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CERTAIN ASPECTS OF $\lambda_{st}^{r}(\mathcal{G})-$CONVERGENCE OF SEQUENCES IN GRADUAL NORMED LINEAR SPACES*

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Abstract

In the present article, we set forth with the new notion of rough $\lambda-$statistical convergence in the gradual normed linear spaces. We produce significant results that present several fundamental properties of this notion. We also introduce the notion of $\lambda_{st}^{r}(\mathcal{G})-$limit set and prove that it is convex, gradually closed, and plays an important role for the gradually $\lambda-$statistical boundedness of a sequence.

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keywords: Gradual number, gradual normed linear space, $\lambda-$density, $\lambda_{st}^{r}(\mathcal{G})-$convergence, $\lambda_{st}^{r}(\mathcal{G})-$limit set.

1 Introduction

The notion of statistical convergence was first presented by Fast [22] and Steinhaus [35] independently in the year 1951. The main idea behind statistical convergence was the notion of natural density. The natural density of a set $A \subseteq \mathbb{N}$ is denoted and defined by

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$$\delta(A) = \lim_n \tfrac{1}{n} \left|\{k \in A : k \leq n\}\right|,$$

where the vertical bars indicate the cardinality of the enclosed set. A real-valued sequence $x = (x_k)$ is said to be statistically convergent to the number x_0 if for each $\eta > 0$,

$$\delta(\{k \in \mathbb{N} : |x_k - x_0| \geq \eta\}) = 0.$$

Later on, statistical convergence was further investigated and worked from the sequence space point of view by Fridy [24, 25], Šalát [34], Tripathy [37, 38], Connor [16], and many others [3, 4, 5, 6, 26].

In 2000, Mursaleen [29] generalized statistical convergence to $\lambda-$statistical convergence as follows:

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ such that

$$\lambda_{n+1} - \lambda_n \leq 1, \quad \lambda_1 = 1.$$

A sequence (x_k) is said to be $\lambda-$statistically convergent to the number x_0 if for each $\eta > 0$,

$$\lim_n \tfrac{1}{\lambda_n} \left|\{k \in I_n : |x_k - x_0| \geq \eta\}\right| = 0.$$

It is obvious that if $\lambda_n = n$, then the above definition reduces to the definition of statistical convergence. The $\lambda-$density of a set $A \subseteq \mathbb{N}$ is denoted and defined by

$$\delta_\lambda(A) = \lim_n \tfrac{1}{\lambda_n} \left|\{k \in A : k \leq n\}\right|.$$

For $\lambda_n = n$, the above definition turns to the definition of natural density. Clearly, $\delta_\lambda(\mathbb{N} \setminus A) + \delta_\lambda(A) = 1$ and $A \subseteq B$ implies $\delta_\lambda(A) \leq \delta_\lambda(B)$. It is obvious that if A is a finite set then $\delta_\lambda(A) = 0$.

In another direction, Phu [31] introduced and investigated the concept of rough convergence in finite dimensional normed spaces. It should be noted that the idea of rough convergence occurs quite naturally in numerical analysis and has interesting applications there. In 2003, Phu [32] further investigated the notion of rough convergence in infinite dimensional normed space setting. Combining the notion of rough convergence and statistical convergence, in 2008, Aytar [10] developed rough statistical convergence. But Akcay and Aytar [2] were the first who introduced and investigated the notion of rough convergence of a sequence of fuzzy numbers. For extensive study in this direction, one may refer to [7, 8, 9, 11, 12, 17, 19, 27, 30], where many more references can be found.

On the other hand, in 1965, the notion of fuzzy sets was introduced by Zadeh [39] as one of the extensions of the classical set-theoretical concept. These days, it has wide applications in different branches of science and engineering. The term "fuzzy number" is important in the study of fuzzy set theory. Fuzzy numbers were essentially the generalization of intervals, not numbers. Indeed fuzzy numbers do not obey a couple of algebraic properties of the classical numbers. So the term "fuzzy number" is debatable to many researchers due to its different behavior. The term "fuzzy intervals" is often used by many authors in place of fuzzy numbers. To overcome the confusion among the researchers, in 2008, Fortin et al. [23] introduced the notion of gradual real numbers as elements of fuzzy intervals. Gradual real numbers are mainly known by their respective assignment function whose domain is the interval $(0,1]$. So, every real number can be thought of as a gradual number with a constant assignment function. The gradual real numbers also obey all the algebraic properties of the classical real numbers and have been used in computation and optimization problems.

In 2011, Sadeqi and Azari [33] were the first to introduce the concept of gradual normed linear space. They studied various properties from both the algebraic and topological points of view. Further development in this direction has been taken place due to Ettefagh et al. [20, 21], Choudhury and Debnath [13, 14], and many others. For an extensive study on gradual real numbers, one may refer to [1, 18, 28, 36].

2 Preliminaries

In this section, we present some existing definitions and results which are crucial for our findings.

Definition 1. *[23] A gradual real number $\tilde{s}$ is defined by an assignment function $\mathcal{S}_{\tilde{s}} : (0,1] \to \mathbb{R}$. The set of all gradual real numbers is denoted by $\mathcal{G}(\mathbb{R})$. A gradual real number $\tilde{s}$ is said to be non-negative if for every $0 < \varphi \leq 1$, $\mathcal{S}_{\tilde{s}}(\varphi) \geq 0$. The set of all non-negative gradual real numbers is denoted by $\mathcal{G}^*(\mathbb{R})$.*

Definition 2. *[23] Let $*$ be any operation in $\mathbb{R}$ and suppose $\tilde{s}_1, \tilde{s}_2 \in \mathcal{G}(\mathbb{R})$ with assignment functions $\mathcal{S}_{\tilde{s}_1}$ and $\mathcal{S}_{\tilde{s}_2}$ respectively. Then, $\tilde{s}_1 * \tilde{s}_2 \in \mathcal{G}(\mathbb{R})$ is defined with the assignment function $\mathcal{S}_{\tilde{s}_1 * \tilde{s}_2}$ given by $\mathcal{S}_{\tilde{s}_1 * \tilde{s}_2}(\varphi) = \mathcal{S}_{\tilde{s}_1}(\varphi) * \mathcal{S}_{\tilde{s}_2}(\varphi)$, $\forall\, 0 < \varphi \leq 1$. In particular, the gradual addition $\tilde{s}_1 + \tilde{s}_2$ and the gradual scalar multiplication $c\tilde{s} (c \in \mathbb{R})$ are defined as follows:*

$$\mathcal{S}_{\tilde{s}_1+\tilde{s}_2}(\varphi) = \mathcal{S}_{\tilde{s}_1}(\varphi) + \mathcal{S}_{\tilde{s}_2}(\varphi) \quad \textit{and} \quad \mathcal{S}_{c\tilde{s}}(\varphi) = c\mathcal{S}_{\tilde{s}}(\varphi),\ \forall\, 0 < \varphi \leq 1.$$

Definition 3. *[33] Let X be a real vector space. The function $\|\cdot\|_{\mathcal{G}} : X \to \mathcal{G}^*(\mathbb{R})$ is said to be a gradual norm on X, if for every $0 < \varphi \leq 1$, the following conditions are true for any $x_0, y_0 \in X$:*

1. *$\mathcal{S}_{\|x_0\|_{\mathcal{G}}}(\varphi) = \mathcal{S}_{\tilde{0}}(\varphi)$ if and only if $x_0 = 0$;*
2. *$\mathcal{S}_{\|\mu x_0\|_{\mathcal{G}}}(\varphi) = |\mu|\mathcal{S}_{\|x_0\|_{\mathcal{G}}}(\varphi)$ for any $\mu \in \mathbb{R}$;*
3. *$\mathcal{S}_{\|x_0+y_0\|_{\mathcal{G}}}(\varphi) \leq \mathcal{S}_{\|x_0\|_{\mathcal{G}}}(\varphi) + \mathcal{S}_{\|y_0\|_{\mathcal{G}}}(\varphi)$.*

The pair $(X, \|\cdot\|_{\mathcal{G}})$ is called a gradual normed linear space (*GNLS*).

Example 1. *[33] Let $X = \mathbb{R}^n$ and for $x_0 = (x_1, x_2, ..., x_n) \in \mathbb{R}^n, 0 < \varphi \leq 1$, define $\|\cdot\|_{\mathcal{G}}$ by*

$$\mathcal{S}_{\|x_0\|_{\mathcal{G}}}(\varphi) = e^{\varphi} \sum_{i=1}^{n} |x_i|.$$

Then, $\|\cdot\|_{\mathcal{G}}$ is a gradual norm on $\mathbb{R}^n$ and $(\mathbb{R}^n, \|\cdot\|_{\mathcal{G}})$ is a GNLS.

Definition 4. *[33] Let $x = (x_k)$ be a sequence in the GNLS $(X, \|\cdot\|_{\mathcal{G}})$. Then, x is said to be gradual convergent to $x_0 \in X$, if for every $0 < \varphi \leq 1$ and $\eta > 0$, there exists $N(= N_\eta(\varphi)) \in \mathbb{N}$ such that*

$$\mathcal{S}_{\|x_k - x_0\|_{\mathcal{G}}}(\varphi) < \eta, \ \forall k \geq N.$$

Symbolically, $x_k \xrightarrow{\|\cdot\|_{\mathcal{G}}} x_0$.

Definition 5. *[21] Let $(X, \|\cdot\|_{\mathcal{G}})$ be a GNLS. Then, a sequence $x = (x_k)$ in X is said to be gradual bounded if for every $0 < \varphi \leq 1$, there exists $M = M(\varphi) > 0$ such that*

$$\mathcal{S}_{\|x_k\|_{\mathcal{G}}}(\varphi) < M, \ \forall k \in \mathbb{N}.$$

Definition 6. *[31] Let r be a non-negative real number. A sequence $x = (x_k)$ in a normed linear space $(X, \|\cdot\|)$ is said to be rough convergent to $x_0 \in X$ with roughness degree r, if for every $\eta > 0$, there exists $N = (N_\eta)$ such that for all $k \geq N$,*

$$\|x_k - x_0\| < r + \eta.$$

Symbolically, it is denoted as $x_k \xrightarrow{r-\|\cdot\|} x_0$.

Definition 7. *[10] Let r be a non-negative real number. A sequence $x = (x_k)$ in a normed linear space $(X, \|\cdot\|)$ is said to be rough statistically convergent to $x_0 \in X$ with roughness degree r, if for every $\eta > 0$,*

$$\delta\left(\{k \in \mathbb{N} : \|x_k - x_0\| \geq r + \eta\}\right) = 0.$$

Definition 8. *[15] Let $x = (x_k)$ be a sequence in the GNLS $\left(X, \|\cdot\|_{\mathcal{G}}\right)$. Then, x is said to be gradually $\lambda-$statistical convergent (in short $\lambda_{st}(\mathcal{G})-$convergent) to $x_0 \in X$ if for every $0 < \varphi \leq 1$ and $\eta > 0$,*

$$\delta_\lambda\left(\left\{k \in \mathbb{N} : \mathcal{S}_{\|x_k - x_0\|_{\mathcal{G}}}(\varphi) \geq \eta\right\}\right) = 0.$$

Symbolically we write, $x_k \xrightarrow{\lambda_{st}(\mathcal{G})} x_0$.

Definition 9. *[15] Let $x = (x_k)$ be a sequence in the GNLS $\left(X, \|\cdot\|_{\mathcal{G}}\right)$. Then, x is said to be gradually $\lambda-$statistical bounded if for every $0 < \varphi \leq 1$, there exists $M(= M(\varphi)) > 0$ such that*

$$\delta_\lambda\left(\left\{k \in \mathbb{N} : \mathcal{S}_{\|x_k\|_{\mathcal{G}}}(\varphi) > M\right\}\right) = 0.$$

3 Main Results

In this section, we present our main findings. We begin with a definitions which will be exclusively used throughout the article.

Definition 10. *Let $x = (x_k)$ be a sequence in the GNLS $\left(X, \|\cdot\|_{\mathcal{G}}\right)$ and r be a non-negative real number. Then, x is said to be gradually $\lambda-$statistically rough convergent (in short $\lambda^r_{st}(\mathcal{G})-$convergent) to $x_0 \in X$, if for every $0 < \varphi \leq 1$ and $\eta > 0$,*

$$\delta_\lambda\left(\left\{k \in \mathbb{N} : \mathcal{S}_{\|x_k - x_0\|_{\mathcal{G}}}(\varphi) \geq r + \eta\right\}\right) = 0.$$

Symbolically we write, $x_k \xrightarrow{\lambda^r_{st}(\mathcal{G})} x_0$.

Here, x_0 is called as the $\lambda^r_{st}(\mathcal{G})-$limit of x, where r is the degree of roughness. For $r = 0$, the above definition turns to the Definition 8. But our main intention is to deal with the case $r > 0$. There are several reasons for such interest. Since a $\lambda_{st}(\mathcal{G})-$convergent sequence $y = (y_k)$ with $y_k \xrightarrow{\lambda_{st}(\mathcal{G})} x_0$ often cannot be measured or calculated accurately, one has to deal with a $\lambda-$statistically approximated sequence $x = (x_k)$ satisfying

$$\delta_\lambda\left(\left\{k \in \mathbb{N} : \mathcal{S}_{\|x_k - y_k\|_{\mathcal{G}}}(\varphi) > r\right\}\right) = 0.$$

Then, no one can guarantee the $\lambda_{st}(\mathcal{G})-$convergence of x, but since for any $\eta > 0$, the following inclusion

$$\Big\{k \in \mathbb{N} : \mathcal{S}_{\|y_k - x_0\|_{\mathcal{G}}}(\varphi) \geq \eta\Big\} \supseteq \Big\{k \in \mathbb{N} : \mathcal{S}_{\|x_k - x_0\|_{\mathcal{G}}}(\varphi) \geq r + \eta\Big\}.$$

holds, one can certainly assure the $\lambda_{st}^r(\mathcal{G})-$convergence of x. We present the following example to illustrate the above fact more preciously.

Example 2. *Let $X = \mathbb{R}^n$ and $\|\cdot\|_{\mathcal{G}}$ be the gradual norm defined in Example 1. Consider the sequence (λ_n) defined by*

$$\lambda_n = \begin{cases} 1, & n = 1 \\ \frac{n}{2}, & n \geq 2. \end{cases}$$

Suppose $y = (y_k)$ in $\mathbb{R}^n$ be defined as

$$y_k = \begin{cases} (0, 0, ..., 0, 0.5), & \text{if } k \text{ is not a perfect square} \\ \left(0, 0, ..., 0, 0.5 + 2 \cdot \frac{(-1)^k}{k}\right), & \text{otherwise.} \end{cases}$$

Then, we have

$$\mathcal{S}_{\|y_k - (0,0,...,0,0.5)\|_{\mathcal{G}}}(\varphi) = \begin{cases} 0, & \text{if } k \text{ is not a perfect square} \\ \frac{2e^{\varphi}}{k}, & \text{otherwise.} \end{cases}$$

Therefore, for any $\eta > 0$, the following inclusion

$$\Big\{k \in \mathbb{N} : \mathcal{S}_{\|y_k - (0,0,...,0,0.5)\|_{\mathcal{G}}}(\varphi) \geq \eta\Big\} \subseteq \{1, 4, 9, ...\}$$

holds and eventually $y_k \xrightarrow{\lambda_{st}(\mathcal{G})} (0, 0, ..., 0, 0.5)$. But for sufficiently large k, it is impossible to calculate y_k exactly by computer but it is rounded to the nearest one. So, for the sake of simplicity, we approximate y_k by $x_k = (0, 0, ..., 0, z)$ at the perfect square positions where z is the integer satisfying $z - 0.5 < y_k < z + 0.5$. Then, the sequence $x = (x_k)$ does not $\lambda_{st}(\mathcal{G})-$converge anymore. But by definition $x_k \xrightarrow{\lambda_{st}^r(\mathcal{G})} (0, 0, ..., 0, 0.5)$ for $r = 0.5$.

Example 3. *Consider the GNLS $\left(\mathbb{R}^n, \|\cdot\|_{\mathcal{G}}\right)$, where $\|\cdot\|_{\mathcal{G}}$ is the gradual norm defined in Example 1 and suppose (λ_n) be the sequence defined in Example 2. Define a sequence $x = (x_k)$ in in $\mathbb{R}^n$ as follows:*

$$x_k = \begin{cases} \left(0, 0, ..., 0, (-1)^k\right), & \text{if } k \text{ is not a perfect square} \\ (0, 0, ..., 0, k), & \text{otherwise.} \end{cases}$$

Then, for $0 < \varphi \leq \ln 2$, each element of the following set

$$\left\{(0, 0, ..., 0, x_0) \in \mathbb{R}^n : x_0 \in \left[1 - \tfrac{2}{e^{\varphi}}, -1 + \tfrac{2}{e^{\varphi}}\right]\right\}$$

is a $\lambda_{st}^{2}(\mathcal{G})-$*limit of* x *and for* $\ln 2 \leq \varphi \leq 1$, *each element of the following set*

$$\left\{(0,0,...,0,y_0) \in \mathbb{R}^n : y_0 \in \left[-1+\frac{2}{e^{\varphi}}, 1-\frac{2}{e^{\varphi}}\right]\right\}$$

is a $\lambda_{st}^{2}(\mathcal{G})-$*limit of* x.

From the above example, it is clear that for $r>0$, the $\lambda_{st}^{r}(\mathcal{G})-$limit of a sequence is not necessarily unique. So our main interest is to investigate the case $r>0$. Therefore, we construct $\lambda_{st}^{r}(\mathcal{G})-$limit set of a sequence $x=(x_k)$ denoted and defined as follows:

$$\lambda_{st} - LIM_x^r(\mathcal{G}) = \left\{x_0 \in X : x_k \xrightarrow{\lambda_{st}^{r}(\mathcal{G})} x_0\right\}.$$

Theorem 1. *Let* (x_k) *and* (y_k) *be two sequences in the GNLS* $\left(X, \|\cdot\|_{\mathcal{G}}\right)$ *such that* $x_k \xrightarrow{\lambda_{st}^{r_1}(\mathcal{G})} x_0$ *and* $y_k \xrightarrow{\lambda_{st}^{r_2}(\mathcal{G})} y_0$. *Then,*
(i) $x_k + y_k \xrightarrow{\lambda_{st}^{(r_1+r_2)}(\mathcal{G})} x_0 + y_0$ *and (ii)* $\mu x_k \xrightarrow{\lambda_{st}^{|\mu| r_1}(\mathcal{G})} \mu x_0$ *for any* $\mu \in \mathbb{R}$.

Proof. (i) Since, $x_k \xrightarrow{\lambda_{st}^{r_1}(\mathcal{G})} x_0$ and $y_k \xrightarrow{\lambda_{st}^{r_2}(\mathcal{G})} y_0$, so for any $0<\varphi\leq 1$ and $\eta>0$,

$$\delta_\lambda(P) = \delta_\lambda(Q) = 0,$$

where

$$P = \left\{k \in \mathbb{N} : \mathcal{S}_{\|x_k - x_0\|_{\mathcal{G}}}(\varphi) \geq r_1 + \frac{\eta}{2}\right\} \text{ and}$$
$$Q = \left\{k \in \mathbb{N} : \mathcal{S}_{\|y_k - y_0\|_{\mathcal{G}}}(\varphi) \geq r_2 + \frac{\eta}{2}\right\}.$$

Now as the inclusion

$$(\mathbb{N} \setminus P) \cap (\mathbb{N} \setminus Q) \subseteq \left\{k \in \mathbb{N} : \mathcal{S}_{\|(x_k+y_k)-(x_0+y_0)\|_{\mathcal{G}}}(\varphi) < r_1 + r_2 + \eta\right\}$$

holds, so we must have

$$\delta_\lambda\left(\left\{k \in \mathbb{N} : \mathcal{S}_{\|(x_k+y_k)-(x_0+y_0)\|_{\mathcal{G}}}(\varphi) \geq r_1 + r_2 + \eta\right\}\right) \leq \delta_\lambda(P \cup Q) = 0;$$

and consequently, $x_k + y_k \xrightarrow{\lambda_{st}^{(r_1+r_2)}(\mathcal{G})} x_0 + y_0$.

(ii) If $\mu = 0$, then there is nothing to prove. So let us assume that $\mu \neq 0$. Now as the conditions

$$\mathcal{S}_{\|x_k - x_0\|_{\mathcal{G}}}(\varphi) \leq r_1 \text{ and } \mathcal{S}_{\|\mu x_k - \mu x_0\|_{\mathcal{G}}}(\varphi) \leq |\mu| r_1$$

are equivalent in gradual normed algebras, so the result follows. □

Remark 1. *[15] Let (x_k) and (y_k) be two sequences in the GNLS $\left(X, \|\cdot\|_{\mathcal{G}}\right)$ such that $x_k \xrightarrow{\lambda_{st}(\mathcal{G})} x_0$ and $y_k \xrightarrow{\lambda_{st}(\mathcal{G})} y_0$. Then,*
(i) $x_k + y_k \xrightarrow{\lambda_{st}(\mathcal{G})} x_0 + y_0$ and (ii) $\mu x_k \xrightarrow{\lambda_{st}(\mathcal{G})} \mu x_0$ for any $\mu \in \mathbb{R}$.

Theorem 2. *Let $x = (x_k)$ be a sequence in a GNLS $\left(X, \|\cdot\|_{\mathcal{G}}\right)$. Then,*

$$diam\left(\lambda_{st} - LIM^r_x(\mathcal{G})\right) = \sup\left\{\mathcal{S}_{\|y-z\|_{\mathcal{G}}}(\varphi) : y, z \in \lambda_{st} - LIM^r_x(\mathcal{G}), \varphi \in [0,1)\right\} \leq 2r.$$

In general, $diam\left(\lambda_{st} - LIM^r_x(\mathcal{G})\right)$ has no smaller bound.

Proof. If possible, let us assume that $diam\left(\lambda_{st} - LIM^r_x(\mathcal{G})\right) > 2r$. Then, there exists $y_0, z_0 \in \lambda_{st} - LIM^r_x(\mathcal{G})$ and $0 < \varphi_0 \leq 1$ such that $\mathcal{S}_{\|y_0-z_0\|_{\mathcal{G}}}(\varphi_0) > 2r$. Choose $\eta > 0$ in such a manner that

$$\eta < \frac{\mathcal{S}_{\|y_0-z_0\|_{\mathcal{G}}}(\varphi_0)}{2} - r. \tag{1}$$

Since, $y_0, z_0 \in \lambda_{st} - LIM^r_x(\mathcal{G})$, so for any $0 < \varphi \leq 1$ and $\eta > 0$, $\delta_\lambda(A) = 0$ and $\delta_\lambda(B) = 0$, where

$$A = \left\{k \in \mathbb{N} : \mathcal{S}_{\|x_k-y_0\|_{\mathcal{G}}}(\varphi) \geq r + \eta\right\} \text{ and}$$
$$B = \left\{k \in \mathbb{N} : \mathcal{S}_{\|x_k-z_0\|_{\mathcal{G}}}(\varphi) \geq r + \eta\right\}.$$

By the property of $\lambda-$density, it is clear that the set $(\mathbb{N} \setminus A) \cap (\mathbb{N} \setminus B)$ is non-empty. Take $p \in (\mathbb{N} \setminus A) \cap (\mathbb{N} \setminus B)$. Then, we have

$$\mathcal{S}_{\|y_0-z_0\|_{\mathcal{G}}}(\varphi_0) \leq \mathcal{S}_{\|x_p-y_0\|_{\mathcal{G}}}(\varphi_0) + \mathcal{S}_{\|x_p-z_0\|_{\mathcal{G}}}(\varphi_0) < 2(r + \eta),$$

which contradicts (1).

For the second part, suppose (x_k) be a sequence in a *GNLS* $\left(X, \|\cdot\|_{\mathcal{G}}\right)$ such that $x_k \xrightarrow{\lambda_{st}(\mathcal{G})} x_0$. Then, for any $0 < \varphi \leq 1$ and $\eta > 0$,

$$\delta_\lambda\left(\left\{k \in \mathbb{N} : \mathcal{S}_{\|x_k-x_0\|_{\mathcal{G}}}(\varphi) \geq \eta\right\}\right) = 0.$$

Now for each $y_0 \in (x_0 + \bar{N}(r, \varphi)) = \left\{x \in X : \mathcal{S}_{\|x_0-x\|_{\mathcal{G}}}(\varphi) \leq r\right\}$, the following inequation

$$\mathcal{S}_{\|x_k-y_0\|_{\mathcal{G}}}(\varphi) \leq \mathcal{S}_{\|x_k-x_0\|_{\mathcal{G}}}(\varphi) + \mathcal{S}_{\|x_0-y_0\|_{\mathcal{G}}}(\varphi) < r + \eta,$$

holds whenever $k \notin \left\{k \in \mathbb{N} : \mathcal{S}_{\|x_k-x_0\|_{\mathcal{G}}}(\varphi) \geq \eta\right\}$. This shows that $y_0 \in \lambda_{st} - LIM^r_x(\mathcal{G})$ and subsequently

$$\lambda_{st} - LIM_x^r(\mathcal{G}) = (x_0 + \bar{N}(r, \varphi))$$

holds. Since, $diam\left(x_0 + \bar{N}(r, \varphi)\right) = 2r$, so in general upper bound $2r$ of the gradual diameter of the set $\lambda_{st} - LIM_x^r(\mathcal{G})$ cannot be decreased anymore. $\square$

Remark 2. *[15] Let $x = (x_k)$ be a sequence in a GNLS $\left(X, \|\cdot\|_{\mathcal{G}}\right)$ such that $x_k \xrightarrow{\lambda_{st}(\mathcal{G})} x_0$. Then, x_0 is uniquely determined.*

Theorem 3. *A sequence $x = (x_k)$ in a GNLS $\left(X, \|\cdot\|_{\mathcal{G}}\right)$ is gradually $\lambda-$statistical bounded if and only if there exists some $r \geq 0$ such that $\lambda_{st} - LIM_x^r(\mathcal{G}) \neq \emptyset$.*

Proof. Let $x = (x_k)$ be gradually $\lambda-$statistical bounded. Then, for every $\varphi \in (0, 1]$, there exists $M(= M(\varphi)) > 0$ such that

$$\delta_\lambda(A) = 0, \text{ where } A = \left\{k \in \mathbb{N} : \mathcal{S}_{\|x_k\|_{\mathcal{G}}}(\varphi) > M\right\}.$$

Suppose

$$r' = \sup\left\{\mathcal{S}_{\|x_k\|_{\mathcal{G}}}(\varphi) : k \in \mathbb{N} \setminus A, \varphi \in [0, 1)\right\}.$$

Then, the set $\lambda_{st} - LIM_x^{r'}(\mathcal{G})$ contains the zero vector of X and eventually

$$\lambda_{st} - LIM_x^{r'}(\mathcal{G}) \neq \emptyset.$$

Conversely suppose that $\lambda_{st} - LIM_x^r(\mathcal{G}) \neq \emptyset$ for some $r \geq 0$. Then, for $x_0 \in \lambda_{st} - LIM_x^r(\mathcal{G})$,

$$\delta_\lambda\left(\left\{k \in \mathbb{N} : \mathcal{S}_{\|x_k - x_0\|_{\mathcal{G}}}(\varphi) \geq r + \eta\right\}\right) = 0$$

holds for any $0 < \varphi \leq 1$ and $\eta > 0$. This means that almost all x_k's are contained in some ball with any radius greater than r. Therefore, x is gradually $\lambda-$statistical bounded. $\square$

Theorem 4. *Let $x = (x_k)$ be a sequence in a GNLS $\left(X, \|\cdot\|_{\mathcal{G}}\right)$. If $y_0 \in \lambda_{st} - LIM_x^{r_0}(\mathcal{G})$ and $y_1 \in \lambda_{st} - LIM_x^{r_1}(\mathcal{G})$, then*

$$y_\sigma = (1 - \sigma)y_0 + \sigma y_1 \in \lambda_{st} - LIM_x^{(1-\sigma)r_0 + \sigma r_1}(\mathcal{G}), \; for\, \sigma \in [0, 1].$$

Proof. Since, $y_0 \in \lambda_{st} - LIM_x^{r_0}(\mathcal{G})$ and $y_1 \in \lambda_{st} - LIM_x^{r_1}(\mathcal{G})$, so for every $0 < \varphi \leq 1$ and $\eta > 0$, $\delta_\lambda(A) = 0$ and $\delta_\lambda(B) = 0$, where

$$A = \left\{k \in \mathbb{N} : \mathcal{S}_{\|x_k - y_0\|_{\mathcal{G}}}(\varphi) \geq r_0 + \eta\right\} \text{ and}$$
$$B = \left\{k \in \mathbb{N} : \mathcal{S}_{\|x_k - y_1\|_{\mathcal{G}}}(\varphi) \geq r_1 + \eta\right\}.$$

Subsequently, for any $k \in (\mathbb{N} \setminus A) \cap (\mathbb{N} \setminus B)$,

$$\begin{aligned}\mathcal{S}_{\|x_k - y_\sigma\|_\mathcal{G}}(\varphi) &\leq (1-\sigma)\mathcal{S}_{\|x_k - y_0\|_\mathcal{G}}(\varphi) + \sigma \mathcal{S}_{\|x_k - y_1\|_\mathcal{G}}(\varphi) \\ &< (1-\sigma)(r_0 + \eta) + \sigma(r_1 + \eta) \\ &= (1-\sigma)r_0 + \sigma r_1 + \eta.\end{aligned}$$

This proves that,

$$\left\{k \in \mathbb{N} : \mathcal{S}_{\|x_k - y_\sigma\|_\mathcal{G}}(\varphi) \geq (1-\sigma)r_0 + \sigma r_1 + \eta\right\} \subseteq A \cup B.$$

Now since the $\lambda-$density of the set in the right-hand side of the above inclusion is zero, so the $\lambda-$density of the set in the left-hand side is also zero. Hence, $y_\sigma \in \lambda_{st} - LIM_x^{(1-\sigma)r_0 + \sigma r_1}(\mathcal{G})$. □

Remark 3. *Let $x = (x_k)$ be a sequence in a GNLS $\left(X, \|\cdot\|_\mathcal{G}\right)$. Then, the set $\lambda_{st} - LIM_x^r(\mathcal{G})$ is convex.*

Theorem 5. *Let $x = (x_k)$ be a sequence in a GNLS $\left(X, \|\cdot\|_\mathcal{G}\right)$. Then, the set $\lambda_{st} - LIM_x^r(\mathcal{G})$ is gradually closed.*

Proof. Let $y = (y_k)$ be a sequence in $\lambda_{st} - LIM_x^r(\mathcal{G})$ such that

$$y_k \xrightarrow{\|\cdot\|_\mathcal{G}} y_0.$$

Then, for every $0 < \varphi \leq 1$ and $\eta > 0$, there exists $N(= N_\eta(\varphi)) \in \mathbb{N}$ such that for all $k \geq N$,

$$\mathcal{S}_{\|y_k - y_0\|_\mathcal{G}}(\varphi) < \tfrac{\eta}{2}.$$

Choose $k_0 \in \mathbb{N}$ such that $k_0 \geq N$. Then, $\mathcal{S}_{\left\|y_{k_0} - y_0\right\|_\mathcal{G}}(\varphi) < \frac{\eta}{2}$. On the other hand, since $(y_k) \subseteq \lambda_{st} - LIM_x^r(\mathcal{G})$, we must have

$$\delta_\lambda\left(\left\{k \in \mathbb{N} : \mathcal{S}_{\left\|x_k - y_{k_0}\right\|_\mathcal{G}}(\varphi) \geq r + \frac{\eta}{2}\right\}\right) = 0. \tag{2}$$

Suppose $p \notin \left\{k \in \mathbb{N} : \mathcal{S}_{\left\|x_k - y_{k_0}\right\|_\mathcal{G}}(\varphi) \geq r + \frac{\eta}{2}\right\}$. Then, $\mathcal{S}_{\left\|x_p - y_{k_0}\right\|_\mathcal{G}}(\varphi) < r + \frac{\eta}{2}$ and eventually

$$\mathcal{S}_{\|x_p - y_0\|_\mathcal{G}}(\varphi) \leq \mathcal{S}_{\left\|x_p - y_{k_0}\right\|_\mathcal{G}}(\varphi) + \mathcal{S}_{\left\|y_{k_0} - y_0\right\|_\mathcal{G}}(\varphi) < r + \eta.$$

This means that $p \notin \left\{k \in \mathbb{N} : \mathcal{S}_{\|x_k - y_0\|_\mathcal{G}}(\varphi) \geq r + \eta\right\}$ and subsequently from (2) we obtain

$$\delta_\lambda\left(\left\{k\in\mathbb{N}:\mathcal{S}_{\|x_k-y_0\|_{\mathcal{G}}}(\varphi)\geq r+\eta\right\}\right)=0.$$

Hence, $y_0\in\lambda_{st}-LIM_x^r(\mathcal{G})$ and the proof ends. □

Theorem 6. *Let $r_1\geq 0$ and $r_2\geq 0$. A sequence $x=(x_k)$ in a GNLS $\left(X,\|\cdot\|_{\mathcal{G}}\right)$ is $\lambda_{st}^{(r_1+r_2)}(\mathcal{G})-$convergent to x_0 if and only if there exists a sequence $y=(y_k)$ such that*

$$y_k\xrightarrow{\lambda_{st}^{r_1}(\mathcal{G})}x_0 \text{ and } \mathcal{S}_{\|x_k-y_k\|_{\mathcal{G}}}(\varphi)\leq r_2$$

for all $k\in\mathbb{N}$.

Proof. Let us assume that $y_k\xrightarrow{\lambda_{st}^{r_1}(\mathcal{G})}x_0$. Then, by definition for any $0<\varphi\leq 1$ and $\eta>0$,

$$\delta_\lambda(P)=0, \text{ where } P=\left\{k\in\mathbb{N}:\mathcal{S}_{\|y_k-x_0\|_{\mathcal{G}}}(\varphi)\geq r_1+\eta\right\}.$$

Now since $\mathcal{S}_{\|x_k-y_k\|_{\mathcal{G}}}(\varphi)\leq r_2$ holds for all $k\in\mathbb{N}$, so for all $k\notin P$,

$$\mathcal{S}_{\|x_k-x_0\|_{\mathcal{G}}}(\varphi)\leq\mathcal{S}_{\|x_k-y_k\|_{\mathcal{G}}}(\varphi)+\mathcal{S}_{\|y_k-x_0\|_{\mathcal{G}}}(\varphi)<r_1+r_2+\eta.$$

This implies that

$$\left\{k\in\mathbb{N}:\mathcal{S}_{\|x_k-x_0\|_{\mathcal{G}}}(\varphi)\geq r_1+r_2+\eta\right\}\subseteq P$$

and eventually by the property of $\lambda-$density,

$$\delta_\lambda\left(\left\{k\in\mathbb{N}:\mathcal{S}_{\|x_k-x_0\|_{\mathcal{G}}}(\varphi)\geq r_1+r_2+\eta\right\}\right)=0.$$

Hence, $x_k\xrightarrow{\lambda_{st}^{(r_1+r_2)}(\mathcal{G})}x_0$.

For the converse part, let us assume that

$$x_k\xrightarrow{\lambda_{st}^{(r_1+r_2)}(\mathcal{G})}x_0. \tag{3}$$

Define $y=(y_k)$ by

$$y_k=\begin{cases}x_0, & if\ \mathcal{S}_{\|x_k-x_0\|_{\mathcal{G}}}(\varphi)\leq r_2\\ x_k+r_2\frac{x_0-x_k}{\mathcal{S}_{\|x_k-x_0\|_{\mathcal{G}}}(\varphi)}, & otherwise.\end{cases}$$

Then, it is easy to observe that $\mathcal{S}_{\|x_k-y_k\|_{\mathcal{G}}}(\varphi)\leq r_2$ for all $k\in\mathbb{N}$.

Moreover,

$$\mathcal{S}_{\|y_k-x_0\|_{\mathcal{G}}}(\varphi)=\begin{cases}0, & if\ \mathcal{S}_{\|x_k-x_0\|_{\mathcal{G}}}(\varphi)\le r_2\\ \mathcal{S}_{\|x_k-x_0\|_{\mathcal{G}}}(\varphi)-r_2, & otherwise.\end{cases}$$

By (3), for every $0<\varphi\le 1$ and $\eta>0$,

$$\delta_\lambda\left(\left\{k\in\mathbb{N}:\mathcal{S}_{\|x_k-x_0\|_{\mathcal{G}}}(\varphi)\ge r_1+r_2+\eta\right\}\right)=0.$$

Now as the inclusion

$$\left\{k\in\mathbb{N}:\mathcal{S}_{\|x_k-x_0\|_{\mathcal{G}}}(\varphi)\ge r_1+r_2+\eta\right\}\supseteq\left\{k\in\mathbb{N}:\mathcal{S}_{\|y_k-x_0\|_{\mathcal{G}}}(\varphi)\ge r_1+\eta\right\}$$

holds, so we must have

$$\delta_\lambda\left(\left\{k\in\mathbb{N}:\mathcal{S}_{\|y_k-x_0\|_{\mathcal{G}}}(\varphi)\ge r_1+\eta\right\}\right)=0.$$

Hence, $y_k\xrightarrow{\lambda_{st}^{r_1}(\mathcal{G})}x_0$ and the proof ends. □

Remark 4. *A sequence $x=(x_k)$ in a GNLS $\left(X,\|\cdot\|_{\mathcal{G}}\right)$ is $\lambda_{st}^{r}(\mathcal{G})-$convergent to $x_0\in X$ with roughness degree $r\ge 0$ if and only if there exists a sequence $y=(y_k)$ in X such that $x_k\xrightarrow{\lambda_{st}(\mathcal{G})}x_0$ and $\mathcal{S}_{\|x_k-y_k\|}\le r$ for all $k\in\mathbb{N}$.*

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