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ON INTEGRO-DIFFERENTIAL EQUATIONS VIA MEIR-KEELER CONDENSING OPERATORS AND THE MEASURE OF NONCOMPACTNESS*

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Abstract

This paper discusses the existence of solution for integro-differential equations via resolvent operators in Banach space. Our approach is based on a new fixed point theorem with respect to Meir-Keeler condensing operators. An example is given to show the application of our result.

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1 Introduction

Integro-differential equations serve as powerful mathematical models for describing a wide range of phenomena in various scientific disciplines, including physics, biology, and engineering, see in [11, 9]. The analysis of such equations poses significant challenges due to the combined presence of differential and integral terms, leading to intricate dynamics and complex solution behaviors. One fundamental aspect in studying integro-differential equations is the identification and analysis of fixed points, which provide valuable insights into the long-term behavior of the system. (For more details see [10, 19, 16, 17, 5, 6, 7]). Various authors have examined qualitative properties such as existence, uniqueness, and stability for many integral, differential and integro-differential equations, (see [4, 5, 15, 21]).

Recently, the resolvent operators play a central role in the analysis of integro-differential equations by transforming the original equation into a more amenable form [11]. They allow for the conversion of an equation with integral terms into an equivalent equation involving differential operators only. This transformation simplifies the analysis and facilitates the application of well-established techniques from the theory of ordinary differential equations, (see for instance [13, 12]).

In addition to resolvent operators, we employ Meir-Keeler condensing operators in our analysis [18]. These operators possess valuable properties that make them particularly suitable for the investigation of fixed points in integro-differential equations. Meir-Keeler condensing operators allow for the characterization of the compactness properties of solution sets, capturing essential aspects of the underlying dynamics and enabling the application of powerful fixed point theorems, for more details see [1, 20].

In [2], and by using the Meir-Keeler condensing operators, the authors considered the following problem:

$$\begin{cases} {}^{c}\mathcal{D}^{\zeta;\omega}_{\kappa_1^+} y(\delta) = \Psi(\delta, y(\delta)), \quad \delta \in [\kappa_1, \kappa_2], \\ y(\kappa_1) = y_{\kappa_1}, \quad y(\kappa_2) = y_{\kappa_2}, \end{cases}$$

where ${}^{c}\mathcal{D}^{\zeta;\omega}_{\kappa_1^+}$ is the ω-Caputo fractional derivative of order $\zeta \in (1, 2], \Psi : [\kappa_1, \kappa_2] \times \Bbbk \longrightarrow \Bbbk$ is a given function, $\Bbbk$ is a Banach space with norm $\|\cdot\|$ and $y_{\kappa_1}, y_{\kappa_2} \in \Bbbk$.

In this article, we discuss the existence of solutions for a new initial value problem with the integro-differential equation in Banach space:

$$\begin{cases} y'(\delta) = Z_1(\delta)y(\delta) + \int_0^\delta Z_2(\delta - s)y(s)ds + \Psi(\delta, y(\delta)); \quad \delta \in J := [0, T], \\ y(0) = y_0 \in \mathbb{k}, \end{cases} \tag{1}$$

where $T > 0, \Psi : J \times \mathbb{k} \to \mathbb{k}$ is a given function, $(\mathbb{k}, \|\cdot\|_{\mathbb{k}})$ is a Banach space, and $\{Z_1(\delta)\}_{\delta > 0}$ is a family of linear closed (not necessarily bounded) operators from $D(Z_1)$ into $\mathbb{k}$ that generates an integro-differential equation system of resolvent operator $\{Y(\delta)\}_{\delta \in J}$ from $\mathbb{k}$ into $\mathbb{k}$.

The structure of this paper is as follows: Section 2 provides general results and preliminary information. In Section 3, we demonstrate the existence of a solution for the problem (1) using the Meir-Keeler fixed point theorem in conjunction with the measure of noncompactness technique. To further illustrate our findings, we present an example in Section 4.

2 Preliminaries

Let $C(J, \mathbb{k})$ be the space of continuous functions from J into $\mathbb{k}$ and $B(\mathbb{k})$ be the space of all bounded linear operators from $\mathbb{k}$ into $\mathbb{k}$, with the norm

$$\|F\|_{B(\mathbb{k})} = \sup_{\|y\|=1} \|F(y)\|_{\mathbb{k}}.$$

A measurable function $y : J \to \mathbb{k}$ is Bochner integrable if and only if $\|y\|_{\mathbb{k}}$ is Lebesgue integrable.

Let $L^1(J, \mathbb{k})$ denotes the Banach space of measurable functions $y : J \to \mathbb{k}$ which are Bochner integrable, with the norm

$$\|y\|_{L^1} = \int_0^T \|y(\delta)\|_{\mathbb{k}} d\delta.$$

and denote by $L^\infty(J)$ the Banach space of measurable function $y : J \to \mathbb{k}$ which are essentially bounded with

$$\|y\|_{L^\infty} = \inf\{c > 0 : \|y(\delta)\|_{\mathbb{k}} \le c, \quad a.e \quad \delta \in J\}.$$

As usual, by $C(J)$ we denote the Banach space of all continuous functions from J into $\mathbb{k}$ with

$$\|y\|_\infty = \sup_{\delta \in J} \|y(\delta)\|_{\mathbb{k}}.$$

We consider the following linear Cauchy problem

$$\begin{cases} y'(\delta) = Z_1 y(\delta) + \int_0^\delta Z_2(\delta - s) y(s) ds; & \text{for } \delta \geq 0, \\ y(0) = y_0 \in \Bbbk. \end{cases} \tag{2}$$

The existence and properties of a resolvent operator has been discussed in [12].

Definition 1 ([12]). *A resolvent operator for a Cauchy problem (2) is a bounded linear operator-valued function $Y \in B(\Bbbk)$ for $\delta \geq 0$, verifying the following conditions:*

(1) $Y(0) = I$ and $\|Y(\delta)\|_{B(\Bbbk)} \leq M e^{\eta\delta}$ for $M > 0$ and $\eta \in \mathbb{R}$.

(2) For each $y \in \Bbbk$, $\delta \to Y(\delta)y$ is strongly continuous for $\delta \geq 0$.

(3) $Y \in B(\Bbbk)$ for $\delta \geq 0$. For $y \in \Bbbk$, $Y(\cdot)y \in C^1(\mathbb{R}_+, \Bbbk) \cap C(\mathbb{R}_+, \Bbbk)$ and

$$\begin{aligned} Y'(\delta)y &= Z_1 Y(\delta) y + \int_0^\delta Z_2(\delta - s) Y(s) y ds \\ &= Y(\delta) Z_1 y + \int_0^\delta Y(\delta - s) Z_2(s) y ds, \end{aligned}$$

for $\delta \geq 0$.

From now on, we assume that:

$(P1)$ The operator Z_1 is the infinitesimal generator of a uniformly continuous semigroup $\{T(\delta)\}_{\delta>0}$.

$(P2)$ For all $\delta \geq 0, Z_2(\delta)$ is closed linear operator from $\mathcal{D}(Z_1)$ to $\Bbbk$ and $Z_2(\delta) \in B(\Bbbk)$. For any $y \in \Bbbk$, the map $\delta \to Z_2(\delta)y$ is bounded, differentiable and the derivative $\delta \to {Z_2}'(\delta)y$ is bounded uniformly continuous on $\mathbb{R}^+$.

Theorem 1 ([12]). *Assume that $(P1)-(P2)$ hold, then there exists a unique resolvent operator for the Cauchy problem (2).*

The resolvent operator gives some results for the existence of solutions for the following integro-differential problem:

$$\begin{cases} y'(\delta) = Z_1 y(\delta) + \int_0^\delta Z_2(\delta - s) y(s) ds + \varphi(\delta); & \text{for } \delta \geq 0, \\ y(0) = y_0 \in \Bbbk. \end{cases} \tag{3}$$

where $\varphi : \mathbb{R}_+ \to \Bbbk$ is a continuous function.

Definition 2 ([12])**.** *A continuous function $y : \mathbb{R}_+ \to \Bbbk$ is said to be a strict solution of equation of problem (3) if $y \in C^1(\mathbb{R}_+, \Bbbk)$ and y satisfies problem (3).*

Theorem 2 ([12])**.** *assume that $(P1) - (P2)$ hold. If y is a strict solution of problem (3), then*

$$y(\delta) = Y(\delta)y_0 + \int_0^\delta Y(\delta - s)\varphi(s)ds \text{ for } \delta \geq 0.$$

Definition 3 ([22])**.** *The function $\Psi : J \times \Bbbk \to \Bbbk$ is said to be Carathéodory if*

(*i*) $\delta \to \Psi(\delta, y)$ *is measurable for each* $y \in \Bbbk$.

(*ii*) $y \to \Psi(\delta, y)$ *is continuous for almost all* $\delta \in J$.

The function Ψ is said to be L^∞-Carathéodory if $(i), (ii)$ and the following condition holds.

(*iii*) *For $k > 0$, there exists $h_k \in L^\infty(J, \mathbb{R}_+)$ where*

$$\|\Psi(\delta, y)\|_\Bbbk \leq h_k(\delta),$$

for all $\|y\|_\Bbbk \leq k$ and almost each $\delta \in J$.

Definition 4 ([3])**.** *Let $\Bbbk$ be a Banach space and Λ a bounded subsets of $\Bbbk$. Then the Hausdorff measure of non-compactness of Λ is defined by*

$$\xi(\Lambda) = \inf \left\{ \begin{matrix} j > 0 : \Lambda \text{ can be covered by} \\ \text{finitely many balls with radius} < j \end{matrix} \right\}.$$

Lemma 1 ([3])**.** *Let $Z_1, \Lambda \subset \Bbbk$ be bounded. Thus, the Hausdorff measure of non-compactness verifies:*

(1) $\xi(Z_1) = 0 \Longleftrightarrow Z_1$ *is relatively compact,*

(2) $Z_1 \subset \Lambda \Longrightarrow \xi(Z_1) \leq \xi(\Lambda)$,

(3) $\xi(Z_1 \cup \Lambda) = \max\{\xi(Z_1), \xi(\Lambda)\}$,

(4) $\xi(Z_1) = \xi(\overline{Z_1}) = \xi(conv(Z_1))$,

(5) $\xi(Z_1 + \Lambda) \leq \xi(Z_1) + \xi(\Lambda)$, *where* $Z_1 + \Lambda = \{x + y : x \in Z_1, y \in \Lambda\}$,

(6) $\xi(\lambda Z_1) \leq |\lambda| \xi(Z_1)$, *for any* $\lambda \in \mathbb{R}$.

Definition 5 ([18]). *Let $(\Bbbk, d)$ be a metric space. Then a mapping N on $\Bbbk$ is said to be a Meir-Keeler contraction if for any $j > 0$, there exists $\wp > 0$ where*

$$j \leq d(\mathfrak{q}, y) < j + \wp \Rightarrow d(N\mathfrak{q}, Ny) < j, \quad \forall \mathfrak{q}, y \in \Bbbk.$$

In [1], the authors defined the notion of Meir-Keeler condensing operators on a Banach space and give some fixed point results.

Definition 6 ([1]). *Let Λ be a nonempty subset of a Banach space $\Bbbk$ and ξ arbitrary measure of noncompactness on $\Bbbk$. We say that an operator $N : \Lambda \to \Lambda$ is a Meir-Keeler condensing operator if for any $j > 0$, there exists $\wp > 0$ such that*

$$j \leq \xi(\Theta) < j + \wp \Rightarrow \xi(N\Theta) < j,$$

for any bounded subset Θ of Λ.

Theorem 3 ([1]). *Let Θ be a nonempty, bounded, closed and convex subset of a Banach space $\Bbbk$. Also, let ξ be an arbitrary measure of noncompactness on $\Bbbk$. If $N : \Theta \to \Theta$ is a continuous and Meir-Keeler condensing operator, then N has at least one fixed point and the set of all fixed points of N in Θ is compact.*

Lemma 2 ([1]). *Let $\Bbbk$ be a Banach space, and let $\Lambda \subset C(J, \Bbbk)$ be bounded and equicontinuous. Then $\xi(\Lambda(\delta))$ is continuous on J, and*

$$\xi_C(\Lambda) = \max_{\delta \in J} \xi(\Lambda(\delta)).$$

Lemma 3 ([8]). *Let $\Bbbk$ be a Banach space and let $\Lambda \subset \Bbbk$ be bounded. Then for each j, there is $\{y_i\}_{i=1}^{\infty} \subset \Lambda$, where*

$$\xi(\Lambda) \leq 2\xi(\{y_i\}_{i=1}^{\infty}) + j.$$

We call $\Lambda \subset L^1(J, \Bbbk)$ uniformly integrable if there exists $b \in L^1(J, \mathbb{R}^+)$ where

$$\|y(\varsigma)\|_{\Bbbk} \leq b(\varsigma), \text{ for all } y \in \Lambda \text{ and a.e. } \varsigma \in J.$$

Lemma 4 ([14]). *If $\{y_i\}_{i=1}^{\infty} \subset L^1(J, \Bbbk)$ is uniformly integrable, then $\delta \mapsto \xi(\{y_i(\delta)\}_{i=1}^{\infty})$ is measurable, and*

$$\xi\left(\left\{\int_a^{\delta} y_i(\varsigma)d\varsigma\right\}_{i=1}^{\infty}\right) \leq 2\int_a^{\delta} \xi(\{y_i(\varsigma)\}_{i=1}^{\infty})d\varsigma.$$

3 Main Results

Definition 7. *We say that a continuous function $y(\cdot) : J \to \Bbbk$ is a mild solution of (1) if y verifies*

$$y(\delta) = Y(\delta)y_0 + \int_0^\delta Y(\delta - s)\Psi(s, y(s))ds, \text{ for each } \delta \in J.$$

The hypotheses:

(H_1) The resolvent operator is uniformly continuous and there exists a constant $\vartheta \geq 1$ such that

$$\|Y(\delta)\|_{B(\Bbbk)} \leq \vartheta \text{ for every } \delta \geq 0.$$

(H_2) The function $\Psi : J \times \Bbbk \to \Bbbk$ is L^∞-Carathéodory.

(H_3) There exist $p_\Psi \in C(J, \mathbb{R}_+)$ and a continuous nondecreasing function $\varpi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ where

$$\|\Psi(\delta, y)\|_\Bbbk \leq p_\Psi(\delta)\varpi(\|y\|_\Bbbk), \quad \text{for a.e. } \delta \in J, \text{ and each } y \in \Bbbk.$$

(H_4) For each bounded set $\Lambda \subset \Bbbk$, and each $\delta \in J$, the following inequality holds,

$$\xi(\Psi(\delta, \Lambda)) \leq p_\Psi(\delta)\xi(\Lambda).$$

(H_5) There exists $\gamma > 0$ such that

$$\gamma \geq \vartheta\|y_0\|_\Bbbk + \vartheta p_\Psi^* \varpi(|\gamma|)T,$$

Theorem 4. *Assume that the hypotheses (H1)–(H5) are satisfied and that*

$$4\ell_\psi < 1,$$

where $\ell_\psi = \vartheta p_\Psi^* T$ and $p_\Psi^* := \sup_{\delta \in J} p_\Psi(\delta)$. Then (1) has at least one mild solution defined on J.

Proof. Consider the operator $N : C(J, \Bbbk) \longrightarrow C(J, \Bbbk)$ defined by:

$$Ny(\delta) = Y(\delta)y_0 + \int_0^\delta Y(\delta - s)\Psi(s, y(s))ds.$$

It is clear that N is well defined due to (H2)-(H3). Then, (1) verifies

$$y = Ny.$$

Define the set

$$\Theta_\gamma = \{y \in C(J, \mathbb{k}) : \|y\|_\infty \leq \gamma\}.$$

We shall prove that N verifies all the requirements of Theorem 3.

Step 1: Suppose that $y \in \Theta_\gamma$. By the assumption (H5), we have

$$\begin{aligned}
\|Ny(\delta)\|_{\mathbb{k}} &\leq \|Y(\delta)\|_{B(\mathbb{k})}\|y_0\|_{\mathbb{k}} + \int_0^\delta \|Y(\delta - s)\|_{B(\mathbb{k})}\|\Psi(s, y(s))\|_{\mathbb{k}} ds \\
&\leq \vartheta\|y_0\|_{\mathbb{k}} + \vartheta \int_0^\delta p_\Psi(s)\varpi(\|y(s)\|_{\mathbb{k}})ds \\
&\leq \vartheta\|y_0\|_{\mathbb{k}} + \vartheta p_\Psi^* \int_0^\delta \varpi(|\gamma|)ds \\
&\leq \vartheta\|y_0\|_{\mathbb{k}} + \vartheta p_\Psi^* \varpi(|\gamma|)T \\
&\leq \gamma.
\end{aligned}$$

Thus,

$$\|Ny\|_{\mathbb{k}} \leq \gamma.$$

Step 2: Suppose that $\{y_i\}$ is a sequence where $y_i \to y$ in Θ_γ as $i \to \infty$. Then, $\Psi(s, y_i(s)) \to \Psi(s, y(s))$, as $i \to +\infty$, due to the Carathéodory continuity of Ψ. For $\delta \in J$ we have

$$\begin{aligned}
\|Ny_i(\delta) - Ny(\delta)\|_{\mathbb{k}} &= \|\int_0^\delta Y(\delta - s)(\Psi(s, y_i(s)) - \Psi(s, y(s)))ds\|_{\mathbb{k}} \\
&\leq \vartheta \int_0^\delta \|\Psi(s, y_i(s)) - \Psi(s, y(s))\|_{\mathbb{k}} ds.
\end{aligned}$$

It follows that $\|Ny_i - Ny\|_{\mathbb{k}} \to 0$ as $i \to +\infty$. Which implies the continuity of the operator N.

Step 3: The set $N(\Theta_\gamma)$ is equicontinuous. For any $0 < \sigma_1 < \sigma_2 < T$ and $y \in \Theta_\gamma$, we get

$$\begin{aligned}
&\|N(y)(\sigma_2) - N(y)(\sigma_1)\|_{\mathbb{k}} \\
&\quad \leq \|Y(\sigma_2) - Y(\sigma_1)\|_{B(\mathbb{k})}\|y_0\|_{\mathbb{k}} \\
&\qquad + \left\| \int_0^{\sigma_1} (Y(\sigma_2 - s) - Y(\sigma_1 - s))\Psi(s, y(s))ds \right\|_{\mathbb{k}} \\
&\qquad + \left\| \int_{\sigma_1}^{\sigma_2} (Y(\sigma_2 - s)\Psi(s, y(s))ds \right\|_{\mathbb{k}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \|Y(\sigma_2)-Y(\sigma_1)\|_{B(\Bbbk)}\|y_0\|_{\Bbbk} \\
&\quad + \int_0^{\sigma_1} \|Y(\sigma_2-s)-Y(\sigma_1-s)\|_{B(\Bbbk)} p_\Psi(\delta)\varpi(\|y\|_{\Bbbk})ds \\
&\quad + \int_{\sigma_1}^{\sigma_2} \|Y(\sigma_2-s)\|_{B(\Bbbk)} p_\Psi(\delta)\varpi(\|y\|_{\Bbbk})ds \\
&\leq \|Y(\sigma_2)-Y(\sigma_1)\|_{B(\Bbbk)}\|y_0\|_{\Bbbk} \\
&\quad + p_\Psi^*\varpi(\|y\|_{\Bbbk}) \int_0^{\sigma_1} \|Y(\sigma_2-s)-Y(\sigma_1-s)\|_{B(\Bbbk)} ds \\
&\quad + \vartheta p_\Psi^*\varpi(\|y\|_{\Bbbk})(\sigma_2-\sigma_1) \\
&\longrightarrow 0, \quad \text{as} \quad \sigma_2 \to \sigma_1.
\end{aligned}
$$

Thus, $N(\Theta_\gamma) \subseteq C(J, \Bbbk)$ is bounded and equicontinuous.

Step 4: $N : \Theta_\gamma \to \Theta_\gamma$ is a Meir-Keeler condensing operator.
We suppose $j > 0$ is given. We will demonstrate that there exists $\wp > 0$ where

$$j \leq \xi_C(\Lambda) < j + \wp \Rightarrow \xi_C(N\Lambda) < j, \quad \text{for any } \Lambda \subset \Theta_\gamma.$$

For every bounded subset $\Lambda \subset \Theta_\gamma$ and $j' > 0$ using Lemma 3 and the properties of ξ, there exists sequence $\{y_i\}_{i=1}^\infty \subset \Lambda$ such that

$$\xi\left(N(\Lambda)(\delta)\right) \leq 2\xi\left(Y(\delta)y_0 + \int_0^\delta Y(\delta-s)\Psi(s, \{y_i(s)\}_{i=1}^\infty)ds\right) + j'.$$

Next, by Lemma 4 and (H1),(H3) we have

$$
\begin{aligned}
\xi\left(N(\Lambda)(\delta)\right) &\leq 4\int_0^\delta Y(\delta-s)\xi\left(\Psi(s,\{y_i(s)\}_{i=1}^\infty)\right)ds + j' \\
&\leq 4\int_0^\delta Y(\delta-s)p_\Psi(\delta)\xi\left(\{y_i(s)\}_{i=1}^\infty\right)ds + j' \\
&\leq 4\vartheta p_\Psi^* \xi_C(\Lambda) + j'.
\end{aligned}
$$

As the last inequality is true, for every $j' > 0$, we infer

$$\xi\left(N(\Lambda)(\delta)\right) \leq 4\ell_\psi \xi_C(\Lambda).$$

As $N(\Lambda) \subset \Theta_\gamma$ is bounded and equicontinuous, by Lemma 2, we have

$$\xi_C\left(N(\Lambda)\right) = \max_{\delta \in J} \xi\left(N(\Lambda)(\delta)\right).$$

Therefore, we have

$$\xi_C\left(N(\Lambda)\right) \leq 4\ell_\psi \xi_C(\Lambda).$$

Observe that from the last estimates

$$\xi_C\left(N(\Lambda)\right) \leq 4\ell_\psi \xi_C(\Lambda) < j \Rightarrow \xi_C(\Lambda) < \frac{1}{4\ell_\psi} j.$$

Consequently, for given $j > 0$, and taking $\wp = \frac{1-4\ell_\psi}{4\ell_\psi} j$ we get the following implication:

$$j \leq \xi_C(\Lambda) < j + \wp \Rightarrow \xi_C(N\Lambda) < j, \quad \text{for any } \Lambda \subset \Theta_\gamma.$$

This means that $N : \Theta_\gamma \to \Theta_\gamma$ is a Meir-Keeler condensing operator. By Theorem 3, (1) has at least one mild solution $y \in \Theta_\gamma$.

□

4 An Example

Consider the following integro-differential equation:

$$\begin{cases} \frac{\partial}{\partial \delta} z(\delta, \mathfrak{p}) = -\frac{\partial}{\partial \mathfrak{p}} z(\delta, \mathfrak{p}) - \pi z(\delta, \mathfrak{p}) - \int_0^\delta \Gamma(\delta - s)(\frac{\partial}{\partial \mathfrak{p}} z(s, \mathfrak{p}) + \pi z(s, \mathfrak{p}))ds \\ \qquad\qquad + \left\{ \frac{1}{e^\delta + 3} \left(\frac{1}{(\delta+1)^2} + \arctan(|z(\delta, \mathfrak{p})|) \right) \right\} \quad \text{if } \delta \in J, \ \mathfrak{p} \in (0,1), \\ z(\delta, 0) = z(\delta, 1) = 0, \quad \delta \in \mathbb{R}^+, \\ z(0, \mathfrak{p}) = e^{\mathfrak{p}}, \ \mathfrak{p} \in (0,1). \end{cases} \tag{4}$$

Let Z_1 be defined by

$$(Z_1 \varphi)(\mathfrak{p}) = -\left(\frac{d}{d\mathfrak{p}} \varphi(\mathfrak{p}) + \pi \varphi(\mathfrak{p}) \right),$$

and

$$D(Z_1) = \{\varphi \in L^2(0,1) \ / \ \varphi, Z_1\varphi \in L^2(0,1) \ \ ; \ \ \varphi(0) = \varphi(1) = 0\}.$$

The operator Z_1 is the infinitesimal generator of a C_0-semigroup on $L^2(0,1)$ with domain $D(Z_1)$, and with more appropriate conditions on operator $Z_2(\cdot) = \Gamma(\cdot) Z_1$, the problem (4) has a resolvent operator $(Y(\delta))_{\delta \geq 0}$ on $L^2(0,1)$ which is norm continuous.

Now, define

$$y(\delta)(\mathfrak{p}) = z(\delta, \mathfrak{p}),$$

and $\Psi : J \times L^2(0,1) \longrightarrow L^2(0,1)$ by

$$\Psi(\delta, y)(\mathfrak{p}) = \left\{ \frac{1}{e^\delta + 3} \left(\frac{1}{(\delta+1)^2} + \arctan(|z(\delta, \mathfrak{p})|) \right) \right\}, \quad \text{for } \delta \in J, \ \mathfrak{p} \in (0,1).$$

With these settings, system (4) can be written in the abstract form

$$\begin{cases} y'(\delta) = Z_1 y(\delta) + \Psi(\delta, y(\delta)) + \int_0^\delta Z_2(\delta - s) y(s) ds, \text{ if } \delta \in J, \\ y(0) = y_0. \end{cases} \tag{5}$$

It is clear that condition (H2) holds, and as

$$\begin{aligned} \|\Psi(\delta, y(\delta))(\mathfrak{p})\|_{B(\Bbbk)} &\leq \frac{1}{e^\delta + 3} (1 + \|y(\delta, \mathfrak{p})\|_{\Bbbk}) \\ &= p_\Psi(\delta) \varpi(\|y(\delta, \mathfrak{p})\|_{\Bbbk}). \end{aligned}$$

Therefore, assumption (H3) of the Theorem 4 is satisfied with

$$p_\Psi(\delta) = \frac{1}{e^\delta + 3}, \ \delta \in J \text{ and } \varpi(\mathfrak{p}) = 1 + \mathfrak{p}, \ \mathfrak{p} \in \mathbb{R}_+.$$

For any bounded set $\Lambda \subset L^2(0,1)$, we get

$$\xi(\Psi(\delta, \Lambda)) \leq p_\Psi(\delta) \xi(\Lambda), \text{ a.e. } \delta \in J.$$

Hence (H4) is satisfied. Now, we shall check that condition $4\ell_\psi \leq 1$ is satisfied. Indeed, we have

$$4\ell_\psi = 0,2203 < 1,$$

and

$$2 + (1+\gamma)\ell_\psi \leq \gamma.$$

Thus,

$$\gamma \geq \frac{2 + \ell_\psi}{1 - \ell_\psi} = 2,1716.$$

Then, we can put $\gamma = 2,5$. All the hypotheses of Theorem 4 are verified and we deduce that (4) has at least one solution $y \in C(J, L^2(0,1))$.

Declarations

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