Ann. Acad. Rom. Sci.
Ser. Math. Appl.
ISSN 2066-6594 Vol. 15, No. 1-2/2023

ON THE NONLINEAR STABILITY FOR QUASI-GEOSTROPHIC FORCED ZONAL FLOWS*

L. Palese†

DOI https://doi.org/10.56082/annalsarscimath.2023.1-2.73

Dedicated to Dr. Dan Tiba on the occasion of his 70^{th} anniversary

Abstract

This paper continues a series of studies providing stability criteria for quasigeostrophic forced zonal flows in in the presence of lateral diffusion and bottom dissipation of the vertical vorticity. We study the Lyapunov stability of a stationary and longitude independent basic flow, obtaining linear and nonlinear stability criteria expressed in terms of the maximum shear of the basic flow and/or its meridional derivative, extending some previous results.

MSC: 76E15, 76E30

keywords: Stability, Energy method

1 Introduction

In dynamic meteorology the large-scale atmospheric motions are characterized by many parameters presenting large spatial variations in the vertical direction. For this reason, we limit our considerations to only a part of

*Accepted for publication on January 17-th, 2023

†lidiarosaria.palese@uniba.it Dipartimento di Matematica, Università di Bari, Via E. Orabona 4, 70125 Bari, Italy; Paper written with financial support of the group "Equazioni di Evoluzione: Analisi Qualitativa e Metodi Numerici", University of Bari, Italy.

the turbulent scale, the synoptic-scale, of interest in short-range weather forecasting, and to a well determined spatial region, taking the rest of the atmosphere into account by means of initial and boundary conditions.

Consequently, we adopt the quasigeostrophic approximation at the synoptic-scale [1]-[5].

The main parameters governing the flow are the Reynolds number related to the lateral dissipation and the parameter r related to the bottom dissipation. The longitude-independent forcing term F includes the wind stress *curl*.

We consider flows ideally placed between rigid walls of a constant latitude which isolate the region, i.e. "channelled flows", of interest from a geophysical point of view, as, in a rotating planet, they maintain themselves without any external forcing.

They are the simplest flow configuration in the presence of longitude-independent force.

Non steady basic flows $\Psi_0(y,t)$, in presence of non stationary forcing, were studied, for barotropic flows, by Kuo [1] [2], Haidvogel and Holland [7], Wolanski [8] and Crisciani and Mosetti [9].

Two dimensional basic flow $\Psi_0(x,y,t)$, suitable for baroclinic instability studies in channel geometry (e.g. for Antarctic flows) or in rectangular closed basins (e.g. for Northern Hemisphere flows) were studied, for various types of basin geometries, by Stommel [10], Munk [11], Mc Williams and Chow [12], Le Provost and Verron [13], Crisciani [14], and Crisciani and Mosetti [15]- [17]. The linear as well as the non linear cases were considered.

In this paper we study the Lyapunov stability of the stationary and longitude independent basic flow $\Psi_0(y)$, corresponding to the forcing $F(y)$. In Section 2 we present the mathematical model (equation and boundary conditions) governing the perturbation.

In Section 3, after introducing a suitable Lyapunov function, we formulate the nonlinear stability problem.

In Section 4 we derive some linear and nonlinear stability criteria in terms of the maximum of local vorticity or its meridional derivative.

In Section 5 the obtained criteria are discussed in physical terms. The stability domains are enlarged with respect to some stability criteria found in literature [15]- [17], [21], [22].

2 Mathematical model:basic state and perturbations

In the quasigeostrophic approximation the mathematical problem governing the stability of zonal flows, for a barotropic flows with longitude independent forcing, leads the following vorticity balance [1]-[5] :

$$\frac{\partial \Delta\Psi}{\partial t} + \mathcal{J}(\Psi, \Delta\Psi + \beta y) = F(y,t) - r\Delta\Psi + A\Delta\Delta\Psi. \tag{1}$$

where Ψ is the stream function (y being latitude and x longitude), $\Delta\Psi$ is the vertical vorticity, i.e. the vertical component of the *curl* of the geostrophic current, β is the planetary vorticity gradient due to latitudinal variation of the Coriolis parameter, F is a forcing term, $-r\Delta\Psi$ is the bottom dissipation term and $A\Delta\Delta\Psi$ is related to lateral vorticity diffusion [4],[5].

In the following both the constants A and r will be assumed to be greater than zero.

To the balance equation (1) we add the boundary conditions on Ψ:

$$\Psi_x = 0 \qquad y = y_1 \quad y = y_2 \tag{2}$$

$$\Delta\Psi = 0 \qquad y = y_1 \quad y = y_2 \tag{3}$$

where (2) is the condition of zero mass flux across the wall latitude, and (3) represents the zero lateral vorticity diffusion. The subscript indicates the differentiation.

If we specify the forcing term $F(y)$, the problem (1)-(2)-(3) has the zonal solution $\Psi_0(y)$ [8]. The basic flow, characterized by the local vorticity $\Delta\Psi_0 = q_0$, is the unique solution of the two-point problem

$$q_0(y_1) = q_0(y_2) = 0$$

for the ordinary differential equation

$$Aq_{0yy} - rq_0 + F(y) = 0.$$

Therefore the perturbation $\phi(x,y,t) = \Psi - \Psi_0$, induced by the perturbation of the initial condition , satisfies the following equation [8] :

$$\Delta\phi_t + \Phi_x\Delta\phi_y - \Phi_y\Delta\phi_x + \Psi_{0x}\Delta\phi_y - \Psi_{0y}\Delta\phi_x + (\Delta\Psi_{0y} + \beta)\phi_x - \Phi_y\Delta\Psi_{0x} +$$

$$r\Delta\phi - A\Delta\Delta\phi = 0, \tag{4}$$

the boundary conditions

$$\phi_x = 0, \quad \Delta\phi = 0 \qquad \text{at} \quad \mathrm{y} = \mathrm{y}_1 \quad \text{and} \quad \mathrm{y} = \mathrm{y}_2, \tag{5}$$

on the rigid walls at constant latitude, and some initial conditions $\phi = \phi_0$ for $t = 0$.

3 Stability problem

Let us introduce the Lyapunov function

$$K(t) = \frac{1}{2}\int_\Omega (\Delta\phi)^2 d\Omega. \tag{6}$$

where Ω is a closed basin we specify later.

The Lyapunov function $K(t)$ can be written also as

$$K(t) = \frac{1}{2}\int_\Omega (v_x - u_y)^2 d\Omega, \tag{7}$$

where (u, v) are the horizontal components of the velocity field $\mathbf{v}$, namely the ratio of the vertical velocity to the horizontal velocity, at the synoptic scale, is of the order 10^{-3}, [4], [5], and the geostrophic vorticity $\zeta_g \equiv (v_x - u_y)$ can be expressed as $\zeta_g \equiv v_x - u_y = \Delta\Phi$.

This equation can be solved to determine ζ_g from a field Φ and, alternatively, to determine Φ from ζ_g. This invertibility allows us to choice, from a physical point of view [5], the Lyapunov function (6).

From now on we shall assume that the closed basin Ω is a periodicity cell [23], $\Omega = \mathcal{V} \times [0, L]$, where $L = y_2 - y_1$, $\mathcal{V} = \left[0, \frac{2\pi}{k_x}\right]$ and k_x is the wave number.

The basic flow Ψ_0 is asimptotically stable if it is stable and $\lim_{t\to\infty} K(t) = 0$.

In order to deduce criteria for asymptotic stability, we need inequalities of the form

$$\frac{dK}{dt} + aK \leq 0, \tag{8}$$

where $a > 0$ is a constant. For stability criteria it is enough that:

$$\frac{dK(t)}{dt} \leq 0. \tag{9}$$

Multiplying the equation (4) by $\Delta\Phi$ and integrating over Ω we obtain:

$$\frac{dK(t)}{dt} = \tag{10}$$

$$-\int_\Omega \Phi_x \Delta\Phi \Delta\Psi_{0y} d\Omega + \int_\Omega \Phi_y \Delta\Phi\Delta\Psi_{0x} d\Omega - r\int_\Omega (\Delta\Phi)^2 d\Omega - A\int_\Omega (\nabla\Delta\Phi)^2 d\Omega.$$

In the energy relation (10) all nonlinear terms disappear, namely

$$\int_\Omega \Phi_x \Delta\Phi\Delta\Phi_y d\Omega - \int_\Omega \Phi_y \Delta\Phi\Delta\Phi_x d\Omega = \int_\Omega \nabla\Phi \cdot \nabla\times\Big(\frac{(\Delta\Phi)^2}{2}\mathbf{k}\Big) d\Omega = \tag{11}$$

$$-\int_\Omega \nabla\cdot\Big[\nabla\Phi\times\Big(\frac{(\Delta\Phi)^2}{2}\mathbf{k}\Big] d\Omega + \int_\Omega \frac{(\Delta\Phi)^2}{2}\mathbf{k}\cdot\nabla\times\nabla\Phi d\Omega =$$

$$-\int_{\partial\Omega} \nabla\Phi\times\Big(\frac{(\Delta\Phi)^2}{2}\mathbf{k}\Big)\cdot \mathbf{n}_e d\Sigma = 0,$$

with $\mathbf{n}_e$ external normal to the boundary $\partial\Omega$.

In a similar way we obtain:

$$\int_\Omega \Psi_{0x}\Delta\Phi\Delta\Phi_y d\Omega - \int_\Omega \Psi_{0y}\Delta\Phi\Delta\Phi_x d\Omega = 0, \quad \int_\Omega \Phi_x\Delta\Phi d\Omega = 0. \tag{12}$$

If we assume the local vorticity $\Delta\Psi_0 = \Delta\Psi_0(y)$, from the energy relation (10) we have:

$$\frac{dK(t)}{dt} = -\int_\Omega \Phi_x\Delta\Phi\Delta\Psi_{0y} d\Omega - r\int_\Omega (\Delta\Phi)^2 d\Omega - A\int_\Omega (\nabla\Delta\Phi)^2 d\Omega. \tag{13}$$

From the boundary conditions (5) it follows:

$$\int_\Omega \Phi_x\Delta\Phi\Delta\Psi_{0y} d\Omega = \int_\Omega \Phi_x(\Phi_{xx}+\Phi_{yy})\Delta\Psi_{0y} d\Omega = \int_\Omega \Big(\frac{\Delta\Psi_{0y}\Phi_x^2}{2}\Big)_x d\Omega +$$

$$\int_\Omega \Phi_x\Phi_{yy}\Delta\Psi_{0y} d\Omega = \int_\Omega \nabla\cdot\Big(\frac{\Delta\Psi_{0y}\Phi_x^2}{2}\mathbf{i}\Big) d\Omega + \int_\Omega \Phi_x\Phi_{yy}\Delta\Psi_{0y} d\Omega =$$

$$\int_{\partial\Omega} \frac{\Delta\Psi_{0y}\Phi_x^2}{2}\mathbf{i}\cdot\mathbf{n}_e d\sigma + \int_\Omega \Phi_x\Phi_{yy}\Delta\Psi_{0y} d\Omega = \int_\Omega \Phi_x\Phi_{yy}\Delta\Psi_{0y} d\Omega, \tag{14}$$

with $\mathbf{i}$ unit vector in x-direction. The energy relation (13) becomes:

$$\frac{dK(t)}{dt} = -\int_\Omega \Phi_x\Phi_{yy}\Delta\Psi_{0y} d\Omega - r\int_\Omega (\Delta\Phi)^2 d\Omega - A\int_\Omega (\nabla\Delta\Phi)^2 d\Omega. \tag{15}$$

We observe that the energy relation (15) is a starting point to study linear Lyapunov stability too, because all nonlinear terms vanish.

4 Energy inequality and stability criteria

In this section we shall derive some stability criteria in terms of the maximum of the local vorticity μ_2 and of its meridional derivative μ_3

$$\mu_2 = max_{y\in[0,1]}|q_0| \qquad \mu_3 = max_{y\in[0,1]}|q_{0y}|$$

In order to derive stability criteria in terms of μ_3 we evaluate the first term on the right hand side of (13) taking into account the Schwarz and Hölder generalized inequalities [18], [19].

$$-\int_\Omega \Phi_x\Phi_{yy}\Delta\Psi_{0y}d\Omega \le \mu_3\int_\Omega |\Phi_x||\Phi_{yy}|d\Omega \le \frac{\mu_3^2}{2\epsilon}\int_\Omega \Phi_x^2 d\Omega + \frac{\epsilon}{2}\int_\Omega \Phi_{yy}^2 d\Omega \quad (16)$$

where $\epsilon > 0$.

From the boundary conditions (3) it follows [19],[20] and [22]

$$\int_\Omega \Phi_x^2 d\Omega \le \frac{1}{\alpha}\int_\Omega \nabla^2\Phi_x d\Omega, \quad (17)$$

where

$$\frac{1}{\alpha} = \frac{L^2}{\pi^2}. \quad (18)$$

It is also easily to show [18], taking into account the boundary conditions (3), that

$$\int_\Omega \nabla^2\Phi_x d\Omega \le \int_\Omega (\Delta\Phi)^2 d\Omega. \quad (19)$$

Substituting (17) in (16),using (19) and differentiating with respect to ϵ, it follows

$$-\int_\Omega \Phi_x\Phi_{yy}\Delta\Psi_{0y}d\Omega \le \frac{\mu_3}{2\sqrt{\alpha}}\int_\Omega (\Delta\Phi)^2 d\Omega, \quad (20)$$

with $\epsilon = \dfrac{\mu_3}{\sqrt{\alpha}}$.

From the energy relation (15) we obtain the inequality:

$$\frac{dK(t)}{dt} \le \Big(\frac{\mu_3}{2\sqrt{\alpha}} - r - A\alpha\Big)\int_\Omega (\Delta\Phi)^2 d\Omega, \quad (21)$$

since, due to the boundary conditions (3), it follows [22]

$$\int_\Omega (\Delta\Phi)^2 d\Omega \le \frac{1}{\alpha}\int_\Omega (\nabla\Delta\Phi)^2 d\Omega. \quad (22)$$

We observe that, taking into account (22), the energy inequality can be written equivalently as:

$$\frac{dK(t)}{dt} \leq \Big(\frac{\mu_3}{2\sqrt{\alpha}} - r - A\alpha\Big)2K(t). \tag{23}$$

The inequality

$$\frac{\mu_3}{2\sqrt{\alpha}} - r - A\alpha < 0 \tag{24}$$

is a sufficient conditions of nonlinear global asymptotical Lyapunov stability of the basic motion. We proved the following

Theorem 1 *The basic motion q_0 is linearly and nonlinearly globally asymptotically stable if:*

$$\mu_3 < 2(r + A\alpha)\sqrt{\alpha}. \tag{25}$$

Let us reconsider, in order to derive stability criteria in terms of μ_2, the first term on the right-hand side of (15),

$$\int_\Omega \Phi_x \Phi_{yy} \Delta\Psi_{0y} d\Omega = -\int_\Omega (\Phi_x \Phi_{yyy} + \Phi_{xy}\Phi_{yy})\Delta\Psi_0 d\Omega \leq$$

$$\frac{\mu_2^2}{2\epsilon_1}\int_\Omega \Phi_{xy}^2 d\Omega + \frac{\epsilon_1}{2}\int_\Omega \Phi_{yy}^2 d\Omega + \frac{\mu_2^2}{2\epsilon_2}\int_\Omega \Phi_x^2 d\Omega + \frac{\epsilon_2}{2}\int_\Omega \Phi_{yyy}^2 d\Omega. \tag{26}$$

Taking into account (17) we have:

$$\int_\Omega \Phi_x \Phi_{yy} \Delta\Psi_{0y} d\Omega \leq$$

$$\Big(\frac{\mu_2^2}{2\epsilon_1} + \frac{\mu_2^2}{2\epsilon_2\alpha}\Big)\int_\Omega \Phi_{xy}^2 d\Omega + \frac{\epsilon_2}{2}\int_\Omega \Phi_{yyy}^2 d\Omega + \frac{\epsilon_1}{2}\int_\Omega \Phi_{yy}^2 d\Omega + \frac{\mu_2^2}{2\epsilon_2\alpha}\int_\Omega \Phi_{xx}^2 d\Omega. \tag{27}$$

From the boundary conditions (3) it follows that

$$\int_\Omega (\nabla\Delta\Phi)^2 d\Omega = \int_\Omega (\nabla\Phi_{xx})^2 \Delta d\Omega + 2\int_\Omega (\nabla\Phi_{xy})^2 d\Omega + \int_\Omega (\nabla\Phi_{yy})^2 d\Omega. \tag{28}$$

Since

$$\int_\Omega \Phi_{xy} d\Omega = 0 \quad \text{and} \quad \Phi xyy = 0, \quad \text{on} \quad y = y_1, y = y_2,$$

it follows that

$$\int_\Omega (\Phi_{xy})^2 d\Omega \leq c_1 \int_\Omega (\nabla\Phi_{xy})^2 d\Omega, \tag{29}$$

where $c_1 = \max\{\frac{L^2}{\pi^2}, k_x^{-2}\}$ [22].

From the boundary conditions (3) follows the Poincaré inequality:

$$\int_\Omega (\Phi_{xx})^2 d\Omega \leq \frac{1}{\alpha}\int_\Omega (\nabla\Phi_{xx})^2 d\Omega. \tag{30}$$

The energy relation (15), using (27) and (28), becomes:

$$\frac{dK(t)}{dt} \leq \Big(\frac{\mu_2^2}{2\epsilon_1}+\frac{\mu_2^2}{2\epsilon_2\alpha}-2r\Big)\int_\Omega \Phi_{xy}^2 d\Omega+\frac{\epsilon_2}{2}\int_\Omega \Phi_{yyy}^2 d\Omega+\Big(\frac{\epsilon_1}{2}-r\Big)\int_\Omega \Phi_{yy}^2 d\Omega+ \tag{31}$$

$$\Big(\frac{\mu_2^2}{2\epsilon_2\alpha}-r\Big)\int_\Omega \Phi_{xx}^2 d\Omega-A\Big(\int_\Omega (\nabla\Phi_{xx})^2\Delta d\Omega+2\int_\Omega (\nabla\Phi_{xy})^2 d\Omega+\int_\Omega (\nabla\Phi_{yy})^2 d\Omega\Big).$$

Using (29) and (30) from (31) it follows:

$$\frac{dK(t)}{dt} \leq \Big(\frac{\mu_2^2}{2\epsilon_1}+\frac{\mu_2^2}{2\epsilon_2\alpha}-2r-\frac{2A}{c_1}\Big)\int_\Omega \Phi_{xy}^2 d\Omega+\Big(\frac{\epsilon_1}{2}-r\Big)\int_\Omega \Phi_{yy}^2 d\Omega+ \tag{32}$$

$$\Big(\frac{\epsilon_2}{2}-A\Big)\int_\Omega \nabla^2\Phi_{yy} d\Omega+\Big(\frac{\mu_2^2}{2\epsilon_2\alpha}-r-A\alpha\Big)\int_\Omega \Phi_{xx}^2 d\Omega.$$

If ϵ_1 satisfies the inequality $\frac{\epsilon_1}{2}-r\leq 0$, from (32) we consider the following term:

$$\Big(\frac{\epsilon_1}{2}-r\Big)\int_\Omega \Phi_{yy}^2 d\Omega+\Big(\frac{\epsilon_2}{2}-A\Big)\int_\Omega \nabla^2\Phi_{yy} d\Omega, \tag{33}$$

to determine ϵ_1, and ϵ_2 such that

$$\Big(\frac{\epsilon_1}{2}-r\Big)\int_\Omega \Phi_{yy}^2 d\Omega+\Big(\frac{\epsilon_2}{2}-A\Big)\int_\Omega \nabla^2\Phi_{yy} d\Omega \leq 0,$$

or

$$\int_\Omega \nabla^2\Phi_{yy} d\Omega \leq \frac{2r-\epsilon_1}{\epsilon_2-2A}\int_\Omega \Phi_{yy}^2 d\Omega, \tag{34}$$

if $\epsilon_2/2-A>0$.

Considering the Poincaré inequality

$$\int_\Omega (\Phi_{yy})^2 d\Omega \leq \frac{1}{\alpha}\int_\Omega (\nabla\Phi_{yy})^2 d\Omega, \tag{35}$$

from (34) and 35) we obtain

$$\int_\Omega \nabla^2\Phi_{yy} d\Omega \leq \frac{2r-\epsilon_1}{(\epsilon_2-2A)\alpha}\int_\Omega \nabla^2\Phi_{yy} d\Omega, \tag{36}$$

it follows

$$\frac{2r - \epsilon_1}{\epsilon_2 - 2A} = \alpha. \tag{37}$$

The energy inequality (32), if ϵ_1 and ϵ_2 satisfy (37), becomes:

$$\frac{dK(t)}{dt} \le \Big(\frac{\mu_2^2}{2\epsilon_1} + \frac{\mu_2^2}{2\epsilon_2\alpha} - 2r - \frac{2A}{c_1}\Big) \int_\Omega \Phi_{xy}^2 d\Omega + \Big(\frac{\mu_2^2}{2\epsilon_2\alpha} - r - A\alpha\Big) \int_\Omega \Phi_{xx}^2 d\Omega. \tag{38}$$

The inequalities

$$\frac{\mu_2^2}{2\epsilon_1} + \frac{\mu_2^2}{2\epsilon_2\alpha} - 2r - \frac{2A}{c_1} < 0 \qquad \frac{\mu_2^2}{2\epsilon_2\alpha} - r - A\alpha < 0, \tag{39}$$

with ϵ_1, ϵ_2 satisfing (37), imply linear and nonlinear global stability of the basic motion.

Deriving ϵ_1 from (37), i.e. $\epsilon_1 = 2(r + A\alpha) - \alpha\epsilon_2$ and substituting in in (39), if $2(r + A\alpha) - \alpha\epsilon_2 > 0$ we have:

$$\mu_2^2 < 2\epsilon_2\alpha(r + A\alpha) \qquad \mu_2^2 < 2(r + Ac_1^{-1})\Big(2\epsilon_2\alpha - \frac{(\alpha\epsilon_2)^2}{r + A\alpha}\Big). \tag{40}$$

To enlarge the stability domain we differentiate the function on right hand side of $(40)_2$ respect to $\alpha\epsilon_2$ obtaining $\alpha\epsilon_2 = r + A\alpha$, and, substituting in (37), $\epsilon_1 = r + A\alpha$. We observe that the inequalities $\epsilon_2/2 - A > 0$ and $\epsilon_1/2 - r < 0$ become

$$2A\alpha < r + A\alpha < 2r,$$

or:

$$A\alpha < r. \tag{41}$$

The inequality (41) is satisfied in physical terms, as we shall see in the next Section. From the inequalities (40), substituting $\epsilon_1 = r + A\alpha = \alpha\epsilon_2$, we obtain:

$$\mu_2^2 < 2(r + A\alpha)^2 \qquad \mu_2^2 < 2(r + Ac_1^{-1})(r + A\alpha). \tag{42}$$

We proved the following

Theorem 2 *The basic motion q_0 is linearly and nonlinearly globally stable if:*

$$\mu_2 < \min\left\{(r + A\alpha)\sqrt{2}, \sqrt{2(r + A\alpha)(r + Ac_1^{-1})}\right\}. \tag{43}$$

5 Discussion and Concluding remarks

We shall summarize our main results in mathematical and physical terms. In this paper we obtained some stability criteria, in terms of the maximum shear of the basic flow.

Let us assume a typical interval for the oceanic values (in S.I. units) of lateral vorticity diffusion

$$10^2 \leq A \leq 10^4, \tag{44}$$

for bottom dissipation r the value

$$r = 10^{-7} \tag{45}$$

and for L the value

$$L = 10^6. \tag{46}$$

Therefore, for the dimensionless variable $\frac{r}{A\alpha}$, considering (18), (44), (45) and (46), we have:

$$\frac{10}{\pi^2} \leq \frac{r}{A\alpha} \leq \frac{10^3}{\pi^2}. \tag{47}$$

Because of (47) it can be easily shown that, if $\frac{r}{A\alpha} > r_2$, where r_2 is the largest positive rooth of the equation :

$$(\frac{r}{A\alpha})^2 + \frac{r}{A\alpha}((\alpha c_1)^{-1} - 3) + (\alpha c_1)^{-1} = 0,$$

(from the definition of c_1 it follows that $c_1 > \frac{1}{\alpha}$, namely $(\alpha c_1)^{-1} < 1$), the stability domain (43) enlarges the previous one obtained in [23], i.e.

$$\mu_2 < 2\sqrt{2rA\alpha} \tag{48}$$

that represents, among the criteria

$$\mu_2 < 2\sqrt{(2-\epsilon)rA\alpha} \qquad \forall\epsilon > 0, \tag{49}$$

the best criterion of linear asymptotic stability, improving the stability criterion of Crisciani and Mosetti [9] $\mu_2 \leq 2\sqrt{rA}$.

The stability criterion (25), expressed in terms of μ_3 only ,

$$\mu_3 < 2(r + A\alpha)\sqrt{\alpha} \tag{50}$$

cannot be compared with some previous results, because in [22] some other linear asymptotic stability regions are defined in terms of the maximum shear of the basic flow and of the maximum of its meridional derivative, in the $(\mu_3, \alpha^{-1}\mu_2)$-plane, leading to considerable improvement of the linear stability domain for cases where functional inequalities at hand are very weak.

However all considerations in [22] were limited by the fact that the terms in the second meridional derivative of basic flow was disregarded, see (13) of [22].

Between those in [22] the largest domain of asymptotic stability is bounded, in the $(\mu_3, \alpha^{-1}\mu_2)$-plane, by the curve Q_0P_0 and the straightline OP_0, with:

$$Q_0P_0 : \alpha^{-1}\mu_2 = \frac{2A}{r}\mu_3 + \frac{r^2}{\mu_3} \qquad 0 \le \mu_3 \le \frac{r}{\sqrt{2}}\sqrt{\frac{r}{A}}$$

$$OP_0 :: \alpha^{-1}\mu_2 = 2\sqrt{2rA} \qquad \mu_3 > \frac{r}{\sqrt{2}}\sqrt{\frac{r}{A}}.$$

References

[1] K.L. Kuo. Dynamic instability of two dimensional non divergent flow in a barotropic atmosphere. *J. Meteor.* 6:105-122, 1949.

[2] K.L. Kuo. Dynamics of quasigeostrophic flow and instability theory. *Adv. Appl. Mech.* 13:247-330, 1973.

[3] M.C. Hendershott. *Single layer models of the general circulation.General Circulation of the Ocean*. H.D.I. Abarbanel, W. R. Young Eds.,Springer Verlag, 1986.

[4] J. Pedlosky. *Geophysical Fluid Dynamics.* 2d ed. Springer Verlag, New York, 1987.

[5] J.R. Holton. *An introduction to Dynamical Meteorology*, International Geophysics series, 88, Elsevier, U.S.A., 2004.

[6] C. Vamos, A. Georgescu. Models of asymptotic approximation for synoptic-scale flow.*Z. Meteorol.* 40 (1):14-20,1990.

[7] D. B. Haidvogel, W.R. Holland. The stability of currents in eddy-resolving general circulation models.*J. Physics Oceanogr* 8(3):393-413,1978.

[8] G. Wolansky. Existence, uniqueness and stability of stationary barotropic flow with forcing and dissipation. *Comm. Pure Appl. Math.* 41:19-46, 1988 .

[9] F. Crisciani, R. Mosetti. Linear asymptotic stability of geophysical channeled flows in the presence of arbitrary longitude -shaped perturbations.*Le Matematiche* 46 (1):107-115, 1991 .

[10] H. Stommel. The westward intensification of wind driven ocean currents.*Trans. Am. Geophys. Union* 29 (2):202-206, 1948.

[11] W. H. Munk. On the wind-driven ocean circulation. *J. Meteor.*7:79-93, 1950 .

[12] J.C. Mc Williams , J.H.S. Chow. Equilibrium geostrophic turbulence I: A reference solution in a beta-plane channel. *J.Phys. Oceanogr*11 (7):921-949, 1981.

[13] C. Le Provost, J. Verron. Wind driven ocean circulation: Transition to barotropic instability. *Dyn. Atmos. Oceans* 11 (2):175-201, 1987.

[14] F. Crisciani. Some a priori properties of unstable flows in the homogeneous model of the ocean. *Il Nuovo Cimento C*11 C:105-113, 1988.

[15] F. Crisciani, R. Mosetti. A stability criterion for time- dependent basic zonal flows in the homogeneous model of the ocean. *Il Nuovo Cimento C* 12 C(2):261-268, 1989.

[16] F. Crisciani, R. Mosetti. On the stability of wind-driven flows in oceanic basins. *J. Physic. Oceanogr.* 20 (11):1787-1790, 1990.

[17] F. Crisciani, R. Mosetti. Stability of analytical solutions of Stommel-Veronis ocean circulation models. *J. Phys. Oceanogr.* 24(1):155-158, 1994.

[18] O.A. Ladyzhenskaya. *The Mathematical Theory of Viscous Incompressible Flows*, Gordon and Breach, New York London,1969.

[19] A. Georgescu.*Hydrodynamic Stability Theory*, Martinus Nijhoff Publishers, Kluwer, Dordrecht, 1985.

[20] B. Straughan. *The Energy Method, Stability and Nonlinear Convection.* 2nd. ed. Ser. in Appl. Math. Sci., 91, Springer, New York, 2003.

[21] F. Crisciani, R.Mosetti. Stability criteria independent from the perturbation wave-number for forced zonal flows. *J. Phys. Oceanogr.* 21 (7):1075-1079, 1991.

[22] A.Georgescu, L. Palese. Stability criteria for quasigeostrophic forced zonal flows I. Asymptotically vanishing linear perturbation theory, *ROMAI Journal* 5(1):63-76, 2009.

[23] E.M.Agee, T.S.Chen, K.E.Dowell, A review of mesoscale cellular convection- *Bulletin of the American Meteorological Society.* 54 (10):1004-1012, 1973.