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# APPROXIMATING OF FIXED POINTS FOR MULTI-VALUED GENERALIZED $\alpha-$NONEXPANSIVE MAPPINGS IN BANACH SPACES*

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**Abstract**

In this paper, we study multi-valued generalized $\alpha$nonexpansive mappings in uniformly convex Banach spaces. We introduce a new multi-valued iterative process and prove some weak and strong convergence results in uniformly convex Banach space. We also study the stability of this iteration process. Further, we provide a numerical example of the multi-valued generalized $\alpha-$nonexpansive mapping. Finally, the convergence of this iteration process to the fixed point for multi-valued generalized $\alpha-$nonexpansive mapping is discussed on this numerical example.

**MSC** 47H010; 54H25.

**keywords:** Generalized $\alpha-$nonexpansive mappings, multi-valued mapping, fixed point, iteration process, uniformly convex Banach spaces.

## 1 Introduction and Preliminaries

Some generalizations of single-valued nonexpansive and the study of related fixed point theorems have been intensively carried out by many authors over

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past decades (see [2], [6], [15], [22]). In 2008, Suzuki [22] defined a class of generalized nonexpansive mappings on a nonempty subset $M$ of a Banach space $X$. Such type of mappings was called the class of mappings satisfying the Condition $(C)$ (also referred as Suzuki generalized nonexpansive mapping), which properly includes the class of nonexpansive mappings. In 2017, Pant and Shukla [15] introduced the following class of nonexpansive type mappings and obtained some fixed point results for this class of mappings.

A mapping $T : M \to M$ is called a generalized $\alpha$-nonexpansive mapping if there exists an $\alpha \in [0, 1)$ and for each $x, y \in M$,

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \quad \text{implies}$$
$$\|Tx - Ty\| \leq \alpha\|x - Ty\| + \alpha\|y - Tx\| + (1 - 2\alpha)\|x - y\|.$$

There are many authors studied with some different iteration processes to find of approximating fixed points of for Suzuki generalized nonexpansive mapping and generalized $\alpha$-nonexpansive mappings (see [5], [8], [16], [23], [24], [26], [27], [29] and so on).

Fixed point theory for multi-valued mappings has many useful applications in various fields, control theory, convex optimization, game theory and mathematical economics. Therefore, it is natural to extend the known fixed point results for single-valued mappings to the setting of multi-valued mappings. The theory of multi-valued nonexpansive mappings is more difficult than the corresponding theory of single-valued nonexpansive mappings. The convergence of a sequence of fixed points of a convergent sequence of set-valued contractions was investigated by [12] and [13]. In the recent years, fixed point theory for multi-valued mappings has been studied by many authors; see [1],[7], [10], [17], [20], [21], [25], [28] and the references therein.

Firstly we give some basic concepts about the multi-valued mappings.

We assume throughout this paper that $(X, \|.\|)$ is a Banach space and $M$ is a nonempty subset of $X$. The set $M$ is called proximinal if for each $x \in X$ , there exists some $y \in M$ such that $d(x, y) = d(x, M)$, where $d(x, M) = inf\,\{d(x, y) : y \in M\}$. In the sequel, the notations $\mathcal{P}_{px}(M), \mathcal{P}_{cb}(M), \mathcal{P}_{cp}(M)$ and $\mathcal{P}(M)$ will denote the families of nonempty proximinal subsets, closed and bounded subsets, compact subsets and all subsets of $M$ , respectively. Let $H(,)$ be the Hausdorff distance on $\mathcal{P}_{cb}(M)$ is defined by

$H(A, B) = max\big\{sup_{x\in A}d(x, B), sup_{y\in B}d(y, A), \forall A, B \in \mathcal{P}_{cb}(M), x \in A, y \in B\big\}$.

Let $T : M \to \mathcal{P}(M)$ be a multivalued mapping. An element $p \in M$ is said to be a fixed point of $T$, if $p \in T(p)$. The set of fixed points of $T$ will be denoted by $F(T)$. A multivalued mapping $T : M \to \mathcal{P}(M)$ is said to be contraction mapping if there exists an $\theta \in [0,1)$ such that $H(Tx,Ty) \leq \theta\|x-y\|$, for all $x,y \in M$, nonexpansive mapping if $H(Tx,Ty) \leq \|x-y\|$, for all $x,y \in M$ , quasi-nonexpansive mapping if $F(T) \neq \emptyset$ and $H(Tx,Tp)) \leq \|x-p\|$, for all $x \in M$ and for all $p \in F(T)$. It is well known that if $M$ is a nonempty closed, bounded and convex subset of a uniformly convex Banach space $X$, then a multivalued nonexpansive mapping $T: M \to \mathcal{P}(M)$ has a fixed point [11]. Iterative techniques for approximating fixed points of nonexpansive multi-valued mappings have been investigated by various authors using the Mann iteration scheme or the Ishikawa iteration scheme (see [17], [20], [21] and so on ). In 2009, Shahzad and Zegeye [20] presented the set $P_T(x) = \{y \in Tx : d(x,Tx) = \|x-y\|\}$ for a multivalued mapping, $T: M \to \mathcal{P}(M)$ and showed that Mann and the Ishikawa iteration processes for multi-valued mappings are well defined. They proved the convergence of these iteration processes for multivalued mappings in a uniformly convex Banach space. In 2011, Abkar and Eslamian [1] extended the notion of Condition $(C)$ to the case of multi-valued mappings.

In 2020, Iqbal et al.[10] introduced a new modified iteration process to approximate fixed points of multi-valued generalized $\alpha-$nonexpansive mappings as follows: for arbitrary $x_1 = x \in M$ construct a sequence $\{x_n\}$ by

$$\begin{cases} v_n = (1-b_n)x_n + b_n\tau_n, \\ w_n \in P_T v_n, \\ x_{n+1} = (1-a_n)\kappa_n + a_n\lambda_n, \forall n \in \mathbb{N}, \end{cases} \tag{1}$$

where $\{a_n\}, \{b_n\} \in (0,1), \kappa_n \in P_T(w_n), \tau_n \in P_T(x_n)$ and $\lambda_n \in P_T(\kappa_n)$.

In 2021, Ullah et al.[28] studied convergence results of M-iterative process for multi-valued generalized $\alpha-$nonexpansive mappings. M-iterative process for multi-valued mappings as follows: for arbitrary $x_1 = x \in M$ construct a sequence $\{x_n\}$ by

$$\begin{cases} v_n = (1-a_n)x_n + a_n\tau_n, \\ w_n = \tau_n^{'}, \\ x_{n+1} = \tau_n^{''}, \forall n \in \mathbb{N}, \end{cases} \tag{2}$$

where $\{a_n\} \in (0,1), \tau_n \in P_T(x_n), \tau_n^{'} \in P_T v_n, \tau_n^{''} \in P_T w_n$.

Motivated by above, we introduce a new iteration process for multi-valued mappings as follows: for arbitrary $x_1 \in M$ construct a sequence $\{x_n\}$

by

$$\begin{cases} v_n = (1-b_n)x_n + b_n\tau_n, \\ s_n \in P_T(v_n), \\ y_n \in P_T(s_n), \\ w_n = (1-a_n)u_n + a_n t_n \\ x_{n+1} = \eta_n, \forall n \in \mathbb{N}, \end{cases} \tag{3}$$

where $\{a_n\}$ and $\{b_n\} \in (0,1)$, $\tau_n \in P_T(x_n)$, $\eta_n \in P_T(w_n)$, $t_n \in P_T(u_n)$, $u_n \in P_T(y_n)$.

In this paper, we prove some weak and strong convergence results using (3)-iteration process for multi-valued generalized $\alpha-$nonexpansive mappings in uniformly convex Banach spaces. Moreover, we present an illustrative numerical example of approximating fixed point of multi-valued generalized $\alpha-$nonexpansive mappings considering the iteration process presented in (3).

Now we recall some notations to be used in main results:

A Banach space $X$ is said to satisfy *Opial's condition* [14] if, for each sequence $\{x_n\}$ in $X$, the condition $x_n \to x$ converges weakly as $n \to \infty$ and for all $y \in X$ with $y \neq x$ imply that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|.$$

In the following we shall give some preliminaries on the concepts of asymptotic radius and asymptotic center which are due to [4].

Let $\{x_n\}$ be a bounded sequence in a Banach space $X$. Then

(i) *The asymptotic radius of* $\{x_n\}$ *at point* $x \in X$ *is the number*

$$r(x, \{x_n\}) = \limsup_{n\to\infty} \|x_n - x\|.$$

(ii) *The asymptotic radius of* $\{x_n\}$ *relative to* $M$ is defined by

$$r(M, \{x_n\}) = \inf\{r(x, \{u_n\}) : x \in M\}.$$

(iii) *The asymptotic center of* $\{x_n\}$ *relative to* $M$ is the set

$$A(M, \{x_n\}) = \{x \in M : r(x, \{x_n\}) = r(M, \{x_n\})\}.$$

It is well known that, in uniformly convex Banach space, $A(M, \{x_n\})$ consists of exactly one-point.

**Lemma 1.** *([18]) Suppose that $X$ is a uniformly convex Banach space and $0 < k \leq t_n \leq m < 1$ for all $n \in \mathbb{N}$. Let $\{x_n\}$ and $\{x_n\}$ be two sequence of $X$ such that $\limsup_{n\to\infty} \|x_n\| \leq c$, $\limsup_{n\to\infty} \|y_n\| \leq c$ and $\limsup_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = c$ hold for $c \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

**Definition 1.** *Let $T : M \to \mathcal{P}_{cb}(M)$. A sequence $\{x_n\}$ in $M$ is called an approximate fixed point sequence (or a.f.p.s) for $T$ provided that $d(x_n, Tx_n) \to 0$ as $n \to \infty$.*

**Definition 2.** *A multivalued mapping $T : M \to \mathcal{P}(M)$ is called demiclosed at $y \in M$ if for any sequence $\{x_n\}$ in $M$ weakly convergent to $x$ and $y_n \in Tx_n$ strongly convergent to $y$, we have $y \in Tx$.*

The following is the multi-valued version of Condition $(I)$ of Senter and Dotson [19].

**Definition 3.** *A multivalued mapping $T : M \to \mathcal{P}(M)$ is said to satisfy* Condition (I), *if there is a nondecreasing function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ and $\varphi(r) > 0$ for all $r \in (0, \infty)$ such that $d(x, Tx)\| \geq \varphi(d(x, F(T))$ for all $x \in M$ .*

**Lemma 2.** *([21]) Let $T : M \to \mathcal{P}_{px}(M)$ and $P_T(x) = \{y \in Tx : d(x, Tx) = \|x - y\|\}$. Then the following are equivalent.*

*(1) $x \in F(T)$.*

*(2) $P_T(x) = \{x\}$.*

*(3) $x \in F(P_T)$.*

*Moreover, $F(T) = F(P_T)$.*

Now we give the definition of multi-valued generalized $\alpha$-nonexpansive mapping:

**Definition 4.** *Let $M$ be a nonempty subset of a Banach space $X$.A mapping $T : M \to \mathcal{P}_{cb}(M)$ is called a multi-valued generalized $\alpha$-nonexpansive if there exists an $\alpha \in [0, 1)$ such that for each $x, y \in M$,*

$$\frac{1}{2}d(x, Tx) \leq \|x - y\| \quad \textit{implies}$$
$$H(Tx, Ty) \leq \alpha d(y, Tx) + \alpha d(x, Ty) + (1 - 2\alpha)\|x - y\|.$$

It is well known that every multivalued mapping satisfying Condition (C) is a multivalued generalized 0 nonexpansive mapping. In comparison to nonexpansive mappingsand mappings satisfying Condition (C), the class of generalized $\alpha-$nonexpansive mappings is larger. We now provide an example of a multivalued mapping T that neither satisfies Condition (C) nor is nonexpansive.

**Example 1.** *Let $(\mathbb{R}^2, \|.\|)$ be a normed space with $\ell_2$-norm and $M = [0,6] \times [0,6]$. Define $T : M \to \mathcal{P}(M)$ by*

$$T\bar{x} = \begin{cases} \{(0,0)\}, & \bar{x} \neq (6,6) \\ \left[\frac{1}{2}, 3\right] \times \left[\frac{1}{2}, 3\right], & \bar{x} = (6,6) \end{cases}$$

*Then, we consider the following cases:*

***Case I:*** *If $\bar{x}, \bar{y} \neq (6,6)$, then*

$$H(P_T(\bar{x}), P_T(\bar{y})) = 0 \leq \frac{1}{2} d(\bar{y}, P_T(\bar{x})) + \frac{1}{2} d(\bar{x}, P_T(\bar{y})).$$

***Case II:*** *If $\bar{x} \neq (6,6)$ and $\bar{y} = (6,6)$ then we have*

$$H(P_T(\bar{x}), P_T(\bar{y})) = H(\{(0,0)\}, \left[\frac{1}{2}, 3\right] \times \left[\frac{1}{2}, 3\right]) = \sqrt{18}.$$

*Thus we have*

$$\frac{1}{2} d(\bar{y}, P_T(\bar{x})) + \frac{1}{2} d(\bar{x}, P_T(\bar{y})) \geq \frac{1}{2}\sqrt{72} = \sqrt{18} = H(P_T(\bar{x}), P_T(\bar{y}))$$

.

***Case III:*** *If $\bar{x}, \bar{y} = (6,6)$ then we have*

$$\begin{aligned} H(P_T(\bar{x}), P_T(\bar{y})) &= H(P_T(\left[\frac{1}{2}, 3\right] \times \left[\frac{1}{2}, 3\right], \left[\frac{1}{2}, 3\right] \times \left[\frac{1}{2}, 3\right]) = 0 \\ &\leq \frac{1}{2} d(\bar{y}, P_T(\bar{x})) + \frac{1}{2} d(\bar{x}, P_T(\bar{y})). \end{aligned}$$

*Therefore, $T$ is a generalized $\frac{1}{2}$-nonexpansive mapping. Observe that $T$ is neither nonexpansive nor satisfies Condition $(C)$ as for $\bar{x} = (3.1, 3.1)$ and $\bar{y} = (6,6)$, we have*

$$\frac{1}{2} d(\bar{x}, P_T(\bar{x})) = \frac{1}{2}\sqrt{19.22} < \sqrt{16.82} = \|(3.1, 3.1) - (6,6)\| = \|\bar{x} - \bar{y}\|$$

*which implies that*

$$H(P_T(\bar{x}), P_T(\bar{y})) = \sqrt{18} > \sqrt{16.82} = \|\bar{x} - \bar{y}\|.$$

**Lemma 3.** *([10]) Let $M$ be a nonempty subset of a Banach space and $T : M \to \mathcal{P}(M)$. If $T$ is generalized $\alpha$-nonexpansive, then $T$ is quasi-nonexpansive.*

**Lemma 4.** *([10]) Let $M$ be a nonempty subset of a Banach space $X$ and $T : M \to \mathcal{P}_{cb}(M)$ be a generalized $\alpha$-nonexpansive. Then*

$$d(x, Ty) = \left(\frac{3+\alpha}{1-\alpha}\right) d(x, Tx) + \|x - y\|$$

*for each $x, y \in M$.*

## 2 Convergence of multi-valued generalized $\alpha$-nonexpansive mappings

In this section, we prove weak and strong convergence theorems for (3)-iterative process of multi-valued generalized $\alpha$-nonexpansive mappings in uniformly convex Banach space.

**Lemma 5.** *Let $M$ be a nonempty closed convex subset of a uniformly convex Banach space $X$. Let $T : M \to \mathcal{P}_{px}(M)$ be a multi-valued mapping such that $F(T) \neq \emptyset$ and $P_T$ is a generalized $\alpha-$nonexpansive mapping. Let $\{x_n\}$ be a sequence generated by (3). Then $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(T)$.*

*Proof.* Let $p \in F(T)$. By Lemma 2, $P_T(p) = \{p\}$ and $F(T) = F(P_T)$. Since $P_T$ is a generalized $\alpha-$nonexpansive mapping, by Lemma 3, then $P_T$ is a quasi-nonexpansive mapping. Now, for any $p \in F(T)$, we have

$$\begin{aligned}
H(P_T(\tau_n), P_T(p)) &\leq \|\tau_n - p\|, \qquad H(P_T(v_n), P_T(p)) \leq \|v_n - p\|,\\
H(P_T(s_n), P_T(p)) &\leq \|s_n - p\|, \qquad H(P_T(y_n), P_T(p)) \leq \|y_n - p\|,\\
H(P_T(w_n), P_T(p)) &\leq \|w_n - p\|, \qquad H(P_T(u_n), P_T(p)) \leq \|u_n - p\|,\\
H(P_T(t_n), P_T(p)) &\leq \|t_n - p\|, \qquad H(P_T(\eta_n), P_T(p)) \leq \|\eta_n - p\|.
\end{aligned}$$

Next by (3), we have

$$\begin{aligned}
\|\tau_n - p\| &\leq d(\tau_n, P_T(p)) \leq H(P_T(x_n), P_T(p)) \\
&\leq \|x_n - p\|.
\end{aligned} \tag{4}$$

By (4), we have

$$\begin{aligned}\|v_n - p\| &= \|(1-b_n)x_n + b_n\tau_n - p\| \\ &\leq (1-b_n)\|x_n - p\| + b_n\|\tau_n - p\| \\ &\leq (1-b_n)\|x_n - p\| + b_n d(\tau_n, P_T(p)) \\ &\leq (1-b_n)\|x_n - p\| + b_n H(P_T(x_n), P_T(p)) \\ &\leq (1-b_n)\|x_n - p\| + b_n\|x_n - p\| = \|x_n - p\| \qquad (5)\end{aligned}$$

and also we have

$$\begin{aligned}\|s_n - p\| &\leq d(s_n, P_T(p)) \leq H(P_T(v_n), P_T(p)) \\ &\leq \|v_n - p\|. \qquad (6)\end{aligned}$$

By (4), (5) and (6), we have

$$\begin{aligned}\|y_n - p\| &\leq d(y_n, P_T(p)) \leq H(P_T(s_n), P_T(p)) \\ &\leq \|s_n - p\|. \qquad (7)\end{aligned}$$

By (4)-(7), we have

$$\begin{aligned}\|w_n - p\| &= \|(1-a_n)u_n + a_n t_n - p\| \\ &\leq (1-a_n)\|u_n - p\| + a_n\|t_n - p\| \\ &\leq (1-a_n)d(u_n, P_T(p)) + a_n d(t_n, P_T(p)) \\ &\leq (1-a_n)H(P_T(y_n), P_T(p)) + a_n H(P_T(u_n), P_T(p)) \\ &\leq (1-a_n)\|y_n - p\| + a_n\|u_n - p\| \\ &\leq (1-a_n)\|s_n - p\| + a_n d(u_n, P_T(p)) \\ &\leq (1-a_n)\|s_n - p\| + a_n H(P_T(y_n), P_T(p)) \\ &\leq (1-a_n)\|s_n - p\| + a_n\|y_n - p\| \\ &\leq (1-a_n)\|v_n - p\| + a_n\|s_n - p\| \\ &\leq (1-a_n)\|x_n - p\| + a_n\|v_n - p\| \\ &\leq (1-a_n)\|x_n - p\| + a_n\|x_n - p\| = \|x_n - p\|. \qquad (8)\end{aligned}$$

By (8), we have

$$\begin{aligned}\|x_{n+1} - p\| &= \|\eta_n - p\| \leq d(\eta_n, P_T(p)) \\ &\leq H(P_T(w_n), P_T(p)) \leq \|w_n - p\| \\ &\leq \|x_n - p\|. \qquad (9)\end{aligned}$$

This implies that $\{\|x_n - p\|\}$ is bounded and non-increasing for all $p \in F(T)$. It follows that $\lim_{n\to\infty}\|x_n - p\|$ exists. □

**Theorem 1.** *Let $M$ be a nonempty closed convex subset of a uniformly convex Banach space $X$. Let $T : M \to \mathcal{P}_{px}(M)$ be a multi-valued mapping and $P_T$ is a generalized $\alpha-$nonexpansive mapping. Let $\{x_n\}$ be a sequence generated by (3). Then $F(T) \neq \emptyset$ if and only if $\{x_n\}$ is bounded and* $\lim_{n\to\infty} \|x_n - \tau_n\| = 0$.

*Proof.* Suppose $F(T) \neq \emptyset$ and let $p \in F(T)$. By Lemma 5, $\lim_{n\to\infty} \|x_n - p\|$ exists and $\{x_n\}$ is bounded. Put

$$\lim_{n\to\infty} \|x_n - p\| = c. \tag{10}$$

From (5)-(8), we have

$$\limsup_{n\to\infty} \|v_n - p\| \leq \limsup_{n\to\infty} \|x_n - p\| \leq c,$$

$$\limsup_{n\to\infty} \|\tau_n - p\| \leq \limsup_{n\to\infty} \|x_n - p\| \leq c. \tag{11}$$

Also

$$\limsup_{n\to\infty} \|s_n - p\| \leq \limsup_{n\to\infty} \|x_n - p\| \leq c,$$

$$\limsup_{n\to\infty} \|y_n - p\| \leq \limsup_{n\to\infty} \|x_n - p\| \leq c,$$

$$\limsup_{n\to\infty} \|w_n - p\| \leq \limsup_{n\to\infty} \|x_n - p\| \leq c. \tag{12}$$

Further, we have the following inequalities

$$\|u_n - p\| \leq H(P_T(y_n), P_T(p)) \leq \|y_n - p\|$$

and

$$\|t_n - p\| \leq H(P_T(u_n), P_T(p)) \leq \|u_n - p\|.$$

On taking $\limsup_{n\to\infty}$ on both sides of the all above inequalities, we obtain that

$$\limsup_{n\to\infty} \|u_n - p\| \leq c, \tag{13}$$

$$\limsup_{n\to\infty} \|t_n - p\| \leq c \tag{14}$$

Next

$$\begin{aligned}\|x_{n+1}-p\| &= \|\eta_n - p\| = H(P_T(w_n), P_T(p)) \\ &\leq \|w_n - p\|\end{aligned}$$

Making $n \longrightarrow \infty$ above inequality, we get

$$c = \limsup_{n\to\infty} \|x_{n+1}-p\| \leq \limsup_{n\to\infty} \|w_n - p\|.$$

Thus by from (12) we have $\limsup\limits_{n\to\infty} \|w_n - p\| = c$. So

$$\begin{aligned}c &= \limsup_{n\to\infty} \|w_n - p\| \\ &= \limsup_{n\to\infty} \|(1-a_n)(u_n - p) + a_n(t_n - p)\|\end{aligned}$$

By Lemma 1, we have

$$\limsup_{n\to\infty} \|t_n - u_n\| = 0. \tag{15}$$

Now

$$\begin{aligned}\|w_n - p\| &= \|((1-a_n)u_n + a_n t_n - p\| \\ &= \|(u_n - p) + a_n(u_n - t_n)\| \\ &\leq \|u_n - p\| + a_n\|u_n - t_n\|.\end{aligned}$$

Making $n \longrightarrow \infty$ above inequality and from (15) we get

$$c = \limsup_{n\to\infty} \|w_n - p\| \leq \limsup_{n\to\infty} \|u_n - p\|.$$

So by (13) we have

$$\limsup_{n\to\infty} \|u_n - p\| = c.$$

Then

$$\|u_n - p\| \leq \|u_n - t_n\| + \|t_n - p\|.$$

Making $n \longrightarrow \infty$ and from (15), we get

$$c \leq \limsup_{n\to\infty} \|t_n - p\|.$$

Hence together with (14) we have

$$c = \lim_{n\to\infty} \|t_n - p\|.$$

Thus

$$\begin{aligned}
c &= \lim_{n\to\infty} \|t_n - p\| \le \lim_{n\to\infty} H(P_T(u_n), P_T(p)) \\
&\le \lim_{n\to\infty} \|u_n - p\| \le \lim_{n\to\infty} H(P_T(y_n), P_T(p)) \\
&\le \lim_{n\to\infty} \|y_n - p\| \le \lim_{n\to\infty} H(P_T(s_n), P_T(p)) \\
&\le \lim_{n\to\infty} \|s_n - p\| \le \lim_{n\to\infty} H(P_T(v_n), P_T(p)) \le \lim_{n\to\infty} \|v_n - p\| \\
&\le \lim_{n\to\infty} \|(1-b_n)x_n + b_n\tau_n - p\| \\
&\le \lim_{n\to\infty} (1-b_n)\|x_n - p\| + b_n\|\tau_n - p\| \le c.
\end{aligned}$$

Consequently

$$\lim_{n\to\infty} \|(1-b_n)(x_n - p) + b_n(\tau_n - p)\| = c. \tag{16}$$

Thus from (10), (11), (16) and by Lemma 1 we have $\lim_{n\to\infty} \|x_n - \tau_n\| = 0$ which implies that $\lim_{n\to\infty} d(x_n, P_T(x_n)) = 0$.

Conversely, suppose that $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(x_n, P_T(x_n)) = 0$. Let $p \in A(K, \{x_n\})$. By Lemma 4, we have

$$d(x_n, P_T(p)) = \left(\frac{3+\alpha}{1-\alpha}\right) d(x_n, P_T(x_n)) + \|x_n - p\|.$$

Using the definition of asymptotic center we have

$$\begin{aligned}
r(Tp, \{x_n\}) &= \limsup_{n\to\infty} d(x_n, P_T(p)) \\
&\le (\frac{3+\alpha}{1-\alpha}) \limsup_{n\to\infty} d(P_T(x_n), x_n)) + \limsup_{n\to\infty} \|x_n - p\| \\
&= \limsup_{n\to\infty} \|x_n - p\| = r(p, \{x_n\}).
\end{aligned}$$

This implies that for $Tp = p \in A(K, \{x_n\})$. Since $X$ is uniformly Banach space, $A(K, \{x_n\})$ is consists of a unique element. Thus, we have $Tp = p$. Hence $F(T) \neq \emptyset$.

□

In the next result, we prove our strong convergence theorems as follows.

**Theorem 2.** *Let $M$ be a nonempty compact convex subset of a uniformly convex Banach space $X$. Let $T : M \to \mathcal{P}_{px}(M)$ be a multi-valued mapping such that $F(T) \neq \emptyset$ and $P_T$ is a generalized $\alpha-$nonexpansive mapping. Let $\{x_n\}$ be a sequence generated by (3). Then $\{x_n\}$ converges strongly to a fixed point of $T$.*

*Proof.* $F(T) \neq \emptyset$, so by Theorem 1, we have $\lim_{n\to\infty} d(x_n, P_T(x_n)) = 0$. Since $M$ is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \longrightarrow q$ as $k \to \infty$ for some $q \in M$. Because $P_T$ is a generalized $\alpha-$nonexpansive mapping, one can find some real constant $\rho = (\frac{3+\alpha}{1-\alpha}) \geq 1$, such that

$$d(x_{n_k}, P_T(q)) \leq \rho d(x_{n_k}, P_T(x_{n_k}) + \|x_{n_k} - q\|.$$

As $F(T) = F(P_T)$, on taking limit as $k \to \infty$, we get $q \in P_T(q)$ i.e. $q \in F(T)$. So $\{x_n\}$ converges strongly to a fixed point of $T$. □

The proof of the following result is elementary and hence omitted.

**Theorem 3.** *Let $M$ be a nonempty closed convex subset of a uniformly convex Banach space $X$. Let $T : M \to \mathcal{P}_{px}(M)$ be a multi-valued mapping such that $P_T$ is a generalized $\alpha-$nonexpansive mapping. $\{x_n\}$ be a sequence generated by (3). If $F(T) \neq \emptyset$ and $\lim_{n\to\infty} d(x_n, F(T)) = 0$, then $\{x_n\}$ converges strongly to a fixed point of $T$.*

**Theorem 4.** *Let $M$ be a nonempty closed convex subset of a uniformly convex Banach space $X$. Let $T : M \to \mathcal{P}_{px}(M)$ be a multi-valued mapping satisfying Condition $(I)$ such that $F(T) \neq \emptyset$. $\{x_n\}$ be a sequence generated by (3). If $P_T$ is a generalized $\alpha-$nonexpansive mapping, then $\{x_n\}$ converges strongly to a fixed point of $T$.*

*Proof.* By Lemma 5, we have $\lim_{n\to\infty} \|x_n - p\|$ exists and for all $p \in F(T)$. Put $c = \lim_{n\to\infty} \|x_n - p\|$ for some $c \geq 0$. If $c = 0$ then the result follows. Suppose that $c > 0$. Then

$$\lim_{n\to\infty} \|x_{n+1} - p\| \leq \lim_{n\to\infty} \|x_n - p\|.$$

It follows that

$$\lim_{n\to\infty} d(x_{n+1}, F(T)) \leq \lim_{n\to\infty} d(x_n, F(T)).$$

$\lim_{n\to\infty} d(x_n, F(T))$ exists. We show that it follows $\lim_{n\to\infty} d(x_n, F(T)) = 0$. From Theorem 1, $\lim_{n\to\infty} d(x_n, P_T(x_n)) = 0$. As $F(T) = F(P_T)$, by Theorem 1 and

Condition (I) we have $0 \leq \lim_{n\to\infty} \varphi(d(x_n, F(T))) \leq \lim_{n\to\infty} d(x_n, P_T(x_n)) = 0$. That is, $\lim_{n\to\infty} \varphi(d(x_n, F(T))) = 0$. Since $\varphi : [0,\infty) \to [0,\infty)$ is a nondecreasing function satisfying $\varphi(0) = 0$ and $\varphi(r) > 0$ for all $r \in (0,\infty)$, we have $\lim_{n\to\infty} d(x_n, F(T)) = 0$. All the conditions of Theorem 3 are satisfied, therefore by its conclusion $\{x_n\}$ converges strongly to a fixed point of $T$. The proof is completed. □

Finally, we prove the weak convergence of the iterative process (3) for multi-valued generalized $\alpha-$nonexpansive mapping in a uniformly convex Banach space satisfying Opial's condition.

**Theorem 5.** *Let $X$ be a real uniformly convex Banach space satisfying Opial's condition and $M$ be a nonempty closed convex subset of $X$. Let $T : M \to \mathcal{P}_{px}(M)$ be a multi-valued mapping such that $F(T) \neq \emptyset$. Suppose $P_T$ is a generalized $\alpha-$nonexpansive mapping and $I - P_T$ is demi-closed with respect to zero. Then $\{x_n\}$ defined by (3) converges weakly to a fixed point of $T$.*

*Proof.* Let $p \in F(T) = F(P_T)$. By Lemma 5, the sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(T)$. Since $X$ is uniformly convex, $X$ is reflexive. By the reflexiveness of $X$ , there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to some $\sigma_1 \in M$. Since $I - P_T$ is demi-closed with respect to zero, $\sigma_1 \in F(P_T) = F(T)$. We prove that $\sigma_1$ is the unique weak limit of $\{x_n\}$. Let one can find another weakly convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with weak limit say $\sigma_2 \in M$ and $\sigma_2 \neq \sigma_1$. Again $\sigma_1 \in F(P_T) = F(T)$. From the Opial's property and Lemma 5, we obtain

$$\begin{aligned} \lim_{n\to\infty} \|x_n - \sigma_1\| &= \lim_{j\to\infty} \|x_{n_j} - \sigma_1\| < \lim_{j\to\infty} \|x_{n_j} - \sigma_2\| = \lim_{n\to\infty} \|x_n - \sigma_2\| \\ &= \lim_{k\to\infty} \|x_{n_k} - \sigma_2\| < \lim_{k\to\infty} \|x_{n_k} - \sigma_1\| = \lim_{n\to\infty} \|x_n - \sigma_1\|, \end{aligned}$$

which is a contradiction. So, $\sigma_1 = \sigma_2$. Therefore $\{x_n\}$ converges weakly to a fixed point of $T$. This completes the proof. □

## 3 Stability for new iterative process

In this section, we analyze the stability of the (3)-iteration process with respect to multivalued contraction mapping. Also we prove that the (3)-iteration process is stable with respect to $T$.

In what follows, we shall make use of the following well-known lemma.

**Lemma 6.** *[30] Let $\{\epsilon_n\}_{n=0}^{\infty}$ and $\{d_n\}_{n=0}^{\infty}$ be nonnegative real sequences satisfying the inequality*

$$d_{n+1} \leq (1-\mu_n)d_n + \epsilon_n, \tag{17}$$

*where $\mu_n \in (0,1)$, for $n = 0,1,2...$, $\sum_{n=0}^{\infty} \mu_n = \infty$ and $\lim_{n\to\infty} \frac{\epsilon_n}{\mu_n} = 0$, then $\lim_{n\to\infty} d_n = 0$.*

Let $T : M \to \mathcal{P}(M)$ be a multi-valued mapping. Harder and Hicks [9] introduced the following concept of $(T)$-stability (see also [3]). Define a fixed point iteration process by $x_{n+1} = f(T, x_n)$, for $n = 0,1,2,...$ such that $x_n$ converges to fixed point $p$ of $T$. Let $\{\upsilon_n\}_{n=0}^{\infty}$ be an arbitrary sequence in $M$ and set $\epsilon_n = \|\upsilon_{n+1} - f(T,\upsilon_n)\|$ for $n = 0,1,2,....$ We shall say that the fixed point iteration process is T-stable or stable with respect to T if and only if $\lim_{n\to\infty} \epsilon_n = 0 \Leftrightarrow \lim_{n\to\infty} \upsilon_n = p$.

**Theorem 6.** *Let $M$ be a nonempty closed convex subset of a uniformly convex Banach space $X$. Let $T : M \to \mathcal{P}_{px}(M)$ be a multi-valued mapping and $P_T$ is multi-valued contraction mapping with $\theta \in (0,1)$. Let $\{x_n\}$ be a sequence generated by (3), where $\{a_n\}, \{b_n\} \in (0,1)$ and $\sum_{n=0}^{\infty} \mu_n = \infty$. Then $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(T)$. Then iteration process (3) is $(T)$-stable.*

*Proof.* The existence of the fixed point of $P_T$ is guaranteed by Nadlers generalization of Banach contraction principle [13]. Now, we show that $\{x_n\}$ converges to some fixed point p (say). It follows from (3), we have,

$$\begin{aligned}
\|\upsilon_n - p\| &= \|(1-b_n)x_n + b_n\tau_n - p\| \\
&\leq (1-b_n)\|x_n - p\| + b_n\|\tau_n - p\| \\
&\leq (1-b_n)\|x_n - p\| + b_n d(\tau_n, P_T(p)) \\
&\leq (1-b_n)\|x_n - p\| + b_n H(P_T(x_n), P_T(p)) \\
&\leq \theta[(1-b_n)\|x_n - p\| + b_n\theta\|x_n - p\|] \\
&= [1 - b_n(1-\theta)]\|x_n - p\|,
\end{aligned} \tag{18}$$

$$\begin{aligned}
\|s_n - p\| &\leq d(s_n, P_T(p)) \leq H(P_T(\upsilon_n), P_T(p)) \\
&\leq \theta\|\upsilon_n - p\|.
\end{aligned} \tag{19}$$

and also we have

$$\begin{aligned}
\|y_n - p\| &\leq d(y_n, P_T(p)) \leq H(P_T(s_n), P_T(p)) \\
&\leq \theta\|s_n - p\|.
\end{aligned} \tag{20}$$

From the inequalities (18), (19) and (20), since $1-b_n(1-\theta)<1$, $\{b_n\}\in(0,1)$ and $\theta\in(0,1)$ then we get

$$\begin{aligned}
\|w_n-p\| &= \|(1-a_n)u_n+a_nt_n-p\| \\
&\leq (1-a_n)\|u_n-p\|+a_n\|t_n-p\| \\
&\leq (1-a_n)d(u_n,P_T(p))+a_nd(t_n,P_T(p)) \\
&\leq (1-a_n)H(P_T(y_n),P_T(p))+a_nH(P_T(u_n),P_T(p)) \\
&\leq (1-a_n)\theta\|y_n-p\|+a_n\theta\|u_n-p\| \\
&\leq (1-a_n)\theta^2\|s_n-p\|+a_n\theta d(u_n,P_T(p)) \\
&\leq (1-a_n)\theta^2\|s_n-p\|+a_n\theta H(P_T(y_n),P_T(p)) \\
&\leq (1-a_n)\theta^2\|s_n-p\|+a_n\theta^2\|y_n-p\| \\
&\leq (1-a_n)\theta^3\|v_n-p\|+a_n\theta^3\|s_n-p\| \\
&\leq (1-a_n)\theta^3[1-b_n(1-\theta)]\|x_n-p\|+a_n\theta^4\|v_n-p\| \\
&\leq (1-a_n)\theta^3[1-b_n(1-\theta)]\|x_n-p\|+a_n\theta^4[1-b_n(1-\theta)]\|x_n-p\| \\
&\leq \theta^3[1-b_n(1-\theta)][1-a_n(1-\theta)]\|x_n-p\| \\
&\leq \theta^3[1-a_n(1-\theta)]\|x_n-p\| \qquad (21)
\end{aligned}$$

From (3) and (21) , we get

$$\begin{aligned}
\|x_{n+1}-p\| &= \|\eta_n-p\|\leq d(\eta_n,P_T(p)) \\
&\leq H(P_T(w_n),P_T(p))\leq\theta\|w_n-p\| \\
&\leq \theta^4[1-a_n(1-\theta)]\|x_n-p\|. \qquad (22)
\end{aligned}$$

By repeating the above process (22), we get,

$$\begin{aligned}
\|x_{n+1}-p\| &\leq \theta^4[1-a_n(1-\theta)]\|x_n-p\| \\
&\;\;. \\
&\;\;. \\
&\;\;. \\
\|x_1-p\| &\leq \theta^4[1-a_0(1-\theta)]\|x_0-p\|.
\end{aligned}$$

Therefore, we obtain

$$\|x_{n+1}-p\| \;\leq\; \theta^{4(n+1)}\prod_{k=0}^{n}[1-a_k(1-\theta)]\|x_0-p\|,$$

$\theta<1$ and $1-\theta<1$ so $(1-\theta)>0$ and $a_k<1$ for $k=0,1,2,....$ Then we have $[1-a_k(1-\theta)]<1$ for $k=0,1,2,....$ So, we know that $1-x\leq e^{-x}$ for all $x \in [0,1]$. Hence we have

$$\|x_{n+1} - p\| \leq \theta^{4(n+1)} e^{-(1-\theta)\sum_{k=0}^{n}[a_k]} \|x_0 - p\|. \tag{23}$$

We can see that $\{x_n\}$ converges to a fixed point of $P_T$ . Since $p \in F(P_T)$, by Lemma 2, we have $p \in F(T)$ and hence $x_n \to p \in F(T)$.

Now we prove that the new iteration defined by (3) is stable with respect to $(T)$.

Let $\{\upsilon_n\}$ be any arbitrary sequence in $M$. Assume that the sequence generated by (3) is $\upsilon_{n+1} = f(T, \upsilon_n)$ converging to a fixed point of $T$. Define $\epsilon_n = \|\upsilon_{n+1} - f(T, \upsilon_n)\|$ for $n = 0, 1, 2, ....$

We have to prove that $\lim\limits_{n\to\infty} \epsilon_n = 0 \Leftrightarrow \lim\limits_{n\to\infty} \upsilon_n = p$.

Suppose $\lim\limits_{n\to\infty} \epsilon_n = 0$. By using (3) and (23) we get

$$\begin{aligned} \|\upsilon_{n+1} - p\| &\leq \|\upsilon_{n+1} - f(T, \upsilon_n)\| + \|f(T, \upsilon_n) - p\| \\ &\leq \epsilon_n + \theta^4[(1 - a_n(1-\theta)]\|\upsilon_n - p\|. \end{aligned} \tag{24}$$

We can easily seen that all conditions of Lemma 6 are fulfilled by above inequality (24). Hence by Lemma 6 we get $\lim\limits_{n\to\infty} \upsilon_n = p$.

Conversely, let $\lim\limits_{n\to\infty} \upsilon_n = p$, we have

$$\begin{aligned} \epsilon_n &= \|\upsilon_{n+1} - f(T, t_n)\| \\ &\leq \|\upsilon_{n+1} - p\| + \|f(T, \upsilon_n) - p\| \\ &\leq \|\upsilon_{n+1} - p\| + \theta^4[(1 - a_n(1-\theta)]\|\upsilon_n - p\|. \end{aligned}$$

This implies that $\lim\limits_{n\to\infty} \epsilon_n = 0$. Hence (3) is stable with respect to $(T)$. □

## 4 Example

In this section, we provide an example of multi-valued mapping for which best approximate operator $P_T$ is a generalized $\alpha-$nonexpansive mapping. Also, using this example, we compare various iterative processes such as (1)-iteration and (2)-iteration processes with our (3)-iteration process to show the numerical efficiency of our results.

**Example 2.** *Let $M = [0, \infty) \subset \mathbb{R}$ endowed with usual norm in $\mathbb{R}$ and $T : M \to \mathcal{P}(M)$ be defined by*

$$Tx = \begin{cases} \{0\}, & 0 \leq x < \frac{1}{2} \\ \left[0, \frac{3x}{7}\right], & x \geq \frac{1}{2} \end{cases}$$

*If $x \in [0, \frac{1}{2})$, then $P_T(x) = \{0\}$. For $x \in [\frac{1}{2}, \infty)$, then $P_T(x) = \{\frac{3x}{7}\}$. We show that $P_T$ is generalized $\frac{1}{4}-$nonexpansive mapping with $F(T)$. We consider the following cases:*

***Case I:*** *Let $x \in [0, \frac{1}{2})$ and $y \in [0, \frac{1}{2})$. We have*

$$\begin{aligned} H(P_T(x), P_T(y)) = 0 &\leq \frac{1}{4}d(y, P_T(x)) + \frac{1}{4}d(x, P_T(y)) + (1 - 2(\frac{1}{4}))\|x - y\| \\ &\leq \frac{1}{4}|x| + \frac{1}{4}|y| + \frac{1}{2}|x - y|. \end{aligned}$$

***Case II:*** *Let $x \in [0, \frac{1}{2})$ and $y \in [\frac{1}{2}, \infty)$. We have*

$$\begin{aligned} &\frac{1}{4}d(y, P_T(x)) + \frac{1}{4}d(x, P_T(y)) + (1 - 2(\frac{1}{4}))\|x - y\| \\ &= \frac{1}{4}|y - \frac{3x}{7}| + \frac{1}{4}|x - \frac{3y}{7}| + \frac{1}{2}|x - y| \\ &\geq \frac{1}{4}|\frac{14x - 14y}{7}| + \frac{1}{2}|x - y| \\ &\geq \frac{1}{2}|x - y| + \frac{1}{2}|x - y| = |x - y| \geq \frac{3}{7}|x - y| = H(P_T(x), P_T(y)) \end{aligned}$$

***Case III:****Let $x \in [\frac{1}{2}, \infty)$ and $y \in [0, \frac{1}{2})$. One has*

$$\begin{aligned} &\frac{1}{4}d(y, P_T(x)) + \frac{1}{4}d(x, P_T(y)) + (1 - 2(\frac{1}{4}))\|x - y\| \\ &= \frac{1}{4}|\frac{3x}{7} - y| + \frac{1}{4}|x| + \frac{1}{2}|x - y| \\ &\geq \frac{1}{4}|\frac{3x}{7} - y| + \frac{1}{2}|x - y| \\ &\geq |\frac{4x}{7}| > \frac{3}{7}|x| = H(P_T(x), P_T(y)) \end{aligned}$$

*Hence for all $x, y \in M = [0, \infty) \subset \mathbb{R}$, we have*

$H(P_T(x), P_T(y)) = 0 \leq \frac{1}{4}d(y, P_T(x)) + \frac{1}{4}d(x, P_T(y)) + (1 - 2(\frac{1}{4}))\|x - y\|$.

*Thus, $P_T$ is generalized $\frac{1}{4}-$nonexpansive mapping with $p = 0$ fixed point.*

*Finally, let us prove that $T$ does not satisfy Condition $(C)$. Indeed, if we take $x = 0.75, v = 0.43$ then*

$$d(x, P_T(x)) = \frac{1}{2}|0.75 - \frac{3 \times 0.75}{7}| = 0.214285 < 0.32 = |x - y|.$$

$$H(P_T(x), P_T(y)) = |\frac{3 \times 0.75}{7} - 0| = 0.321428 > 0.320000 = |x - y|.$$

*Thus $P_T$ does not satisfy Condition $(C)$.*

Let $a_n = b_n = 0.75$ for all $n \in \mathbb{N}$. Assume $x_1 = 200$. We compute that the sequence $\{x_n\}$ generated by (1), (2) and (3) iterative processes converge to a fixed point 0 of the multi-valued generalized $\alpha-$nonexpansive mapping defined in Example 2 which is shown by the Table 1. Also we compute that the sequence $\{x_n\}$ generated by (3)-iterative process converges to fixed point 0 of the multi-valued generalized $\alpha-$nonexpansive mapping defined in Example 2 which is shown by the Table 1 and Figure 1.

Table 1: Sequences generated by (1)-iteration, (2)-iteration and (3)-iteration for the multi-valued generalized $\alpha-$nonexpansive mapping defined in Example 2.

| | (1)-iteration | (2)-iteration | (3)-iteration |
|---|---|---|---|
| $x_1$ | 200 | 200 | 200 |
| $x_2$ | 11.995002082465639 | 20.991253644314867 | 2.203163647799812 |
| $x_3$ | 0.719400374791775 | 2.203163647799811 | 0 |
| $x_4$ | 0 | 0.231235834754499 | 0 |
| $x_5$ | 0 | 0 | 0 |

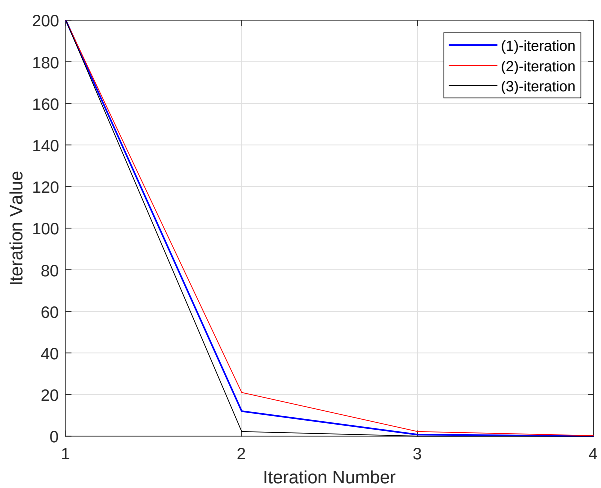


Figure 1: Convergences of (1)-iteration, (2)-iteration and (3)-iteration to the fixed point 0 of the multi-valued generalized $\alpha-$nonexpansive mapping defined in Example 2.

## 5 Conclusion

We study the convergence of (3)-iteration process to fixed for the multi-valued generalized $\alpha-$nonexpansive mapping in uniformly convex Banach space. Moreover, we give an illustrative numerical example that is multi-valued generalized $\alpha-$nonexpansive mapping but is not Suzuki generalized nonexpansive mapping, as in Example 2 of this paper. From Table 1 and Figure 1, we see that (3)-iteration process converges faster than (1)-iteration and (2)-iteration processes.

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