-walks of G with inner vertices the ends of their common edge.
With an analogous reasoning on every edge of G, all the 3-walks of G are obtained.
The number of 3-paths follows excluding 3-walks with the same ends: in fact such walks belong to cycle subgraphs C_3 of G, each of which has three distinct.
Moreover, let $\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]$, $\alpha \ge \beta \ge 2$, be the multiplicity of the rows R_{i_2}, R_{i_3} of M_G to which the non-zero entries of the column Γ_j belong.
Observe that there exist, on the row R_{i_2} of M_G, $\alpha-1$ pairs having the first entry always belonging to the column Γ_j, the other one to the column Γ_h, $h \ne j$; analogously, on the row R_{i_3}, there exist $\beta-1$ pairs having the first entry always belonging to Γ_j, the other one to the column Γ_k, $k \ne j$.
Combining in all possible ways one of the above pairs on the row R_{i_2} with one of those on the row R_{i_3}, $(\alpha–1)\,(\beta–1)$ distinct 3-walks of G are obtained, each having inner vertices x_{i_2}, x_{i_3}.
The ends x_{i_1}, x_{i_4} of such 3-walks are related to the row R_{i_1} of M_G on which the remaining non-zero entry of the column Γ_h stays, and to the row R_{i_4} to which the other non-zero entry of Γ_k belongs.
When the rows R_{i_1} and R_{i_4} of M_G are distinct, the first assertion is proved.
From the definition of generalized graph ideal, the generators of $I_4(G)$ are all the 3-paths of G obtained as above for every choice of Γ_j, up to permutation of indices. □

Example 1 Consider connected graphs on 3 and 4 vertices and determine all their $(q-1)$-paths, $2 \le q \le 4$.

∗ 3 vertices

- ∘ 2 edges (complete bipartite graph $K_{1,2}$)

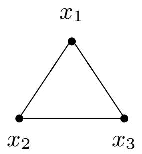


$$M_G = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

There is 1 2-path: $x_2 \, x_1 \, x_3$.

So $L_3 (K_{1,2}) = I_1 \, J_2 = (x_1 \, x_2 \, x_3)$.

- ∘ 3 edges (cycle graph C_3, complete graph K_3)



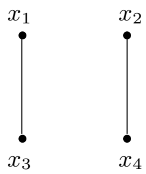
$$M_G = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

There are 3 2-paths: $x_1 \, x_2 \, x_3, \; x_1 \, x_3 \, x_2, \; x_2 \, x_1 \, x_3$.

So $I_3 (K_3) = (x_1 \, x_2 \, x_3)$.

∗ 4 vertices

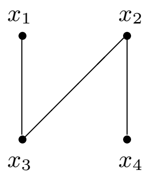
- ∘ 2 edges (bipartite graph)



$$M_G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

There are no paths of length more than 1.

- ∘ 3 edges (bipartite graph)

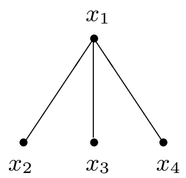


$$M_G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

There are 2 2-paths: $x_1 \, x_3 \, x_2, \; x_3 \, x_2 \, x_4$;
1 3-path: $x_1 \, x_3 \, x_2 \, x_4$.

So $I_3 = (x_1 \, x_2 \, x_3, \; x_2 \, x_3 \, x_4)$; $I_4 = (x_1 \, x_2 \, x_3 \, x_4)$.

3 edges (complete bipartite graph $K_{1,3}$)

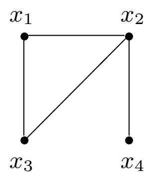


$$M_G = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

There are 3 2-paths: $x_2\,x_1\,x_3,\ x_2\,x_1\,x_4,\ x_3\,x_1\,x_4\,.$

So $L_3\,(K_{1,3}) = I_1\,J_2 = (x_1\,x_2\,x_3,\ x_1\,x_2\,x_4,\ x_1\,x_3\,x_4\,)\,.$

- 4 edges



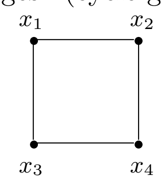
$$M_G = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

There are 5 2-paths: $x_1\,x_2\,x_3,\ x_1\,x_2\,x_4,\ x_1\,x_3\,x_2,$
$x_2\,x_1\,x_3,\ x_3\,x_2\,x_4\,;$

2 3-paths: $x_1\,x_3\,x_2\,x_4,\ x_3\,x_1\,x_2\,x_4\,.$

So $I_3 = (x_1\,x_2\,x_3,\ x_1\,x_2\,x_4,\ x_2\,x_3\,x_4)\,;\quad I_4 = (x_1\,x_2\,x_3\,x_4)\,.$

4 edges (cycle graph C_4)



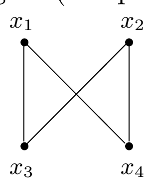
$$M_G = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

There are 4 2-paths: $x_1\,x_2\,x_4,\ x_1\,x_3\,x_4,\ x_2\,x_1\,x_3,\ x_2\,x_4\,x_3\,;$

4 3-paths: $x_1\,x_2\,x_4\,x_3,\ x_1\,x_3\,x_4\,x_2,$
$x_2\,x_1\,x_3\,x_4,\ x_3\,x_1\,x_2\,x_4\,.$

So $I_3(C_4) = (x_1\,x_2\,x_3,\ x_1\,x_2\,x_4,\ x_1\,x_3\,x_4,\ x_2\,x_3\,x_4)\,;$
$I_4(C_4) = (x_1\,x_2\,x_3\,x_4)\,.$

4 edges (complete bipartite graph $K_{2,2}$)



$$M_G = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

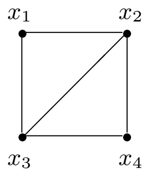
There are 4 2-paths: $x_1\,x_3\,x_2,\ x_1\,x_4\,x_2,\ x_3\,x_1\,x_4,\ x_3\,x_2\,x_4\,;$

4 3-paths: $x_1\,x_3\,x_2\,x_4,\ x_1\,x_4\,x_2\,x_3,$

$$x_2\,x_3\,x_1\,x_4,\ x_2\,x_4\,x_1\,x_3\,.$$

So $L_3\,(K_{2,\,2}) = I_2\,J_1 + I_1\,J_2 =$
$= (x_1\,x_2\,x_3,\ x_1\,x_2\,x_4,\ x_1\,x_3\,x_4,\ x_2\,x_3\,x_4)\,;$
$L_4(K_{2,\,2}) = I_2\,J_2 = (x_1\,x_2\,x_3\,x_4)\,.$

- 5 edges

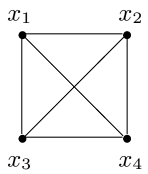


$$M_G = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

There are 8 2-paths: $x_1\,x_2\,x_3,\ x_1\,x_2\,x_4,\ x_1\,x_3\,x_2,\ x_1\,x_3\,x_4,$
$x_2\,x_1\,x_3,\ x_2\,x_3\,x_4,\ x_2\,x_4\,x_3,\ x_3\,x_2\,x_4\,;$
6 3-paths: $x_1\,x_2\,x_3\,x_4,\ x_1\,x_2\,x_4\,x_3,\ x_1\,x_3\,x_2\,x_4,$
$x_1\,x_3\,x_4\,x_2,\ x_2\,x_1\,x_3\,x_4,\ x_3\,x_1\,x_2\,x_4\,.$

So $I_3 = (x_1\,x_2\,x_3,\ x_1\,x_2\,x_4,\ x_1\,x_3\,x_4,\ x_2\,x_3\,x_4)\,;$
$I_4 = (x_1\,x_2\,x_3\,x_4)\,.$

- 6 edges (complete graph K_4)



$$M_G = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

There are 12 2-paths: $x_1\,x_2\,x_3,\ x_1\,x_2\,x_4,\ x_1\,x_3\,x_2,\ x_1\,x_3\,x_4,$
$x_1\,x_4\,x_2,\ x_1\,x_4\,x_3,\ x_2\,x_1\,x_3,\ x_2\,x_1\,x_4,$
$x_2\,x_3\,x_4,\ x_2\,x_4\,x_3,\ x_3\,x_1\,x_4,\ x_3\,x_2\,x_4\,;$
12 3-paths: $x_1\,x_2\,x_3\,x_4,\ x_1\,x_2\,x_4\,x_3,\ x_1\,x_3\,x_2\,x_4,$
$x_1\,x_3\,x_4\,x_2,\ x_1\,x_4\,x_2\,x_3,\ x_1\,x_4\,x_3\,x_2,$
$x_2\,x_1\,x_3\,x_4,\ x_2\,x_1\,x_4\,x_3,\ x_2\,x_3\,x_1\,x_4,$
$x_2\,x_4\,x_1\,x_3,\ x_3\,x_1\,x_2\,x_4,\ x_3\,x_2\,x_1\,x_4\,.$

So $I_3(K_4) = (x_1\,x_2\,x_3,\ x_1\,x_2\,x_4,\ x_1\,x_3\,x_4,\ x_2\,x_3\,x_4)\,;$
$I_4(K_4) = (x_1\,x_2\,x_3\,x_4)\,.$

– **4-paths of G and generators of $I_5(G)$**

Proposition 4 *Let G be a connected graph having $n \geq 5$ vertices $v_1, \ldots, v_n$, m edges, s cycle subgraphs C_3, r cycle subgraphs C_4, and incidence matrix M_G. Let d_h be the number of vertices of G adjacent to C_3, $h = 1, \ldots, s$. For every $(n \times 2)$-submatrix A_ℓ of M_G with only one row of 1's, let $\begin{bmatrix} \alpha_\ell \\ \beta_\ell \end{bmatrix}$, $\alpha_\ell \geq \beta_\ell \geq 2$, be the multiplicity of the rows of M_G corresponding to the rows of A_ℓ with a unique entry 1.*

Then G has $\displaystyle\sum_\ell \begin{bmatrix} \alpha_\ell \\ \beta_\ell \end{bmatrix} - \sum_{h=1}^{s} (3 + 2\, d_h) - 4\, r$ *4-paths.*

Proof. A 4-path of G can be thought as a pair of 3-paths having a common 2-path and the remaining two edges without a common vertex.
According to Proposition 2, the inner vertex and the endpoints of any 2-path of G characterize a $(n \times 2)$-submatrix A_ℓ of M_G having only one row of 1's, and these vertices correspond in A_ℓ to the row of 1's and to the pair of rows with a unique entry 1, respectively.
The number of such submatrices is $\displaystyle\sum_{i=1}^{n} \binom{\deg v_i}{2}$.
If m_ℓ is the multiplicity of the rows in M_G that correspond to the rows of A_ℓ with a unique entry 1, every pair of 3-paths having a common 2-path determines m_ℓ walks of length 4 in G having as inner vertices the three vertices of their common 2-path.
For $\ell = 1, \ldots, \displaystyle\sum_{i=1}^{n} \binom{\deg v_i}{2}$, all these walks of length 4 in G are found. The assertion follows excluding walks having some repeated vertex, that is:

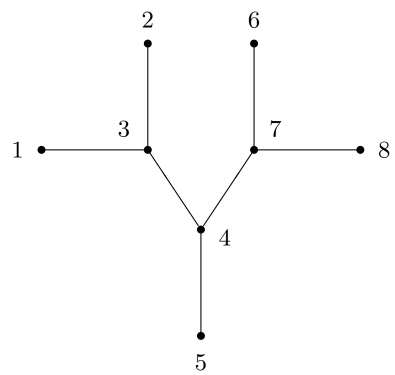
- for every cycle subgraph C_3 of G, there are
  - 3 distinct walks, having the same start and end edges,
  - twice the sum of the degrees of the vertices of C_3 minus twice the sum of the degrees of the vertices of a triangular cycle graph distinct walks, having one pair of equal vertices not at both the ends,
- for every cycle subgraph C_4 of G, there are
  - 4 distinct walks, having the same start and end edges. □

Theorem 4 *Let G, M_G, and A_ℓ be as in the Proposition 4, and R_{i_3} be the row of 1's in any A_ℓ.*
Let R_{i_2}, R_{i_4} denote the rows of M_G relative to the rows of A_ℓ with a unique entry 1, and R_{i_1}, R_{i_5} be the rows of M_G on which the remaining entry 1

of the columns, not belonging to A_ℓ, located by an entry 1 *in R_{i_2} and an entry* 1 *in R_{i_4} lies.*
Then all the 4*-paths of G are of the type $x_{i_1}\,x_{i_2}\,x_{i_3}\,x_{i_4}\,x_{i_5}$, $1 \leq i_1 \neq i_2 \neq \neq i_3 \neq i_4 \neq i_5 \leq n$, where the vertices correspond to the above rows of M_G, and the generators of $I_5(G)$ are the* 4*-paths of G different from one another for at least an index.*

Proof. To construct 4-paths in G, let's start from a 2-path whose middle vertex is given by the row R_{i_3} of 1's in any $(n{\times}2)$-submatrix A_ℓ of M_G, and whose ends by the two rows of A_ℓ with a unique 1.
These three vertices represent the inner vertices of the 4-paths can be obtained from A_ℓ.
To determine the ends of such 4-paths, let's consider all the entries 1 lying on each of the rows R_{i_2}, R_{i_4} of M_G relative to the rows of A_ℓ with a unique entry 1. If one of these rows in M_G contains only the entry 1 of the correspondent row in A_ℓ, no 4-path is formed.
Otherwise, let S_p, $p \geq 1$, denote every set whose elements are two pairs of entries 1, a pair on R_{i_2}, the other one on R_{i_4}, such that an entry of each pair always lies on A_ℓ. If Γ_h and Γ_k, $h \neq k$, are the columns of M_G to which the entry not lying on A_ℓ in each pair of any S_p belongs, let R_{i_1}, R_{i_5} be the rows of M_G on which the remaining entry 1 of Γ_h and Γ_k lies.
When R_{i_1}, R_{i_5} are different from each other and from R_{i_2}, R_{i_3}, R_{i_4}, they give the ends of the 4-paths in G that come from A_ℓ.
Such 4-paths, for every choice of A_ℓ, have the form $x_{i_1}\,x_{i_2}\,x_{i_3}\,x_{i_4}\,x_{i_5}$.
The last assertion derives from the definition of generalized graph ideal. □

Example 2 Consider the following tree G on 8 nodes



Vertices of G are the generators of $I_1(G) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$.
Edges of G are the generators of

$I_2(G) = \{x_1\,x_3,\ x_2\,x_3,\ x_3\,x_4,\ x_4\,x_5,\ x_4\,x_7,\ x_6\,x_7,\ x_7\,x_8\}\,.$

The incidence matrix of G is $M_G = \begin{pmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 1&1&1&0&0&0&0\\ 0&0&1&1&1&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&1&1&1\\ 0&0&0&0&0&0&1 \end{pmatrix}$.

Observe that the maximal length of the paths of G is 4;
therefore the generalized graph ideal $I_q(G)$ exists for $2 \le q \le 5$.

By Theorem 2, the number of 2-paths of G is $\binom{3}{2} + \binom{3}{2} + \binom{3}{2} = 9$.

The 2-paths of G are:

$$x_1\,x_3\,x_2,\ x_1\,x_3\,x_4,\ x_2\,x_3\,x_4,\ x_3\,x_4\,x_5,\ x_3\,x_4\,x_7,$$
$$x_4\,x_7\,x_6,\ x_4\,x_7\,x_8,\ x_5\,x_4\,x_7,\ x_6\,x_7\,x_8\,.$$

So $I_3(G) = (\,x_1\,x_3\,x_2,\ x_1\,x_3\,x_4,\ x_2\,x_3\,x_4,\ x_3\,x_4\,x_5,\ x_3\,x_4\,x_7,$
$x_4\,x_7\,x_6,\ x_4\,x_7\,x_8,\ x_5\,x_4\,x_7,\ x_6\,x_7\,x_8\,)$
is generated by 9 2-paths.

By Theorem 3, the number of 3-paths of G is $\left[{3 \atop 3}\right] + \left[{3 \atop 3}\right] = 2 \cdot 4 = 8$.

The 3-paths of G are:

$$x_1\,x_3\,x_4\,x_5,\ x_1\,x_3\,x_4\,x_7,\ x_2\,x_3\,x_4\,x_5,\ x_2\,x_3\,x_4\,x_7,$$
$$x_3\,x_4\,x_7\,x_6,\ x_3\,x_4\,x_7\,x_8,\ x_5\,x_4\,x_7\,x_6,\ x_5\,x_4\,x_7\,x_8\,.$$

So $I_4(G) = (\,x_1\,x_3\,x_4\,x_5,\ x_1\,x_3\,x_4\,x_7,\ x_2\,x_3\,x_4\,x_5,\ x_2\,x_3\,x_4\,x_7,$
$x_3\,x_4\,x_7\,x_6,\ x_3\,x_4\,x_7\,x_8,\ x_5\,x_4\,x_7\,x_6,\ x_5\,x_4\,x_7\,x_8\,)$
is generated by 8 3-paths.

By Proposition 4, the number of 4-paths of G is $\left[{3 \atop 3}\right] = 4$.

By Theorem 4, the 4-paths of G are:

$$x_1\,x_3\,x_4\,x_7\,x_6,\ x_1\,x_3\,x_4\,x_7\,x_8,\ x_2\,x_3\,x_4\,x_7\,x_6,\ x_2\,x_3\,x_4\,x_7\,x_8\,.$$

So $I_5(G) = (\,x_1\,x_3\,x_4\,x_7\,x_6,\ x_1\,x_3\,x_4\,x_7\,x_8,\ x_2\,x_3\,x_4\,x_7\,x_6,\ x_2\,x_3\,x_4\,x_7\,x_8\,)$
is generated by 4 4-paths.

– **5-paths of G and generators of $I_6(G)$**

Proposition 5 *Let G be a connected graph having $n \ge 6$ vertices $v_1, \ldots, v_n$, m edges, s cycle subgraphs C_3, r cycle subgraphs C_4, p cycle subgraphs C_5, and incidence matrix M_G.*

For any C_3 in G, let d_h be the number of vertices v_{i_λ} of G adjacent to C_3, $h = 1, \dots, s$, and ρ be the number of pairs of C_3 with a common edge.
For any C_4 in G, let δ_k be the number of vertices of G, not belonging to C_4, adjacent to C_4, $k = 1, \dots, r$.
For every $(n{\times}3)$-submatrix B_ℓ of M_G having exactly two rows both with two 1's, let $\begin{bmatrix}\alpha_\ell\\ \beta_\ell\end{bmatrix}$, $\alpha_\ell \geq \beta_\ell \geq 2$, be the multiplicity of the rows of M_G corresponding to the rows of B_ℓ with a unique entry 1. Then the number of 5-paths of G is

$$\sum_\ell \begin{bmatrix}\alpha_\ell\\ \beta_\ell\end{bmatrix} - 2\left(\sum_{\lambda=1}^{d_h} (\deg v_{i_\lambda} - 1)\right) + 2\rho - \sum_{k=1}^{r} (4 + 2\,\delta_k) - 5\,p\,.$$

Proof. A 5-path of G can be thought as a pair of 4-paths having a common 3-path and the remaining two edges without a common vertex.
According to Proposition 3, the two inner vertices and the endpoints of any 3-path of G characterize a $(n\times 3)$-submatrix B_ℓ of M_G having exactly two rows both with two 1's, and these vertices correspond in B_ℓ to the rows with two 1's and to the pair of rows with a unique entry 1, respectively.
The number of such submatrices is $\sum \nu_j - 3\,s$, where ν_j is the multiplicity of the rows of M_G on which the entries 1 of its j-th column lie, $j = 1, \dots, t$.
If m_ℓ is the multiplicity of the rows in M_G that correspond to the rows of B_ℓ with a unique entry 1, every pair of 4-paths having a common 3-path determine m_ℓ 5-walks in G having as inner vertices the four vertices of their common 3-path.
For $\ell = 1, \dots, \sum \nu_j - 3\,s$, all such 5-walks in G are found. The assertion follows excluding walks with some repeated vertex. In particular:

- for every cycle subgraph C_3 of G, by considering all the vertices of G adjacent to each node of C_3, there are
  - twice the sum of the degrees of such vertices minus 1 distinct walks, having at least one pair of equal vertices not at both the ends,

  but if a vertex of G is adjacent to a pair of nodes of C_3, another triangular cycle subgraph of G that has a common edge with C_3 is formed, so 2 walks, whose middle edge is the one in common, are obtained twice, in the procedure of C_3 as well as in the other one, then for each pair of cycle subgraphs C_3 of G with a common edge, 2 walks from the above computation are needed to be taken off;
- for every cycle subgraph C_4 of G, there are
  - 4 distinct walks, having the same start and end edges,

- twice the sum of the degrees of the vertices of C_4 minus twice the sum of the degrees of the vertices of a squared cycle graph distinct walks, having at least one pair of equal vertices not at both the ends,

but if an edge of G joins two non-consecutive vertices of C_4, a pair of cycle subgraphs C_3 contained in C_4 arise, so 4 walks, having two pairs of equal vertices, are the same walks obtained in the single procedures of such C_3 ,
then for each of the above edges of G, 4 walks from the above computation are needed to be taken off;

◦ for every cycle subgraph C_5 of G, there are
  - 5 distinct walks, having the same start and end edges. □

Theorem 5 *Let G, M_G, and B_ℓ be as in the Proposition 5, and R_{i_3}, R_{i_4} be the rows both with two 1's in any B_ℓ.*
Let R_{i_2}, R_{i_5} denote the rows of M_G relative to the rows of B_ℓ with a unique entry 1, and R_{i_1}, R_{i_6} be the rows of M_G on which the remaining entry 1 of the columns, not belonging to B_ℓ, located by an entry 1 in R_{i_2} and an entry 1 in R_{i_5} lies.
Then all the 5-paths of G are of the type $x_{i_1}\, x_{i_2}\, x_{i_3}\, x_{i_4}\, x_{i_5}\, x_{i_6}$, $1 \le i_1 \ne i_2 \ne i_3 \ne i_4 \ne i_5 \ne i_6 \le n$, where the vertices correspond to the above rows of M_G, and the generators of $I_6(G)$ are the 5-paths of G different from one another for at least an index.

Proof. To construct 5-paths in G, let's start from a 3-path whose inner vertices are given by the rows R_{i_3}, R_{i_4} with two 1's in any $(n{\times}3)$-submatrix B_ℓ of M_G, and whose ends by the rows R_{i_2}, R_{i_5} of B_ℓ with a unique 1.
These four vertices represent the inner vertices of the 5-paths can be obtained from B_ℓ.
By similar reasoning as in the proof of Theorem 4, the ends of the 5-paths in G that come from B_ℓ are given by well-determined rows R_{i_1}, R_{i_6} of M_G, different from each other and from R_{i_2}, R_{i_3}, R_{i_4}, R_{i_5}.
Such 5-paths, for every choice of B_ℓ, have the form $x_{i_1}\, x_{i_2}\, x_{i_3}\, x_{i_4}\, x_{i_5}\, x_{i_6}$.
The last assertion derives from the definition of generalized graph ideal. □

3 Computing paths of significant graphs

In this section we examine important classes of connected graphs and calculate the totality of their paths, highlighting the structure of the generalized graph ideals $I_q(G)$, q integer at most 6.

Property 1 The number of paths of any length and the generators of the generalized graph ideals for lengths less than 6 related to cycle graphs, complete graphs and complete bipartite graphs are the following ones

1. Cycle graphs $C_n\,,\ n \geq 3$.

Vertices in C_n have degree 2 ; the incidence matrix M_{C_n} is an $(n{\times}n)$-matrix.
C_n has n edges (or 1-paths) ;
C_n has n 2-paths (n , number of the rows of M_{C_n}) ;
C_n has n 3-paths (n , number of the columns of M_{C_n}) ;
. . . .
C_n has n $(n{-}1)$-paths .

When $3{\leq}q{\leq}6$, Theorems 2, 3 4, 5 give the structure of $(q{-}1)$-paths of C_n and the generators of the generalized graph ideals $I_q(C_n)$.

2. Complete graphs $K_n\,,\ n \geq 3$.

Vertices in K_n have degree $n{-}1$; there are $\dfrac{(k{-}1)\,!}{2}\dbinom{n}{k}$ cycles $C_k,\ k \geq 3$; the incidence matrix M_{K_n} is an $\left(n\times\dfrac{n\,(n{-}1)}{2}\right)$-matrix.

K_n has $\dbinom{n}{2} = \dfrac{n\,(n{-}1)}{2}$ edges (or 1-paths) ;

K_n has $n\dbinom{n{-}1}{2} = \dfrac{n\,(n{-}1)\,(n{-}2)}{2}$ 2-paths ;

K_n has $\dbinom{n}{2}\begin{bmatrix} n{-}1 \\ n{-}1\end{bmatrix} - 3\dbinom{n}{3} = \dfrac{n\,(n{-}1)\,(n{-}2)\,(n{-}3)}{2}$ 3-paths ;

K_n has $n\dbinom{n{-}1}{2}\begin{bmatrix} n{-}1 \\ n{-}1\end{bmatrix} - (3+2\cdot 3\,(n-3))\dbinom{n}{3} - 4\cdot 3\dbinom{n}{4} =$
$= \dfrac{n\,(n{-}1)\,(n{-}2)\,(n{-}3)\,(n{-}4)}{2}$ 4-paths ;

K_n has $\dfrac{n\,(n{-}1)\,(n{-}2)\,(n{-}3)}{2}\begin{bmatrix} n{-}1 \\ n{-}1\end{bmatrix} - 2\cdot 3\,(n-2)\,(n-3)\dbinom{n}{3} +$
$+2\dbinom{n}{2}\dbinom{n-2}{2} - 3\,(4+2\cdot 4\,(n-4))\dbinom{n}{4} - 5\cdot 12\dbinom{n}{5} =$
$= \dfrac{n\,(n{-}1)\,(n{-}2)\,(n{-}3)\,(n{-}4)\,(n{-}5)}{2}$ 5-paths ;

. . . .

K_n has $\dfrac{n\,(n{-}1)\,(n{-}2)\ \cdots\ 2\cdot 1}{2} = \dfrac{n\,!}{2}$ $(n{-}1)$-paths .

When $3{\leq}q{\leq}6$, Theorems 2, 3 4, 5 give the structure of $(q{-}1)$-paths of K_n and the generators of the generalized graph ideals $I_q(K_n)$.

3. Complete bipartite graphs $K_{m,n}\,,\ m{+}n \geq 3$.

m vertices of $K_{m,n}$ have degree n ; n vertices have degree m ;
in $K_{m,n}$ there are $\dfrac{k\,!\,k\,!}{2\,k}\dbinom{m}{k}\dbinom{n}{k}$ cycles $C_{2\,k}$, $k \geq 2$;
the incidence matrix $M_{K_{m,n}}$ is an $((m+n)\times m\,n)$-matrix.

$K_{m,n}$ has $m\,n$ edges (or 1-paths) ;

$K_{m,n}$ has $m\dbinom{n}{2} + n\dbinom{m}{2}$ 2-paths ;

$K_{m,n}$ has $m\,n\begin{bmatrix} m \\ n \end{bmatrix} = m\,(m-1)\,n\,(n-1) = 2\dbinom{m}{2}2\dbinom{n}{2}$ 3-paths ;

$K_{m,n}$ has
$$m\binom{n}{2}\begin{bmatrix} m \\ m \end{bmatrix} + n\binom{m}{2}\begin{bmatrix} n \\ n \end{bmatrix} - 4\,\frac{2\,!\,2\,!}{4}\binom{m}{2}\binom{n}{2} =$$
$$= 2\binom{m}{2}2\binom{n}{2}\frac{m-1}{2} + 2\binom{m}{2}2\binom{n}{2}\frac{n-1}{2} - 2\binom{m}{2}2\binom{n}{2} =$$
$$= m\,(m-1)\,n\,(n-1)\,\frac{(m-2)+(n-2)}{2} =$$
$$= 3\,!\binom{m}{3}\binom{n}{2} + \binom{m}{2}3\,!\binom{n}{3} \quad \text{4-paths ;}$$

$K_{m,n}$ has
$$m\,n\begin{bmatrix} m \\ n \end{bmatrix}\begin{bmatrix} m \\ n \end{bmatrix} - (4 + 2\,(2\,(m-2) + 2\,(n-2)))\,\frac{2\,!\,2\,!}{4}\binom{m}{2}\binom{n}{2} =$$
$$= 2\binom{m}{2}2\binom{n}{2}\,((m-1)\,(n-1) - (1 + (m-2) + (n-2))) =$$
$$= m\,(m-1)\,n\,(n-1)\;(\,m\,n - m - n + 1 - m - n + 3\,) =$$
$$= m\,(m–1)\,(m–2)\,n\,(n–1)\,(n–2) = 3\,!\binom{m}{3} + 3\,!\binom{n}{3} \quad \text{5-paths ;}$$

. . . .

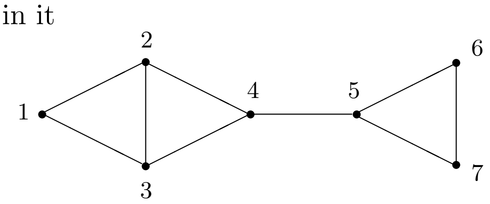
$K_{m,n}$ has $\dfrac{h\,!}{2}\dbinom{m}{h}(h{+}1)\,!\dbinom{n}{h+1} + (h{+}1)\,!\dbinom{m}{h+1}\dfrac{h\,!}{2}\dbinom{n}{h}$ $(m{+}n{–}1)$-paths
if $m{+}n = 2\,h+1$ is odd ,

$h\,!\dbinom{m}{h}h\,!\dbinom{n}{h}$ $(m{+}n-1)$-paths if $m{+}n = 2\,h$ is even .

(see [2, Corollary 4]).

When $3 \leq q \leq 6$, Theorems 2, 3 4, 5 give the structure of $(q-1)$-paths of $K_{m,n}$ and the generators of the generalized graph ideals $L_q(K_{m,n})$. □

Example 3 Consider the following graph G with triangular and square cycle subgraphs in it



and compute the number of $(q-1)$-paths of G and the generators of the generalized graph ideal $I_q(G)$, $q>2$, using only the incidence matrix of G.

Vertices of G are the generators of $I_1(G) = (x_1, x_2, x_3, x_4, x_5, x_6, x_7)$.

Edges of G are the generators of

$I_2(G) = (x_1 x_2, x_1 x_3, x_2 x_3, x_2 x_4, x_3 x_4, x_4 x_5, x_5 x_6, x_5 x_7, x_6 x_7)$.

The incidence matrix of G is $M_G = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$.

Observe that the maximal length of the paths of G is 6;
therefore the generalized graph ideal $I_q(G)$ exists for $q \le 7$.

The number of 2-paths of G is:

$$\binom{2}{2}+\binom{3}{2}+\binom{3}{2}+\binom{3}{2}+\binom{3}{2}+\binom{2}{2}+\binom{2}{2} = 15\,.$$

Such 2-paths are: $x_1 x_2 x_3,\ x_1 x_3 x_2,\ x_1 x_2 x_4,\ x_1 x_3 x_4,\ x_2 x_1 x_3,$
$x_2 x_3 x_4,\ x_2 x_4 x_3,\ x_2 x_4 x_5,\ x_3 x_2 x_4,\ x_3 x_4 x_5,$
$x_4 x_5 x_6,\ x_4 x_5 x_7,\ x_5 x_6 x_7,\ x_5 x_7 x_6,\ x_6 x_5 x_7$.

So $I_3(G) = (x_1 x_2 x_3,\ x_1 x_2 x_4,\ x_1 x_3 x_4,\ x_2 x_3 x_4,\ x_2 x_4 x_5,\ x_3 x_4 x_5,$
$x_4 x_5 x_6,\ x_4 x_5 x_7,\ x_5 x_6 x_7)$ is generated by 9 2-paths.

The number of 3-paths is:

$$\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}3\\3\end{bmatrix}+\begin{bmatrix}3\\3\end{bmatrix}+\begin{bmatrix}3\\3\end{bmatrix}+\begin{bmatrix}3\\3\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}2\\2\end{bmatrix}-3\cdot 3 =$$

$$= 4\begin{bmatrix}3\\3\end{bmatrix}+4\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}2\\2\end{bmatrix}-9 = 16+8+1-9 = 16\,.$$

Such 3-paths are: $x_1 x_2 x_3 x_4,\ x_1 x_2 x_4 x_3,\ x_1 x_2 x_4 x_5,\ x_1 x_3 x_2 x_4,$
$x_1 x_3 x_4 x_2,\ x_1 x_3 x_4 x_5,\ x_2 x_1 x_3 x_4,\ x_2 x_3 x_4 x_5,$
$x_2 x_4 x_5 x_6,\ x_2 x_4 x_5 x_7,\ x_3 x_1 x_2 x_4,\ x_3 x_2 x_4 x_5,$
$x_3 x_4 x_5 x_6,\ x_3 x_4 x_5 x_7,\ x_4 x_5 x_6 x_7,\ x_4 x_5 x_7 x_6$.

So $I_4(G) = (x_1 x_2 x_3 x_4,\ x_1 x_2 x_4 x_5,\ x_1 x_3 x_4 x_5,\ x_2 x_3 x_4 x_5,\ x_2 x_4 x_5 x_6,$
$x_2 x_4 x_5 x_7,\ x_3 x_4 x_5 x_6,\ x_3 x_4 x_5 x_7,\ x_4 x_5 x_6 x_7)$
is generated by 9 3-paths.

To determine the number of 4-paths of G, consider the (7×2)-submatrices of M_G having one row of 1's and other two rows both with a unique 1:

$$\begin{pmatrix}1&1\\1&0\\0&1\\0&0\\0&0\\0&0\\0&0\end{pmatrix},\begin{pmatrix}1&0\\1&1\\0&1\\0&0\\0&0\\0&0\\0&0\end{pmatrix},\begin{pmatrix}1&0\\1&1\\0&0\\0&1\\0&0\\0&0\\0&0\end{pmatrix},\begin{pmatrix}1&0\\0&1\\1&1\\0&0\\0&0\\0&0\\0&0\end{pmatrix},\begin{pmatrix}1&0\\0&0\\1&1\\0&1\\0&0\\0&0\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&1\\1&0\\0&1\\0&0\\0&0\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\\1&1\\0&1\\0&0\\0&0\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\\0&1\\1&1\\0&0\\0&0\\0&0\end{pmatrix},$$

$$\begin{pmatrix}0&0\\1&0\\0&0\\1&1\\0&1\\0&0\\0&0\end{pmatrix},\begin{pmatrix}0&0\\0&0\\1&0\\1&1\\0&1\\0&0\\0&0\end{pmatrix},\begin{pmatrix}0&0\\0&0\\0&0\\1&0\\1&1\\0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\0&0\\0&0\\1&0\\1&1\\0&0\\0&1\end{pmatrix},\begin{pmatrix}0&0\\0&0\\0&0\\0&0\\1&1\\1&0\\0&1\end{pmatrix},\begin{pmatrix}0&0\\0&0\\0&0\\0&0\\1&0\\1&1\\0&1\end{pmatrix},\begin{pmatrix}0&0\\1&0\\0&1\\0&0\\1&0\\0&1\\1&1\end{pmatrix}.$$

Then it is: $\begin{bmatrix}3\\3\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}3\\3\end{bmatrix}+\begin{bmatrix}3\\3\end{bmatrix}+\begin{bmatrix}3\\3\end{bmatrix}+\begin{bmatrix}3\\3\end{bmatrix}+\begin{bmatrix}3\\3\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+$

$$+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}2\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}-(3+4)-(3+6)-(3+2)-4=$$

$$=6\begin{bmatrix}3\\3\end{bmatrix}+8\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}2\\2\end{bmatrix}-7-9-5-4=24+16+1-25=16\,.$$

Such 4-paths are:

$$\begin{aligned}&x_1\,x_2\,x_3\,x_4\,x_5,\;\; x_1\,x_2\,x_4\,x_5\,x_6,\;\; x_1\,x_2\,x_4\,x_5\,x_7,\;\; x_1\,x_3\,x_2\,x_4\,x_5,\\ &x_1\,x_3\,x_4\,x_5\,x_6,\;\; x_1\,x_3\,x_4\,x_5\,x_7,\;\; x_2\,x_1\,x_3\,x_4\,x_5,\;\; x_2\,x_3\,x_4\,x_5\,x_6,\\ &x_2\,x_3\,x_4\,x_5\,x_7,\;\; x_2\,x_4\,x_5\,x_6\,x_7,\;\; x_2\,x_4\,x_5\,x_7\,x_6,\;\; x_3\,x_1\,x_2\,x_4\,x_5,\\ &x_3\,x_2\,x_4\,x_5\,x_6,\;\; x_3\,x_2\,x_4\,x_5\,x_7,\;\; x_3\,x_4\,x_5\,x_6\,x_7,\;\; x_3\,x_4\,x_5\,x_7\,x_6\,.\end{aligned}$$

So $I_5(G)=(\,x_1\,x_2\,x_3\,x_4\,x_5,\;\; x_1\,x_2\,x_4\,x_5\,x_6,\;\; x_1\,x_2\,x_4\,x_5\,x_7,$
$x_1\,x_3\,x_4\,x_5\,x_6,\;\; x_1\,x_3\,x_4\,x_5\,x_7,\;\; x_2\,x_3\,x_4\,x_5\,x_6,$
$x_2\,x_3\,x_4\,x_5\,x_7,\;\; x_2\,x_4\,x_5\,x_6\,x_7,\;\; x_3\,x_4\,x_5\,x_6\,x_7\,)$
is generated by 9 4-paths.

To determine the number of 5-paths of G, consider the (7×3)-submatrices of M_G having just two rows both with two 1's and other two rows both with a unique 1:

$$\begin{pmatrix}1&1&0\\1&0&1\\0&1&0\\0&0&1\\0&0&0\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}1&1&0\\1&0&0\\0&1&1\\0&0&1\\0&0&0\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}1&0&0\\1&1&0\\0&1&1\\0&0&1\\0&0&0\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}1&0&0\\1&1&0\\0&0&1\\0&1&1\\0&0&0\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}1&0&0\\1&1&0\\0&0&0\\0&1&1\\0&0&1\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}1&0&0\\0&1&1\\1&1&0\\0&0&1\\0&0&0\\0&0&0\\0&0&0\end{pmatrix},$$

$$\begin{pmatrix}1&0&0\\0&1&0\\1&0&1\\0&1&1\\0&0&0\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}1&0&0\\0&0&0\\1&1&0\\0&1&1\\0&0&1\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\1&1&0\\1&0&0\\0&1&1\\0&0&1\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\1&0&0\\1&1&0\\0&1&1\\0&0&1\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\1&0&0\\0&0&0\\1&1&0\\0&1&1\\0&0&1\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\1&0&0\\0&0&0\\1&1&0\\0&1&1\\0&0&0\\0&0&1\end{pmatrix},$$

$$\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\\1&1&0\\0&1&1\\0&0&1\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&0\\1&0&0\\1&1&0\\0&1&1\\0&0&0\\0&0&1\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\\1&0&0\\1&1&0\\0&1&1\\0&0&1\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\\1&0&0\\1&1&0\\0&0&1\\0&1&1\end{pmatrix}.$$

Then it is:

$$\begin{bmatrix}3\\3\end{bmatrix}+\begin{bmatrix}3\\3\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}3\\3\end{bmatrix}+\begin{bmatrix}3\\3\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+$$
$$+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}-2\,(\,(2+2)+(1+1+2)+2\,)+2\cdot 1-(4+6-2\cdot2)=$$
$$=4\begin{bmatrix}3\\3\end{bmatrix}+12\begin{bmatrix}3\\2\end{bmatrix}-2\cdot10+2-6=16+24-24=16\,.$$

Such 5-paths are:

$$\begin{gathered}x_1\,x_2\,x_3\,x_4\,x_5\,x_6,\;\; x_1\,x_2\,x_3\,x_4\,x_5\,x_7,\;\; x_1\,x_2\,x_4\,x_5\,x_6\,x_7,\;\; x_1\,x_2\,x_4\,x_5\,x_7\,x_6,\\ x_1\,x_3\,x_2\,x_4\,x_5\,x_6,\;\; x_1\,x_3\,x_2\,x_4\,x_5\,x_7,\;\; x_1\,x_3\,x_4\,x_5\,x_6\,x_7,\;\; x_1\,x_3\,x_4\,x_5\,x_7\,x_6,\\ x_2\,x_1\,x_3\,x_4\,x_5\,x_6,\;\; x_2\,x_1\,x_3\,x_4\,x_5\,x_7,\;\; x_2\,x_3\,x_4\,x_5\,x_6\,x_7,\;\; x_2\,x_3\,x_4\,x_5\,x_7\,x_6,\\ x_3\,x_1\,x_2\,x_4\,x_5\,x_6,\;\; x_3\,x_1\,x_2\,x_4\,x_5\,x_7,\;\; x_3\,x_2\,x_4\,x_5\,x_6\,x_7,\;\; x_3\,x_2\,x_4\,x_5\,x_7\,x_6\,.\end{gathered}$$

So $I_6(G) = (\,x_1\,x_2\,x_3\,x_4\,x_5\,x_6,\;\; x_1\,x_2\,x_3\,x_4\,x_5\,x_7,\;\; x_1\,x_2\,x_4\,x_5\,x_6\,x_7,$
$x_1\,x_3\,x_4\,x_5\,x_6\,x_7,\;\; x_2\,x_3\,x_4\,x_5\,x_6\,x_7\,)$
is generated by 5 5-paths.

Since in G there not exist paths of length greater than 6, the generalized graph ideal $I_7(G)$ is generated by 1 6-path, namely $x_1\,x_2\,x_3\,x_4\,x_5\,x_6\,x_7$.

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