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VORONOVSKAYA-TYPE THEOREM FOR POSITIVE LINEAR OPERATORS BASED ON LAGRANGE INTERPOLATION*

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Dedicated to Dr. Dan Tiba on the occasion of his 70^{th} anniversary

Abstract

Since the classical asymptotic theorems of Voronovskaya-type for positive and linear operators are in fact based on the Taylor's formula which is a very particular case of Lagrange-Hermite interpolation formula, in the recent paper Gal [3], I have obtained semi-discrete quantitative Voronovskaya-type theorems based on other Lagrange-Hermite interpolation formulas, like Lagrange interpolation on two and three simple knots and Hermite interpolation on two knots, one simple and the other one double. In the present paper we obtain a semi-discrete quantitative Voronovskaya-type theorem based on Lagrange interpolation on arbitrary $p+1$ simple distinct knots.

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1 Introduction

Let us consider the well-known Bernstein polynomials defined by $B_n(f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right)$, with $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, $f \in C[0,1]$, $x \in [0,1]$, $n \in \mathbb{N}$.

One of the most important result in approximation theory is the asymptotic Voronovskaya's result on Bernstein polynomials obtained in [14] (see also, e.g., the book of Lorentz [9], formula (1), p. 22) :

Theorem 1.1. *If* $f \in C^2[0,1]$, *then*

$$\lim_{n\to\infty} n[B_n(f)(x) - f(x)] = \frac{x(1-x)}{2} \cdot f''(x),$$

uniformly on $[0,1]$. *Here* $C^p[0,1]$ *denotes the space of all real functions having a continuous derivative of order* $p \in \mathbb{N} \cup \{0\}$ *on* $[0,1]$.

In [1] (see also, e.g., the book of Lorentz [9], formula (4), p. 23), Bernstein gave the following generalization :

Theorem 1.2. *If* $p \in \mathbb{N}$ *is even and* $f \in C^p[0,1]$, *then*

$$\lim_{n\to\infty} n^{p/2} \left(B_n(f)(x) - \sum_{r=0}^{p} B_n((\cdot - x)^r)(x) \cdot \frac{f^{(r)}(x)}{r!} \right) = 0,$$

uniformly on $[0,1]$.

This result was proved for all $p \in \mathbb{N}$ by Gavrea and Ivan [4] and Tachev [12].

In [10], Mamedov extended Theorem 1.2 to positive linear operators, as follows.

Theorem 1.3. *Suppose that* $p \in \mathbb{N}$ *is even,* $f \in C^p[0,1]$ *and let* $L_n : C[0,1] \to C[0,1]$, $n \in \mathbb{N}$, *be a sequence of positive linear operators preserving the constants and satisfying*

$$L_n((\cdot - x)^{p+2j})(x) = o(L_n((\cdot - x)^p)(x)), \text{ as } n \to \infty,$$

for some $x \in [0,1]$ *and for at least one integer* $j > 0$.

Then, denoting

$$R_p(L_n(f))(x) = L_n(f)(x) - \sum_{r=0}^{p} L_n((\cdot - x)^r)(x) \cdot \frac{f^{(r)}(x)}{r!},$$

we have

$$R_p(L_n(f))(x) = o(L_n((\cdot - x)^p))(x), \text{ as } n \to \infty.$$

Here $a_n = o(b_n)$ *means that there exists* $c_n \to 0$ *such that* $a_n = c_n b_n$, $n \in \mathbb{N}$.

A complete asymptotic expansion in a quantitative form in Theorem 1.3 was already given some 40 years ago by Sikkema-van der Meer [11].

The first quantitative estimates in Theorem 1.1 were obtained for $f \in C^3[0,1]$ by Ditzian-Ivanov in [2], for $f \in C^4[0,1]$ by Gonska-Rasa in [8], pointwise estimates in terms of the modulus of continuity by Videnskij in [13], p. 19, Theorem 15.2 and in terms of the least concave majorant by Gonska-Pitul-Rasa in [7].

Also, quantitative estimates in Theorem 1.2 were obtained in terms of the least concave majorant and a K-functional by Theorem 3.2 in Gonska [6] and by Gavrea-Ivan in [5].

A general characteristic of all the above results is that their proofs are based on the Taylor's formula with remainder. But because the Taylor's formula is nothing else that a particular Lagrange-Hermite interpolation formula with only one multiple knot, it is natural to seek for asymptotic formulas based on Lagrange-Hermite interpolation formula with several simple or multiple knots.

In the recent paper Gal [3], I have obtained semi-discrete quantitative Voronovskaya-type theorems based on other Lagrange-Hermite interpolation formulas, like Lagrange interpolation on 2 and 3 simple knots and Hermite interpolation on two knots, one simple and the other one double. In the present paper we obtain a semi-discrete quantitative Voronovskaya-type theorem based on Lagrange interpolation on arbitrary $p \in \mathbb{N}$ simple knots.

Section 2 contains some preliminaries on Lagrange interpolation. In Section 3 we obtain a semi-discrete quantitative Voronovskaya-type theorem based on Lagrange interpolation on arbitrary $p+1$ simple knots.

2 Preliminaries on Lagrange interpolation

Let us consider the Lagrange interpolation formula on $p+1$ simple distinct knots $y_0, y_1, y_2, ..., y_p$, i.e. $f(t) = H_p(f)(t) + R_p(f, y_0, ..., y_p)(t)$, with $H_p(f)(t)$ written in the Newton's form

$$H_p(t) = f(y_0) + \sum_{j=1}^{p} (t - y_0) \cdot ... \cdot (t - y_{j-1})[y_0, ..., y_j; f],$$

with $[y_0, ..., y_j; f]$ the divided difference of f on the knots $y_0, ..., y_j$ and with the remainder $R_p(f, y_0, ..., y_p)(t) = (t - y_0) \cdot ... \cdot (t - y_p) \cdot \frac{f^{(p+1)}(\xi)}{p!}$, with ξ belonging to the convex hull of the points $t, y_0, ..., y_p$.

Denoting above $y_0 := x$, we can write

$$H_p(t) = f(x)+(t-x)[x,y_1;f]+\sum_{j=2}^{p}(t-x)(t-y_1)\cdot\ldots\cdot(t-y_{j-1})[x,y_1,...,y_j;f],$$

with the remainder

$$R_p(f)(t,x,y_1,...,y_p) = (t-x)(t-y_1)\cdot\ldots\cdot(t-y_p)\cdot\frac{f^{(p+1)}(\xi)}{p!},$$

where

$$|\xi - x| \le \max\{|t-x|,|y_1-x|,...,|y_p-x|\} \le |t-x| + \sum_{j=1}^{p}|y_j - x|.$$

3 Main results

In this section we obtain a general semi-discrete quantitative Voronosvkaya-type theorem for positive linear operators based on Lagrange interpolation on an arbitrary number of knots.

Denoting by $\omega(f;\delta) = \sup\{|f(u)-f(v)|; u,v \in [0,1], |u-v| \le \delta\}$ the modulus of continuity, we can state the following result.

Theorem 3.1. *Suppose that $p \in \mathbb{N}$, $f \in C^{p+1}[0,1]$ and let $L_n : C[0,1] \to C[0,1]$, $n \in \mathbb{N}$, be a sequence of positive linear operators preserving the constants. For all $n \in \mathbb{N}$ and all distinct knots $x, y_1 ..., y_p \in [0,1]$ we have*

$$\begin{aligned}
&\Big|L_n(f)(x) - f(x) - L_n((e_1-x))(x)\cdot[x,y_1;f] \\
&-\sum_{j=2}^{p} L_n((e_1-x)\Pi_{k=1}^{j-1}(e_1-y_k))(x)\cdot[x,y_1,...,y_j;f] \\
&- L_n((e_1-x)\Pi_{j=1}^{p}(e_1-y_j))(x)\cdot\frac{f^{(p+1)}(x)}{(p+1)!}\Big| \\
&\le \frac{2L_n(|e_1-x|\cdot\Pi_{j=1}^{p}|e_1-y_j|)(x)}{(p+1)!} \\
&\cdot\omega\left(f^{(p+1)}; \frac{L_n((e_1-x)^2\cdot\Pi_{j=1}^{p}|e_1-y_j|)(x)}{L_n(|e_1-x|\cdot\Pi_{j=1}^{p}|e_1-y_j|)(x)} + \sum_{j=1}^{p}|y_j-x|\right).
\end{aligned}$$

Proof. For all $t, y_j \in [0,1]$, $j=1,...,p$, we can write

$$\begin{aligned}|f(t)-H_p(f)(t)| =&|f(t)-f(x)-(t-x)\cdot[x,y_1;f]\\&-\sum_{j=2}^{p}(t-x)(t-y_1)\cdot ...\cdot(t-y_{j-1})\cdot[x,y_1,...,y_j;f]\\&-(t-x)\Pi_{j=1}^{p}(t-y_j)\frac{f^{(p+1)}(x)}{3!}|\\=&\frac{|t-x|\cdot\Pi_{j=1}^{p}|t-y_j|}{(p+1)!}|f^{(p+1)}(\xi)-f^{(p+1)}(x)|\end{aligned}$$

$$\begin{aligned}\leq&\frac{|t-x|\cdot\Pi_{j=1}^{p}|t-y_j|}{(p+1)!}\cdot\omega\left(f^{(p+1)};\delta\cdot\frac{|t-x|+\sum_{j=1}^{p}|y_j-x|}{\delta}\right)\\\leq&\frac{|t-x|\cdot\Pi_{j=1}^{p}|t-y_j|}{(p+1)!}\left[1+\frac{|t-x|+\sum_{j=1}^{p}|y_j-x|}{\delta}\right]\cdot\omega\left(f^{(p+1)};\delta\right).\end{aligned}$$

Applying L_n to the above inequality, we immediately obtain

$$\begin{aligned}&|L_n(f)(x)-f(x)-L_n((e_1-x))(x)\cdot[x,y_1;f]\\&-\sum_{j=2}^{p}L_n((e_1-x)\Pi_{k=1}^{j-1}(e_1-y_k))(x)\cdot[x,y_1,...,y_j;f]\\&-L_n((e_1-x)\Pi_{j=1}^{p}(e_1-y_j))(x)\cdot\frac{f^{(p+1)}(x)}{(p+1)!}\Bigg|\end{aligned}$$

$$\begin{aligned}\leq&L_n\left(\left(|e_1-x|\cdot\Pi_{j=1}^{p}|e_1-y_j|\right)\left[1+\frac{|e_1-x|+\sum_{j=1}^{p}|y_j-x|}{\delta}\right]\right)(x)\\&\cdot\frac{1}{(p+1)!}\cdot\omega\left(f^{(p+1)};\delta\right)\\=&\Bigg[L_n(|e_1-x|\cdot\Pi_{j=1}^{p}|e_1-y_j|)(x)\\&+\frac{L_n\left(|e_1-x|\cdot\Pi_{j=1}^{p}|e_1-y_j|\cdot\left[|e_1-x|+\sum_{j=1}^{p}|y_j-x|\right]\right)(x)}{\delta}\Bigg]\\&\cdot\frac{1}{(p+1)!}\cdot\omega\left(f^{(p+1)};\delta\right)(x).\end{aligned}$$

Choosing here

$$\delta = \frac{L_n\left(|e_1 - x| \cdot \Pi_{j=1}^{p}|e_1 - y_j| \cdot \left[|e_1 - x| + \sum_{j=1}^{p}|y_j - x|\right]\right)(x)}{L_n(|e_1 - x| \cdot \Pi_{j=1}^{p}|e_1 - y_j|)(x)}$$
$$= \frac{L_n((e_1 - x)^2 \cdot \Pi_{j=1}^{p}|e_1 - y_j|)(x)}{L_n(|e_1 - x| \cdot \Pi_{j=1}^{p}|e_1 - y_j|)(x)} + \sum_{j=1}^{p}|y_j - x|,$$

we easily arrive at the desired formula. □

Remark 3.2. Notice that for $y_j \to x$ for all $j = 1, ..., p$, since $[x, y_1; f] \to f'(x)$, $[x, y_1, y_2; f] \to \frac{f''(x)}{2!}$, $[x, y_1, y_2, ..., y_j; f] \to \frac{f^{(j)}(x)}{j!}$ and so one, it is easy to see that the formula in Theorem 3 becomes the following asymptotic quantitative Voronovskaya-type theorem

$$\left|L_n(f)(x) - f(x) - \sum_{j=1}^{p+1} L_n((e_1 - x)^j)(x) \cdot \frac{f^{(j)}(x)}{j!}\right|$$
$$\leq \frac{2L_n(|e_1 - x|^{p+1})(x)}{(p+1)!} \cdot \omega\left(f^{(p+1)}; \frac{L_n((e_1 - x)^{p+2})(x)}{L_n(|e_1 - x|^{p+1})(x)}\right),$$

recapturing thus the general Theorem 3.2 in Gonska [6].

For $p = 2$ we recapture the left-hand side in Theorem 3.3 and Remark 3.4 in Gal [3], but with different estimates on the right-hand side.

The above Theorem 3.1 and Remark 3.2 are applicable, for example, to the classical Bernstein polynomials considered at the beginning of Introduction.

Remark 3.3. Applying Theorem 3.1 to Bernstein polynomials, we get the following semi-discrete quantitative Voronovskaya-type formula

$$\Bigg|B_n(f)(x) - f(x) - \sum_{j=2}^{p} B_n((e_1 - x)\Pi_{k=1}^{j-1}(e_1 - y_k))(x) \cdot [x, y_1, ..., y_j; f]$$
$$- B_n((e_1 - x)\Pi_{j=1}^{p}(e_1 - y_j))(x) \cdot \frac{f^{(p+1)}(x)}{(p+1)!}\Bigg|$$
$$\leq \frac{2B_n(|e_1 - x| \cdot \Pi_{j=1}^{p}|e_1 - y_j|)(x)}{(p+1)!}$$
$$\cdot \omega\left(f^{(p+1)}; \frac{B_n((e_1 - x)^2 \cdot \Pi_{j=1}^{p}|e_1 - y_j|)(x)}{B_n(|e_1 - x| \cdot \Pi_{j=1}^{p}|e_1 - y_j|)(x)} + \sum_{j=1}^{p}|y_j - x|\right),$$

since $B_n((e_1 - x))(x) = 0$. Also, since the moments of Bernstein polynomials $B_n((e_1 - x)^j)(x)$, $j = 0, 1, 2, 3, ...$, can be explicitly calculated (see, e.g., Lorentz [9], page 4 where $B_n(1)(x) = 1, B_n(e_1 - x)(x) = 0$, $B_n((e_1 - x)^2)(x) = \frac{x(1-x)}{n}$, $B_n((e_1 - x)^3)(x) = \frac{(1-2x)x(1-x)}{n^2}$, so one), and since, for example we get

$$\begin{aligned} &B_n((e_1 - y)^2)(x) = B_n((e_1 - x + x - y)^2)(x) \\ =&B_n((e_1 - x)^2)(x) + 2(x - y)B_n(e_1 - x)(x) + (x - y)^2 \\ =&B_n((e_1 - x)^2)(x) + (x - y)^2 = \frac{x(1-x)}{n} + (x - y)^2, \end{aligned}$$

$$\begin{aligned} &B_n((e_1 - x)(e_1 - y)^2)(x) = B_n((e_1 - x)(e_1 - x + x - y)^2)(x) \\ =&B_n((e_1 - x)((e_1 - x)^2 + 2(x - y)(e_1 - x) + (x - y)^2))(x) \\ =&B_n((e_1 - x)^3)(x) + 2(x - y)B_n((e_1 - x)^2)(x) \\ =&B_n((e_1 - x)^3)(x) + 2(x - y) \cdot \frac{x(1-x)}{n} \\ =&\frac{x(1-x)}{n}\left(\frac{1-2x}{n} + 2(x - y)\right), \end{aligned}$$

so one, it follows that for various values of p we can obtain explicit formulas in Theorem 3.1 for Bernstein polyomials.

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