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FROM PROPAGATION SYSTEMS TO TIME DELAYS AND BACK. CRITICAL CASES*

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Dedicated to Dr. Dan Tiba on the occasion of his 70^{th} anniversary

Abstract

The paper originates from the early ideas of A. D. Myshkis and his co-workers and of K. L. Cooke and his co-worker. These ideas send to a one-to-one correspondence between lossless and/or distortionless propagation described by nonstandard boundary value problems and a system of coupled differential and difference equations with deviated argument. In this way any property obtained for one mathematical object is automatically projected back on the other one. This approach is considered here for certain engineering applications. The common feature of these applications is the critical stability of the difference operator associated with the system with deviated argument obtained for each of the aforementioned applications. In fact the associated systems are of neutral type and, according to the assumption of Hale, only strong stability of the difference operator ensures robust asymptotic stability with respect to the delays. If the difference operator is in the critical case, the stability becomes fragile with respect to the delays. Based on some old results in the field, a conjecture concerning the (quasi)-critical modes of the system is stated; also a connection with the so called *dissipative boundary conditions* is suggested.

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1 Introduction. The methodology

Those who know the early history of the differential equations with deviated argument in the XXth century know also the very first book on such equations - the book of of A. D. Myshkis [1], more precisely its first edition of 1951, because a more recent revised edition was published in 1972. This book contains no reference to applications of such equations but another book published after 4 years does [2]. It is however not our intention to draw a history of these equations over the last 7 decades, but to point out a certain line of research and studies. It is firstly interesting to point out a series of three papers [3, 4, 5] where dynamics of controlled structures of thermal power engineering incorporated rather long steam or oil pipes inducing propagation effects. Linearization and use of the Laplace transform suggested characteristic quasi-polynomials which were specific to equations with deviated arguments with pointwise delays *of neutral type*.

Much later, in the 60ies of XXth century, a certain research made at IBM under the guidance of R. K. Brayton displayed similar aspects but for electrical circuits containing lossless transmission lines [6, 7, 8, 9]. Later it was pointed out [10] that pointwise delays are modeling what is called *lossless and/or distortionless propagation*.

The mathematical theory connected to those aspects grew in parallel. Firstly, the papers of Myshkis and his co-workers were published [11, 12]. In these papers there were considered hyperbolic partial differential equations in the plane - having as independent variables time and one space variables - in fact $1D$ systems. Their boundary conditions were non-standard since they contained Volterra operators. Somehow later the papers of K. L. Cooke were published: the first one, co-authored by D. W. Krumme, had clear reference to electrical circuits containing lossless transmission lines [13]. The second one [14] - less circulated (it looked more as a seminar exposition, with incomplete proofs) has little reference to applications. Worth mentioning that the papers of Cooke contain boundary problems of non-standard type but whose boundary conditions contain only differential equations, thus being less general than those considered by Myshkis and his co-workers.

Now, regardless the generality of the non-standard boundary conditions, *the methodology* of the two groups of papers is the same. Namely, making use of the fact that the Riemann invariants of the problems are constant along the characteristics, a system of functional equations is associated to the boundary value problem for hyperbolic partial differential equations. The type of the functional equations is induced by the type of operators involved in the non-standard boundary conditions.

This association of the system of functional equations is *one-to-one* in the sense that a one-to-one correspondence is established between the solutions of the boundary value problem for the hyperbolic partial differential equations (with given initial conditions also) and the initial value (Cauchy) problem for the associated functional equations. The aforementioned one-to-one correspondence is far going since any property obtained for one mathematical object is automatically projected back on the other. Some advantages of this aspect will be discussed in the next section.

2 A basic theorem and its consequences

We shall refer here to [14]. The basic result of it is given only with a sketch of half-proof and, in our opinion, lacked rigor. For this reason we re-considered it in [15], endowed with a complete rigorous proof. Here we shall reproduce what is necessary to understand the methodology mentioned in the previous section. Consider the following initial boundary value problem

$$
\begin{aligned}
&\frac{\partial u^+}{\partial t} + \tau^+(\lambda,t)\frac{\partial u^+}{\partial \lambda} = \Phi^+(\lambda,t) \\
&\frac{\partial u^-}{\partial t} + \tau^-(\lambda,t)\frac{\partial u^-}{\partial \lambda} = \Phi^-(\lambda,t),\ 0 \le \lambda \le 1,\ t \ge t_0, \\
&\sum_{k=0}^{m}\left[a_k^+(t)\frac{\mathrm{d}^k}{\mathrm{d}t^k}u^+(0,t) + a_k^-(t)\frac{\mathrm{d}^k}{\mathrm{d}t^k}u^-(0,t)\right] = f_0(t) \\
&\sum_{k=0}^{m}\left[b_k^+(t)\frac{\mathrm{d}^k}{\mathrm{d}t^k}u^+(1,t) + b_k^-(t)\frac{\mathrm{d}^k}{\mathrm{d}t^k}u^-(1,t)\right] = f_1(t) \\
&u^\pm(\lambda,t_0) = \omega^\pm(\lambda)\ ,\ 0 \le \lambda \le 1\ ;\ \tau^+(\lambda,\tau) > 0\ ,\ \tau^-(\lambda,\tau) < 0
\end{aligned}
\tag{1}
$$

and the two families of characteristics

$$
\frac{\mathrm{d}t}{\mathrm{d}\lambda} = \frac{1}{\tau^\pm(\lambda,t)},\ \tau^+(\lambda,t) > 0\ ,\ \tau^-(\lambda,t) < 0 \tag{2}
$$

In (1) and (2) we have $\tau^+ : [0,1] \times [t_0,\infty) \to \mathbb{R}^+$ and $\tau^- : [0,1] \times [t_0,\infty) \to \mathbb{R}^-$. Let $t^\pm(\sigma;\lambda,t)$ be the two characteristics crossing some point (λ,t) of the strip $[0,1]\times[t_0,t_1]$. As it will appear in the sequel, the current variable σ is in $[0,1]$ or in some sub-interval of it. Define

$$T^+(t) := t^+(1;0,t) - t\ ,\ T^-(t) := t^-(0;1,t) - t, \tag{3}$$

$t\in[t_0,t_1]$, called *propagation times along the characteristics* or *forward and backward* propagation time respectively. Also (2) shows that $t^+(\cdot;\lambda,t)$ is increasing hence it can be extended "to the right" up to $\sigma=1$; the other characteristic $t^-(\cdot;\lambda,t)$ is strictly decreasing hence it can be extended "to the left" up to $\sigma=0$. Considering the "progressive (forward) wave" $u^+(\lambda,t)$ along the increasing characteristic and the "reflected (backward) wave" $u^-(\lambda,t)$ along the decreasing one, then integrating from λ to 1 and from 0 to λ respectively, the following is obtained

$$\begin{aligned} u^+(\lambda,t) &= u^+(1,t^+(1;\lambda,t)) - \int_\lambda^1 \frac{\Phi^+(\sigma,t^+(\sigma;\lambda,t))}{\tau^+(\sigma,t^+(\sigma;\lambda,t))}\mathrm{d}\sigma \\ u^-(\lambda,t) &= u^-(0,t^-(0;\lambda,t)) + \int_0^\lambda \frac{\Phi^-(\sigma,t^-(\sigma;\lambda,t))}{\tau^-(\sigma,t^-(\sigma;\lambda,t))}\mathrm{d}\sigma \end{aligned} \tag{4}$$

When the two characteristics can be extended on the entire segment $[0,1]$ i.e. $t^+(\cdot;\lambda,t)$ - "to the left" and $t^-(\cdot;\lambda,t)$ "to the right", the pair (4) becomes

$$\begin{aligned} u^+(0,t) &= u^+(1,t+T^+(t)) - \int_0^1 \frac{\Phi^+(\sigma,t^+(\sigma;0,t))}{\tau^+(\sigma,t^+(\sigma;0,t))}\mathrm{d}\sigma \\ u^-(1,t) &= u^-(0,t+T^-(t)) + \int_0^1 \frac{\Phi^-(\sigma,t^-(\sigma;1,t))}{\tau^-(\sigma,t^-(\sigma;1,t))}\mathrm{d}\sigma \end{aligned} \tag{5}$$

(with (3) also taken into account). Equalities (5) are in fact some functional relations between the boundary values of the forward and backward waves. Denoting

$$\begin{aligned} y^+(t) &:= u^+(1,t)\ ,\ \Psi^+(t) := \int_0^1 \frac{\Phi^+(\sigma,t^+(\sigma;0,t))}{\tau^+(\sigma,t^+(\sigma;0,t))}\mathrm{d}\sigma \\ y^-(t) &:= u^-(0,t)\ ,\ \Psi^-(t) := \int_0^1 \frac{\Phi^-(\sigma,t^-(\sigma;1,t))}{\tau^-(\sigma,t^-(\sigma;1,t))}\mathrm{d}\sigma \end{aligned} \tag{6}$$

it follows that $(y^+(t), y^-(t))$ thus defined satisfy the following system of differential equations with deviated argument

$$\sum_{k=0}^{m}\left[a_k^+(t)\frac{\mathrm{d}^k}{\mathrm{d}t^k}y^+(t+T^+(t)) + a_k^-(t)\frac{\mathrm{d}^k}{\mathrm{d}t^k}y^-(t)\right] = f_0(t) + \sum_{k=0}^{m}a_k^+(t)\frac{\mathrm{d}^k}{\mathrm{d}t^k}\Psi^+(t)$$
$$\sum_{k=0}^{m}\left[b_k^+(t)\frac{\mathrm{d}^k}{\mathrm{d}t^k}y^+(t) + b_k^-(t)\frac{\mathrm{d}^k}{\mathrm{d}t^k}y^-(t+T^-(t))\right] = f_1(t) - \sum_{k=0}^{m}b_k^-(t)\frac{\mathrm{d}^k}{\mathrm{d}t^k}\Psi^-(t) \tag{7}$$

Considering (7) as an independent mathematical object, its solutions can be constructed by steps for $t > t_0 + \max\{T^-(t_0), T^+(0)\}$. Some initial conditions are needed for this procedure: they can be obtained starting from the initial conditions of (1) as follows.

Consider the case when $t^+(\cdot;\lambda,t)$ cannot be extended any longer up to $\lambda = 0$ but only to the point where the characteristic curve crosses the abscissa line $t = t_0$ i.e. $\hat{\sigma} = \hat{\lambda}(\lambda, t, t_0)$ deduced from $t^+(\hat{\sigma};\lambda,t) = t_0$. Consider next the first equality of (4) and write it down for $\lambda = \hat{\lambda}$, $t = t_0$

$$u^+(\hat{\lambda}, t_0) = u^+(1, t^+(1;\hat{\lambda}, t_0)) - \int_{\hat{\lambda}}^{1}\frac{\Phi^+(\sigma, t^+(\sigma;\hat{\lambda}, t_0))}{\tau^+(\sigma, t^+(\sigma;\hat{\lambda}, t_0))}\mathrm{d}\sigma$$

From here, using (6) and the initial conditions, it follows that:

$$y_0^+(t^+(1;\lambda,t_0)) = \omega^+(\lambda) + \int_{\lambda}^{1}\frac{\Phi^+(\sigma, t^+(\sigma;\lambda,t_0))}{\tau^+(\sigma, t^+(\sigma;\lambda,t_0))}\mathrm{d}\sigma \tag{8}$$

for $0 \le \lambda \le 1$, $t_0 \le t^+(1;\lambda,t_0) \le t_0 + T^+(t_0)$. Analogously

$$y_0^-(t^-(0;\lambda,t_0)) = \omega^-(\lambda) - \int_{0}^{\lambda}\frac{\Phi^-(\sigma, t^-(\sigma;\lambda,t_0))}{\tau^-(\sigma, t^-(\sigma;\lambda,t_0))}\mathrm{d}\sigma \tag{9}$$

for $0 \le \lambda \le 1$, $t_0 \le t^-(0;\lambda,t_0) \le t_0 + T^-(t_0)$. Obviously $y_0^{\pm}(\cdot)$ should be viewed together with their derivatives and this requires sufficiently smooth initial conditions $\omega^{\pm}(\cdot)$ and sufficiently smooth functions $\tau^{\pm}(\lambda,t)$. A discussion on the smoothness aspects will follow when discussing the type of the associated equations with deviated arguments.

Conversely, equalities (4) suggest the representation formulae

$$u^+(\lambda,t) = y^+(t^+(1;\lambda,t)) - \int_{\lambda}^{1}\frac{\Phi^+(\sigma, t^+(\sigma;\lambda,t))}{\tau^+(\sigma, t^+(\sigma;\lambda,t))}\mathrm{d}\sigma$$
$$u^-(\lambda,t) = y^-(t^-(0;\lambda,t)) + \int_{0}^{\lambda}\frac{\Phi^-(\sigma, t^-(\sigma;\lambda,t))}{\tau^-(\sigma, t^-(\sigma;\lambda,t))}d\sigma \tag{10}$$

for the solutions of (2). The following result is thus valid

Theorem 1. *Consider the initial boundary value problem (2) with* $\tau^{\pm}(\lambda, t)$ *sufficiently smooth e.g. of class* $\mathcal{C}^{m+1}$ *on* $[0, 1] \times [t_0, t_1]$ *and* $\omega^{\pm}(\cdot)$ *on* $[0, 1]$. *Then, if* $u^{\pm}(\lambda, t)$ *is a classical solution of (1), then* $y^{\pm}(t)$ *defined by (6) is a solution of (7) with the initial conditions defined by (8), (9). Conversely, let* $y^{\pm}(t)$ *be a sufficiently smooth solution of (7) with some initial conditions* $y_0^{\pm}(t)$ *defined on* $[t_0, t_0 + T^{\pm}(t_0)]$. *Then* $u^{\pm}(\lambda, t)$ *defined by (10) is a (possibly discontinuous) classical solution of (7) with the initial conditions defined also by (10) computed at* $t = t_0$.

The following remark of [14] is quite suggestive. Define the integers

$$\begin{aligned} & L^+ = \max\{k : a_k^+(t) \neq 0\}\ ,\ L^- = \max\{k : b_k^-(t) \neq 0\} \\ & K^+ = \max\{k : b_k^+(t) \neq 0\}\ ,\ K^- = \max\{k : a_k^-(t) \neq 0\} \\ & M = L^+ + L^- - (K^+ + K^-) \end{aligned} \tag{11}$$

According to the sign of M, system (7) belongs to one of the three classes of systems with deviated arguments: if $M > 0$, it is of delayed type; if $M < 0$, it is of advanced type; if $M = 0$, it is of neutral type. This assertion is consistent with all definitions and classifications from the standard reference [16, 17, 18], especially with the classification of G. A. Kamenskii (see in [17]). The aforementioned facts allow the assertion the *the most natural source of equations with deviated argument is represented by the boundary value problems for hyperbolic partial differential equations.* The reference list as well as the motivating Section 1, Chapter 1 of [18] speak for this statement. Worth mentioning also that the very first paper on a certain equation with deviated argument, belonging to J. Bernoulli [19], refers to the equation of the vibrating string - a hyperbolic partial differential equation.

It is now the place to discuss the smoothness problems mentioned in the text of Theorem 1. The statement of the theorem is general enough to view this theorem as giving an approach of associating the functional differential equations of a certain type to the boundary value problem with derivative boundary conditions.If the equations result of the retarded type, the solutions will be in any case smoother in time. The smoothness of the initial conditions $\omega^{\pm}(\lambda)$ has to be sufficient to ensure e.g. piecewise continuity on the first interval. In the neutral case, smoothness is only preserved hence it has to be imposed from the beginning. In the advanced type, smoothness is diminishing and its choice will depend on the existence interval required for the solution.It should be also clear that here, for simplicity, only classical solutions for the boundary value problem are considered. Obviously, these aspects can be specified according to the tackled problem.

As it will appear in the sequel, most systems with deviated argument associated to boundary value problems for hyperbolic partial differential equations modeling engineering applications are of neutral type. Consequently their solutions preserve the initial piecewise smoothness and they display discontinuities which propagate - as resulting from mismatched initial and boundary conditions.

We end this section by briefly considering the special case when $\tau^{\pm}(\lambda,t)\equiv \tau^{\pm}(\lambda)$ i.e time invariant propagation coefficients. Therefore the characteristic curves will be expressed as

$$t^{\pm}(\sigma;\lambda,t)=t+\int_{\lambda}^{\sigma}\frac{\mathrm{d}\xi}{\tau^{\pm}(\xi)}\ ,\ T^{\pm}=\pm\int_{0}^{1}\frac{\mathrm{d}\xi}{\tau^{\pm}(\xi)}>0 \tag{12}$$

the propagation times being constant. Equations (7) will display constant argument deviations - the attribute of lossless and/or distortionless propagation [10].

3 Two significant applications. A stability condition

3.1 The first application we consider significant arises from combined heat electricity generation. Two types of models are considered - linearized [20, 21] and nonlinear [22]. For the aims of this paper, the linearized model is sufficient. In fact even this model contains bilinear terms which are not important for the sequel. The equations are as follows

$$\begin{aligned}
&T_a\frac{\mathrm{d}s}{\mathrm{d}t}=\alpha\pi_1+(1-\alpha)\pi_2-\nu_g\\
&T_1\frac{\mathrm{d}\pi_1}{\mathrm{d}t}=\mu_1(t)-\pi_1\\
&T_p\frac{\mathrm{d}\pi_s}{\mathrm{d}t}=\pi_1-\beta_1\mu_2(t)\pi_s-\beta_2\xi_w(0,t)\\
&T_2\frac{\mathrm{d}\pi_2}{\mathrm{d}t}=\mu_2(t)\pi_s-\pi_2\\
&T_c\partial_t\xi_p+\partial_\lambda\xi_w=0\ ,\ \psi_c^2T_c\partial_t\xi_w+\partial_\lambda\xi_p=0\ ;\ 0\le\lambda\le1\ ,\ t>0\\
&\xi_w(0,t)+\alpha_p\xi_p(0,t)=\alpha_p\pi_s(t)\ ,\ \xi_w(1,t)-\psi_s\xi_p(1,t)=0
\end{aligned} \tag{13}$$

with the significance of the notations as in [20, 22]. Let us follow the procedure of the previous section. But we have to start by pointing out the

Riemann invariants of the problem

$$\xi^{\pm}(\lambda,t)=\frac{1}{2}\left[\pm\xi_w(\lambda,t)+\frac{1}{\psi_c}\xi_p(\lambda,t)\right] \tag{14}$$

to obtain the transformed boundary value problem

$$\begin{aligned}
&T_a\frac{\mathrm{d}s}{\mathrm{d}t}=\alpha\pi_1+(1-\alpha)\pi_2-\nu_g\\
&T_1\frac{\mathrm{d}\pi_1}{\mathrm{d}t}=\mu_1(t)-\pi_1\\
&T_p\frac{\mathrm{d}\pi_s}{\mathrm{d}t}=\pi_1-\beta_1\mu_2(t)\pi_s-\beta_2[\xi^+(0,t)-\xi^-(0,t)]\\
&T_2\frac{\mathrm{d}\pi_2}{\mathrm{d}t}=\mu_2(t)\pi_s-\pi_2\\
&\psi_c T_c\partial_t\xi^{\pm}\pm\partial_\lambda\xi^{\pm}=0\\
&(1+\alpha_p\psi_c)\xi^+(0,t)-(1-\alpha_p\psi_c)\xi^-(0,t)=\alpha_p\pi_s(t)\\
&(1+\psi_s\psi_c)\xi^-(1,t)-(1-\psi_s\psi_c)\xi^+(1,t)=0
\end{aligned} \tag{15}$$

With reference to system (1), here $K^+=L^+=K^-=L^-=0$ hence $M=0$. Proceeding as in the previous section we shall find

$$\begin{aligned}
&y^+(t):=\xi^+(1,t)\ \Rightarrow\ \xi^+(0,t)=y^+(t+\psi_cT_c)\\
&y^-(t):=\xi^-(0,t)\ \Rightarrow\ \xi^-(1,t)=y^-(t+\psi_cT_c)
\end{aligned} \tag{16}$$

and denoting $w^{\pm}(t):=y^{\pm}(t+\psi_cT_c)$, the following system of differential and difference equations is obtained

$$\begin{aligned}
&T_a\frac{\mathrm{d}s}{\mathrm{d}t}=\alpha\pi_1+(1-\alpha)\pi_2-\nu_g\\
&T_1\frac{\mathrm{d}\pi_1}{\mathrm{d}t}=\mu_1(t)-\pi_1\\
&T_p\frac{\mathrm{d}\pi_s}{\mathrm{d}t}=\pi_1-\left(\beta_1\mu_2(t)+\frac{\beta_2\alpha_p}{1+\alpha_p\psi_c}\right)\pi_s+\frac{2\beta_2\alpha_p}{1+\alpha_p\psi_c}w^-(t-\psi_cT_c)\\
&T_2\frac{\mathrm{d}\pi_2}{\mathrm{d}t}=\mu_2(t)\pi_s-\pi_2\\
&w^+(t)=\frac{1-\alpha_p\psi_c}{1+\alpha_p\psi_c}w^-(t-\psi_cT_c)+\frac{\alpha_p}{1+\alpha_p\psi_c}\pi_s(t)\\
&w^-(t)=\frac{1-\psi_s\psi_c}{1+\psi_s\psi_c}w^+(t-\psi_cT_c)
\end{aligned} \tag{17}$$

In stability studies, mainly the behavior for large $t > 0$ is important; in turn $w^-(t)$ can be eliminated and the equations where it was involved are re-written as follows

$$\begin{aligned} T_p\frac{\mathrm{d}\pi_s}{\mathrm{d}t} &= \pi_1 - \left(\beta_1\mu_2(t) + \frac{\beta_2\alpha_p}{1+\alpha_p\psi_c}\right)\pi_s + \frac{2\beta_2\alpha_p}{1+\alpha_p\psi_c}\cdot\frac{1-\psi_s\psi_c}{1+\psi_s\psi_c}w^+(t-2\psi_cT_c) \\ w^+(t) &= \frac{1-\alpha_p\psi_c}{1+\alpha_p\psi_c}\cdot\frac{1-\psi_s\psi_c}{1+\psi_s\psi_c}w^+(t-2\psi_cT_c) + \frac{\alpha_p}{1+\alpha_p\psi_c}\pi_s(t) \end{aligned} \tag{18}$$

We focus on the difference operator in the second equation of (18) and its role in stability. We consider the inherent stability of (18) thus taking firstly $\pi_1(t) \equiv 0$, $\mu_2(t) \equiv 0$. Then we make some notations and introduce a transformed state variable

$$\rho_1 := \frac{1-\alpha_p\psi_c}{1+\alpha_p\psi_c}\ ,\ \rho_2 := \frac{1-\psi_s\psi_c}{1+\psi_s\psi_c}\ ,\ T_p' := \frac{2\psi_cT_p}{1-\rho_1}\ ;\ \hat{\pi}_s := \frac{1-\rho_1}{2\psi_c}\pi_s, \tag{19}$$

to obtain the transformed system

$$\begin{aligned} T_p'\frac{\mathrm{d}\hat{\pi}_s}{\mathrm{d}t} &= -\beta_2\hat{\pi}_s + \beta_2(1-\rho_1)\rho_2w^+(t-2\psi_cT_c) \\ w^+(t) &= \rho_1\rho_2w^+(t-2\psi_cT_c) + \hat{\pi}_s(t) \end{aligned} \tag{20}$$

At its turn system (20) can be given the form of a standard neutral functional differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}(w^+(t) - \rho_1\rho_2w^+(t-2\psi_cT_c)) = -\frac{\beta_2}{T_p'}(w^+(t) - \rho_2w^+(t-2\psi_cT_c)) \tag{21}$$

This equation is of the type considered in the introductory Chapter 1, Section 1.7 of [23] - equation (1.7.5). Its characteristic equation belongs to the type

$$\lambda(1-d\mathrm{e}^{-\lambda r}) - a - b\mathrm{e}^{-\lambda r} = 0 \tag{22}$$

where $d = \rho_1\rho_2$, $a = -\beta_2/T_p'$, $b = a\rho_2$, $r = 2\psi_cT_c$. The role of d in (22) is displayed by Lemma 1.7.1 of [23], reproduced here for the sake of completeness

Lemma 1. *(Lemma 1.7.1 of [23]) There exists $\alpha \in \mathbb{R}$ such that all roots of (22) are subject to $\Re e(\lambda) < \alpha$. If $d \neq 0$, all solutions of (22) lie in a vertical strip $\beta < \Re e(\lambda) < \alpha$ in $\mathbb{C}$. If $d \neq 0$ and there is a sequence $\{\lambda_j\}_j$ of roots of (22) such that $|\lambda_j| \to \infty$ as $j \to \infty$, then there exists a sequence $\{\lambda_j'\}_j$ of roots of*

$$1 - d\mathrm{e}^{-\lambda r} = 0 \tag{23}$$

such that $\lambda_j - \lambda_j' \to 0$ *for* $j \to \infty$. *Moreover, there exists such a sequence* $\{\lambda_j\}_j$ *whenever* $d \neq 0$.

The following assertions are important. Equation (23) is the characteristic equation of the difference operator $\mathcal{D}\phi(\cdot) := \phi(0) - d\phi(-r)$, associated to the neutral functional differential equation (21) with the above notations. Its importance in stability analysis follows from the fact that the roots of (23) with $d \neq 0$ are given by

$$\lambda_k' = \begin{cases} \frac{1}{r}\ln\ d + \imath\frac{2k\pi}{r}\ ,\ k = 0, \pm 1 \pm 2, \dots,\ \text{if}\ d > 0 \\ \frac{1}{r}\ln\ |d| + \imath\frac{(2k+1)\pi}{r}\ ,\ k = 0, \pm 1 \pm 2, \dots,\ \text{if}\ d < 0 \end{cases} \tag{24}$$

From (24) it follows that if $\ln\ |d| > 0$ (for either $d < 0$ or $d > 0$), then Lemma 1 implies existence of infinitely many solutions of the basic equation

$$\frac{\mathrm{d}}{\mathrm{d}t}(y(t) - dy(t-r)) = ay(t) + by(t-r) \tag{25}$$

approaching ∞ exponentially. Therefore stability is never obtained. However, system (20) or, equivalently, equation (21), have $d = \rho_1\rho_2 \in (0,1)$ and stability can be obtained.

It appears that such aspects motivated the well known and popular assumption concerning the exponential stability of the difference operator associated to neutral functional differential equations, assumption which covered all studies of these equations and their applications. Most probably this assumption was introduced in [26] and [27].

3.2 We shall turn here to another important application arising from mechanical Engineering - oscillation quenching in distributed parameter oil drillstrings. The dynamics of oilwell drillstrings for vibration quenching became of increased interest in the last two - three decades when control approaches started to be involved in this matter. We mention here only two survey references - [24] and [25] and consider the model we deduced in [15] using the variational principle of Hamilton for distributed parameter systems. In the particular case of constant parameters and zero distributed damping,

$$\begin{aligned} &\rho\theta_{tt} - G\theta_{ss} = 0\ ;\ 0 < s < L\ ,\ t > 0 \\ &c_\ell\dot{\theta}_m + GI_p\theta_s(0,t) = 0\ ;\ J_m\ddot{\theta}_m + c_0\dot{\theta}_m + \dot{\theta}(0,t) = \tau(t) \\ &J_b\ddot{\theta}(L,t) + T(\dot{\theta}(L,t)) + GI_p\theta_s(L,t) = 0, \end{aligned} \tag{26}$$

$\theta(s,t)$ and $\theta_m(t)$ are cyclic variables. Denoting $\omega(s,t) := \theta_t(s,t)$, $\omega_m(t) = \dot{\theta}_m(t)$, system (26) becomes

$$\begin{aligned} &\rho\omega_t - G\theta_{ss} = 0\ ;\ c_\ell\omega_m = -GI_p\theta_s(0,t) \\ &J_m\dot{\omega}_m + c_0\omega_m + c_\ell\omega(0,t) = \tau(t) \\ &J_b\dot{\omega}(L,t) + T(\omega(L,t)) + GI_p\theta_s(L,t) = 0 \end{aligned} \tag{27}$$

In practice, control of the drillstring vibrations is ensured by controlling the rotating speed $\omega(0,t)$ at the surface; this goal is achieved by varying the motor torque $\tau(t)$ at the surface. The controller is designed as

$$\tau(t) - \bar{\tau} = -g_0(\omega_m(t) - \bar{\omega}_m) \tag{28}$$

where $g_0(\sigma)/\sigma > -c_0$ is a sector restricted nonlinearity. The controller thus introduces an additional damping at $s = 0$, the speed reference $\bar{\omega}_m$ equals $\bar{\omega}$ - the steady state rotating speed of the drillstring (physical reasons - the integrity of the string). The closed loop equations, obtained by combining (26) and (28) read

$$\begin{aligned} &\rho\theta_{tt} - G\theta_{ss} = 0\ ;\ c_\ell\dot{\theta}_m + GI_p\theta_s(0,t) = 0 \\ &J_m\ddot{\theta}_m + c_0\dot{\theta}_m + \dot{\theta}(0,t) = \bar{\tau}(\bar{\omega}) - g_0(\dot{\theta}_m - \bar{\omega}_m) \\ &J_b\ddot{\theta}(L,t) + T(\dot{\theta}(L,t)) + GI_p\theta_s(L,t) = 0, \end{aligned} \tag{29}$$

where $\bar{\tau}(\bar{\omega}) = c_0(\bar{\omega} - T(\bar{\omega})/c_\ell)$ - see [15]. For this system it was constructed the control Lyapunov functional

$$\begin{aligned} \mathcal{V}(\dot{\theta}_m, \dot{\theta}(L,t), \theta_t(\cdot,t), \theta_s(\cdot,t)) = \frac{1}{2}\Big\{ & J_m(\dot{\theta}_m - \bar{\omega})^2 + J_b(\dot{\theta}(L,t) - \bar{\omega})^2 + \\ & + I_p\int_0^L [\rho(\theta_t(s,t) - \bar{\omega})^2 + G(\theta_s(s,t) - \bar{\theta}_s(s))^2]\mathrm{d}s \Big\} \end{aligned} \tag{30}$$

whose derivative results negative semi-definite - see [15]. This fact ensures Lyapunov stability of the steady state, in the sense of the metrics induced by the Lyapunov functional (30) itself. To obtain asymptotic stability we use the approach of associating the system of functional differential equations following the methodology of Section 2 - see also [15]. We skip the intermediate development and give below the associated system

$$
\begin{aligned}
&(J_m I_p/c_\ell)\frac{\mathrm{d}}{\mathrm{d}t}(\eta^+(t)-\eta^-(t-L\sqrt{\rho/G}))+(I_p c_0/c_\ell)\sqrt{\rho G}(\eta^+(t)-\\
&-\eta^-(t-L\sqrt{\rho/G}))+c_\ell(\eta^+(t)+\eta^-(t-L\sqrt{\rho/G}))+\\
&+g_0((I_p/c_\ell)(\eta^+(t)-\eta^-(t-L\sqrt{\rho/G})))=0\\
&J_b\frac{\mathrm{d}}{\mathrm{d}t}(\eta^-(t)+\eta^+(t-L\sqrt{\rho/G}))+T(\bar\omega+\eta^-(t)+\eta^+(t-L\sqrt{\rho/G}))-\\
&-T(\bar\omega)+I_p\sqrt{\rho G}(\eta^-(t)-\eta^+(t-L\sqrt{\rho/G}))=0
\end{aligned}
\tag{31}
$$

This system is obviously of neutral type: its difference operator is defined by

$$
\mathcal{D}\begin{pmatrix}\eta^+(\cdot)\\ \eta^-(\cdot)\end{pmatrix}=\begin{pmatrix}\eta^+(0)\\ \eta^-(0)\end{pmatrix}+\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\cdot\begin{pmatrix}\eta^+(-L\sqrt{\rho/G})\\ \eta^-(-L\sqrt{\rho/G})\end{pmatrix}
\tag{32}
$$

The matrix D of the operator has its eigenvalues $\pm\imath$ and the characteristic equation of the difference operator reads

$$
1+\mathrm{e}^{-\lambda r}=0\ ,\ r=L\sqrt{\rho/G}
\tag{33}
$$

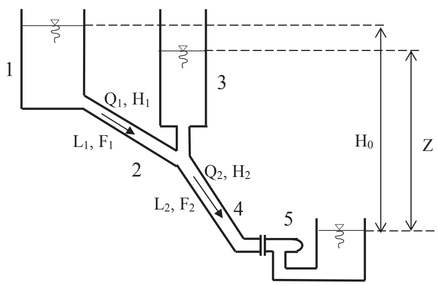
with its roots $\lambda_k=\imath(2k+1)\pi/(2r)$, $k=0,\pm1,\pm2,\ldots$.

If we have in mind a generalization of Lemma 1 to the vector case, then at least the linearized version of (31) will have chains of eigenvalues approaching the roots λ_k for $k\to\infty$. Consequently the linearized system (31) can have an infinity of solutions displaying oscillatory behavior. The nonlinear system (31) has two sector restricted nonlinearities - $T(\cdot)$ and $g_0(\cdot)$. The absolute stability theory in its frequency domain version [28] does not cover the critical case with infinitely many roots of the linear part located on the imaginary axis $\imath\mathbb{R}$. We do not know anything within the framework of the Lyapunov function(al) approach.

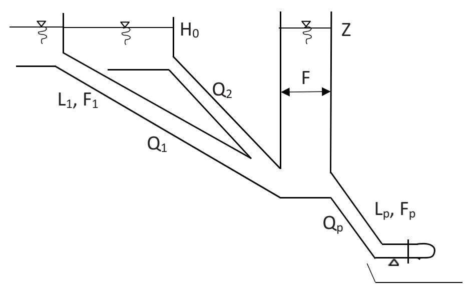
We end this section by mentioning that all applications arising from Mechanical Engineering, described in [15], display the same critical case of the difference operators associated to the systems of neutral functional differential equations involved in the aforementioned systems. A direct consequence of this property is that Theorem 9.8.2 of [23] - the Barbashin Krasovskii LaSalle invariance principle - cannot be applied, see also [29, 30]

4 Two applications in water hammer - hydraulic engineering

Water hammer is an abnormal regime of the hydraulic power plants occurring after a sudden large load discharge of the hydraulic turbine. It consists of large water mass oscillations which may result in water conduit breaking, flood a.o. Its regulating "device" (in fact - construction) is the surge tank - an energy dissipator also. For the purpose of the paper we shall consider the two plant configurations of figure 1a,b



a) Hydroelectric plant with a single tunnel:1. Lake 2. Tunnel 3. Surge tank 4. Penstock 5. Hydraulic turbine



b) Hydroelectric plant with two tunnels and common lake.

Figure 1: Hydroelectric plants with surge tank

The mathematical model of the water hammer processes is based on the Saint Venant partial differential equations under the following assumptions which are covering from the engineering point of view since they refer to energy dissipation: i) the dynamic (velocity) heads and the distributed (Darcy-Weisbach) losses are neglected; ii) the water hammer is generated by the complete shutdown of the hydraulic turbine.

The models of the analyzed hydroelectric power plants have been considered and processed by introducing rated (*per unit* - p.u) variables, also a rated independent variable - the "time". Details, as well as a list of variables and parameters significance, can be found in our previously published [31, 32, 33, 34, 35].

4.1 Consider firstly the "standard" configuration of figure 1a. Its model appeared in [36], next in [37]. The model as resulting from its processing as previously described reads as follows

$$
\begin{aligned}
&\theta_{wi}\partial_\tau q_i + \partial_{\xi_i} h_i = 0 \ , \ \delta_i^2\theta_{wi}\partial_\tau h_i + \partial_{\xi_i} q_i = 0 \ , \ i = 1,2 \ ; \ 0 < \xi_i < 1 \ , \ t > 0 \\
&h_1(0,\tau) \equiv 1 \ ; \ \theta_s \frac{\mathrm{d}z}{\mathrm{d}\tau} = q_1(1,\tau) - q_2(0,\tau) \\
&h_1(1,\tau) = 1 + z(\tau) + \lambda_s \frac{\mathrm{d}z}{\mathrm{d}\tau} = h_2(0,\tau) \ ; \ q_2(1,\tau) \equiv 0
\end{aligned}
\tag{34}
$$

The boundary condition $q_2(1,\tau) \equiv 0$ accounts for the complete shutdown of the hydraulic turbine. The parameter λ_s shows that the surge tank has throttling. Briefly the things are as follows. The turbine shutdown imposes a new steady state of (34) defined by

$$
\begin{aligned}
&\bar{h}_i(\xi_i) \equiv \text{const} \ , \ \bar{q}_i(\xi_i) \equiv \text{const} \ ; \ \bar{h}_1(0) = 1 \ , \ \bar{q}_1(1) = \bar{q}_2(0) \\
&\bar{h}_1(1) = 1 + \bar{z} = \bar{h}_2(0) \ , \ \bar{q}_2(1) = 0
\end{aligned}
\tag{35}
$$

hence $\bar{h}_1 = \bar{h}_2 = 1$, $\bar{z} = 0$, $\bar{q}_1 = \bar{q}_2 = 0$. Introducing the deviations $\chi_i(\xi_i,\tau) = h_i(\xi_i,\tau) - 1$, the following system in deviations is obtained

$$
\begin{aligned}
&\theta_{wi}\partial_\tau q_i + \partial_{\xi_i} \chi_i = 0 \ , \ \delta_i^2\theta_{wi}\partial_\tau \chi_i + \partial_{\xi_i} q_i = 0 \ , \ i = 1,2 \ ; \ 0 < \xi_i < 1 \ , \ t > 0 \\
&\chi_1(0,\tau) \equiv 0 \ ; \ \theta_s \frac{\mathrm{d}z}{\mathrm{d}\tau} = q_1(1,\tau) - q_2(0,\tau) \\
&\chi_1(1,\tau) = z(\tau) + \lambda_s \frac{\mathrm{d}z}{\mathrm{d}\tau} = \chi_2(0,\tau) \ ; \ q_2(1,\tau) \equiv 0
\end{aligned}
\tag{36}
$$

For this system the energy identity suggests the following Lyapunov functional (written as a state functional)

$$
\mathcal{V}(z,\phi_i(\cdot),\psi_i(\cdot)) = \frac{1}{2}\left\{\theta_s z^2 + \sum_1^2 \theta_{wi}\int_0^1 [\phi_i^2(\xi_i) + \delta_i^2\psi_i^2(\xi_i)]\mathrm{d}\xi_i\right\} \tag{37}
$$

This functional is written along the solutions of (36) as $\mathcal{V}^\star(z(\tau), q_i(\cdot,\tau), \chi_i(\cdot,\tau))$. Differentiating it along the solutions and taking into account the energy identity, the following inequality is obtained

$$
\frac{\mathrm{d}\mathcal{V}^\star}{\mathrm{d}\tau} = -\lambda_s \left(\frac{\mathrm{d}z}{\mathrm{d}\tau}\right)^2 \le 0 \tag{38}
$$

Inequality (38) displays Lyapunov stability of (36), in the sense of the metrics induced by the Lyapunov functional itself. For the asymptotic stability, application of the Barbashin Krasovskii LaSalle invariance principle is suitable. This principle holds for neutral functional differential equations (Theorem 9.8.2 of [23]). Consequently we apply the methodology of Section 2 and associate the following system with deviated argument

$$
\begin{aligned}
&(1+(\delta_1+\delta_2)\lambda_s)\theta_s\frac{\mathrm{d}z}{\mathrm{d}\tau}=-(\delta_1+\delta_2)z(\tau)-2\eta_1^-(\tau-2\delta_1\theta_{w1})+\\
&\qquad\qquad +2\eta_2^+(t-2\delta_2\theta_{w2})\\
&(1+(\delta_1+\delta_2)\lambda_s)\eta_1^-(\tau)=\delta_1 z(\tau)+(1+(\delta_2-\delta_1)\lambda_s)\eta_1^-(\tau-2\delta_1\theta_{w1})+\\
&\qquad\qquad +2\delta_1\lambda_s\eta_2^+(t-2\delta_2\theta_{w2})\\
&(1+(\delta_1+\delta_2)\lambda_s)\eta_2^+(\tau)=\delta_2 z(\tau)-2\delta_2\lambda_s\eta_1^-(\tau-2\delta_1\theta_{w1})-\\
&\qquad\qquad -(1+(\delta_1-\delta_2)\lambda_s\eta_2^+(t-2\delta_2\theta_{w2})
\end{aligned}
\tag{39}
$$

On the set where the derivative of of the Lyapunov functional vanishes we shall have

$$
z(\tau)=\frac{2}{\delta_1+\delta_2}[-\eta_1^-(\tau-2\delta_1\theta_{w1})+\eta_2^+(t-2\delta_2\theta_{w2})] \tag{40}
$$

and the difference subsystem of (39) considered on this set will be

$$
\begin{aligned}
&(\delta_1+\delta_2)\eta_1^-(\tau)=(\delta_2-\delta_1)\eta_1^-(\tau-2\delta_1\theta_{w1})+2\delta_1\eta_2^+(t-2\delta_2\theta_{w2})\\
&-2\delta_2\eta_1^-(\tau-2\delta_1\theta_{w1})-(\delta_1-\delta_2)\eta_2^+(t-2\delta_2\theta_{w2})
\end{aligned}
\tag{41}
$$

The invariant set included in the set where $\mathrm{d}\mathcal{V}^\star/\mathrm{d}\tau=0$ consists of the constant solutions of (40)-(41). The unique constant solution of (41) is $\bar{\eta}_1^-=\bar{\eta}_2^+=0$ hence $\bar{z}=0$. Applying Theorem 9.8.2 of [23] we obtain global asymptotic stability hence of (36) - due to the one-to-one correspondence between the solutions of the two systems. However, application of Theorem 9.8.2 of [23] is conditioned by the asymptotic stability of the difference operator. Here the difference operator displays two delays

$$
\begin{aligned}
\mathcal{D}\begin{pmatrix}\eta_1^-(\cdot)\\ \eta_2^+(\cdot)\end{pmatrix}&=\begin{pmatrix}\eta_1^-(0)\\ \eta_2^+(0)\end{pmatrix}-\begin{pmatrix}\rho_1 & 0\\ -(1-\rho_2) & 0\end{pmatrix}\begin{pmatrix}\eta_1^-(-2\delta_1\theta_{w1})\\ \eta_2^+(-2\delta_1\theta_{w1})\end{pmatrix}-\\
&-\begin{pmatrix}0 & 1-\rho_1\\ 0 & \rho_2\end{pmatrix}\begin{pmatrix}\eta_1^-(-2\delta_2\theta_{w2})\\ \eta_2^+(-2\delta_2\theta_{w2})\end{pmatrix}
\end{aligned}
\tag{42}
$$

where

$$\rho_1 = \frac{1 + (\delta_2 - \delta_1)\lambda_s}{1 + (\delta_1 + \delta_2)\lambda_s}\ ,\ \rho_1 = \frac{1 + (\delta_1 - \delta_2)\lambda_s}{1 + (\delta_1 + \delta_2)\lambda_s}$$

The characteristic equation of the difference operator reduces to

$$\begin{aligned} & z^{\nu+1} + \rho_2 z^{\nu} - \rho_1 z + 1 - \rho_1 - \rho_1 = 0; \\ & \nu := (\delta_1\theta_{w1})(\delta_2\theta_{w2})^{-1} + 1 - \rho_1 - \rho_2 = 0 \end{aligned} \tag{43}$$

and asymptotic stability means its roots are subject to $|z| < 1$. However, if there are no roots such that $|z| > 1$, for $\nu = p/q$ rational and irreducible, if p is odd, all roots are subject to $|z| < 1$. If p is even (and q - odd) then one root equals -1 *and the system is in critical case*. We have thus asymptotic stability of the difference operator - and, therefore, of systems (39) and (36) - only for a countable set of values of the delay ratio. Referring now to Theorem 9.6.1 of [23], dealing with *strong stability* of the difference operator, it is stated there that existence of at least one set of rationally independent delays i.e. *one* irrational ν for asymptotic stability of the difference operator is equivalent to its spectral radius less than 1. In the case of (42) this means for the equation with complex coefficients

$$(z - \rho_1)(z + \rho_2 e^{\imath\theta}) + (1 - \rho_1)(1 - \rho_2)e^{\imath\theta} = 0 \tag{44}$$

to have its roots subject to $|z| < 1$ for $\forall\theta \in [0, 2\pi)$. It is quite obvious however that for $\theta = \pi$, one root is $z = 1$ while the other is $z = (\rho_1 + \rho_2 - 1) \in (0, 1)$. Consequently, according to Theorem 9.6.1 of [23], there is *no irrational* ν to ensure asymptotic stability of the difference operator.

We deduce that asymptotic stability of system (39) - and (36) - is possible for a countable set of delay ratios - a rational number with *odd numerator*. We call this property, firstly signaled in [36], *fragile stability*.

4.2 The plant structure of figure 1b was considered in an early paper [38] under lumped parameters and re-analyzed in [37] since it is very much alike to the "Tismana" hydroelectric power plant in Romania. The model with distributed parameters, corresponding to fast water mass oscillations, has been proposed in a seminar discussion as a challenge. The idea was to tackle it under the same assumptions as (34) in order to point out (possibly) the same "arithmetic conditions" of stability. The assumptions being similar to those leading to (34), the rating of the state variables and the parameters being the same, the equations are as follows

$$
\begin{aligned}
&\theta_{wi}\partial_\tau q_i + \partial_{\xi_i} h_i = 0\ ,\ \delta_i^2\theta_{wi}\partial_\tau h_i + \partial_{\xi_i} q_i = 0\ ,\ i=1,2\ ;\ 0<\xi_i<1\ ,\ t>0\\
&h_1(0,\tau) = h_2(0,\tau) \equiv 1\ ;\ \theta_s\frac{\mathrm{d}z}{\mathrm{d}\tau} = q_1(1,\tau) + q_2(0,\tau) - q_p(0,\tau)\\
&h_1(1,\tau) = h_2(1,\tau) = 1 + z(\tau) + \lambda_s\frac{\mathrm{d}z}{\mathrm{d}\tau} = h_p(0,\tau)\ ;\ q_p(1,\tau)\equiv 0
\end{aligned}
\tag{45}
$$

Here, as in figure 1b, the indexes $i = 1, 2$ account for the two tunnels while $i = p$ designates the common penstock. The last boundary condition shows that, in this case also, the water hammer is generated by complete turbine shutdown. Also in this case, the turbine shutdown defines a new steady state described by

$$
\begin{aligned}
&\bar{h}_i(\xi_i) \equiv \text{const}\ ,\ \bar{q}_i(\xi_i) \equiv \text{const}\ ;\ \bar{h}_1(0) = \bar{h}_2(0) = 1\ ,\ \bar{q}_p = \bar{q}_1 + \bar{q}_2\\
&\bar{h}_1(1) = \bar{h}_2(1) = 1 + \bar{z} = \bar{h}_p(1)\ ,\ \bar{q}_p = 0
\end{aligned}
\tag{46}
$$

The result is

$$
\bar{h}_1 = \bar{h}_2 = \bar{h}_p = 1\ ,\ \bar{z} = 0\ ;\ \bar{q}_p = 0\ ,\ \bar{q}_1 + \bar{q}_2 = 0 \tag{47}
$$

and the first remark is that $\bar{q}_1$ and $\bar{q}_2$ are not uniquely determined. Observe that $\bar{q}_1 = \bar{q}_2 = 0$ is one of the possible solutions. Moreover, $\bar{q}_1 = -\bar{q}_2$ suggests the possibility of the flow back into the lake. This senseless situation occurs from the fact that the model is completely lossless hence a flow upstream is not impossible.

To complete this preliminary analysis, let us consider the slow water mass oscillations by assuming $\delta_i^2\theta_{wi}$ a small time constant (usually $\delta_i \ll 1$) and introducing formally the singular perturbations. We obtain $q_i(\xi_i,\tau) \equiv q_i(\tau)$ and integrate with respect to ξ_i to obtain

$$
\begin{aligned}
&\theta_{wi}\frac{\mathrm{d}q_i}{\mathrm{d}\tau} + h_i(1,\tau) - h_i(0,\tau)\ ,\ i=1,2\ ;\ \theta_{wp}\frac{\mathrm{d}q_p}{\mathrm{d}\tau} + h_p(1,\tau) - h_p(0,\tau)\\
&q_p(\tau)\equiv 0\ ;\ \theta_s\frac{\mathrm{d}z}{\mathrm{d}\tau} = q_1(\tau) + q_2(\tau) - q_p(\tau)
\end{aligned}
\tag{48}
$$

If the boundary conditions on h_i are taken into account, equations (48) become

$$
\begin{aligned}
&\theta_{w1}\frac{\mathrm{d}q_1}{\mathrm{d}\tau} + (\lambda_s/\theta_s)(q_1+q_2) + z = 0\\
&\theta_{w2}\frac{\mathrm{d}q_2}{\mathrm{d}\tau} + (\lambda_s/\theta_s)(q_1+q_2) + z = 0\ ;\ \theta_s\frac{\mathrm{d}z}{\mathrm{d}\tau} = q_1 + q_2
\end{aligned}
\tag{49}
$$

If $\lambda_s = 0$ (the surge tank without throttling), the case of [38] is re-discovered but without the local hydraulic losses in the first two equations. The steady state solution is again $\bar{q}_1 + \bar{q}_2 = 0$, $\bar{z} = 0$. The conclusion is as follows. *Introduction of at least local hydraulic losses is necessary to ensure uniqueness of the steady state.* Too much idealization is *harmful*! However, system (49) has as invariant set (in fact as *prime integral*)

$$\theta_{w1} q_1(\tau) - \theta_{w2} q_2(\tau) \equiv \text{const}$$

allowing a detailed stability analysis. This prime integral *might exist also for system (45)* but discovering it is outside the purpose of this paper.

Since system (45) is linear, we can introduce the deviations with respect to *some* steady state and discuss stability. Let $\bar{h}_i = 1$, $\bar{z} = 0$, $\bar{q}_i = 0$ be this steady state and consider the system in deviations

$$\begin{aligned}
&\theta_{wi}\partial_\tau q_i + \partial_{\xi_i}\chi_i = 0\ ,\ \delta_i^2\theta_{wi}\partial_\tau\chi_i + \partial_{\xi_i} q_i = 0\ ,\ i = 1,2\ ;\ 0 < \xi_i < 1\ ,\ t > 0 \\
&\chi_1(0,\tau) = \chi_2(0,\tau) \equiv 0\ ;\ \theta_s \frac{\mathrm{d}z}{\mathrm{d}\tau} = q_1(1,\tau) + q_2(0,\tau) - q_p(0,\tau) \\
&\chi_1(1,\tau) = \chi_2(1,\tau) = z(\tau) + \lambda_s \frac{\mathrm{d}z}{\mathrm{d}\tau} = \chi_p(0,\tau)\ ;\ q_p(1,\tau) \equiv 0
\end{aligned} \tag{50}$$

The energy identities being the same, we consider the Lyapunov functional similar to (37) namely

$$\mathcal{V}(z, \phi_i(\cdot), \psi_i(\cdot)) = \frac{1}{2}\left\{\theta_s z^2 + \sum_i \theta_{wi}\int_0^1 [\phi_i^2(\xi_i) + \delta_i^2 \psi_i^2(\xi_i)]\mathrm{d}\xi_i\right\}, \tag{51}$$

its derivative along the solutions of (50) having *exactly* the form (38). As in the previous case, Lyapunov stability is obtained, in the sense of the metrics induced by the Lyapunov functional itself.

Again, for the asymptotic stability, application of the Barbashin Krasovskii LaSalle principle is suitable. This principle being proven for neutral functional differential equations [23] (Theorem 9.8.2), we use again the methodology of Section 2 and associate the following system of delay differential and difference equations

$$
\begin{aligned}
&(1+(\delta_1+\delta_2+\delta_p)\lambda_s)\theta_s\frac{\mathrm{d}z}{\mathrm{d}\tau}=-(\delta_1+\delta_2+\delta_p)z(\tau)- \\
&\qquad -2(w_1^-(\tau-2\delta_1\theta_{w1})+w_2^-(\tau-2\delta_2\theta_{w2})-w_p^-(\tau-2\delta_p\theta_{wp})) \\
&(1+(\delta_1+\delta_2+\delta_p)\lambda_s)\begin{pmatrix} w_1^-(\tau) \\ w_2^-(\tau) \\ w_p^+(\tau)\end{pmatrix}=\begin{pmatrix}\delta_1 \\ \delta_2 \\ \delta_p\end{pmatrix}z(\tau)+ \\
&+\begin{pmatrix} 1+(\delta_2+\delta_p-\delta_1)\lambda_s & -2\delta_1\lambda_s & 2\delta_1\lambda_s \\ -2\delta_2\lambda_s & 1+(\delta_1+\delta_p-\delta_2)\lambda_s & 2\delta_2\lambda_s \\ -2\delta_p\lambda_s & -2\delta_p\lambda_s & -(1+(\delta_1+\delta_2-\delta_p)\lambda_s)\end{pmatrix}\times \\
&\times\begin{pmatrix} w_1^-(\tau-2\delta_1\theta_{w1}) \\ w_2^-(\tau-2\delta_2\theta_{w2}) \\ w_p^-(\tau-2\delta_p\theta_{wp})\end{pmatrix}
\end{aligned}
\tag{52}
$$

This system is associated with the representation formulae

$$
\begin{aligned}
&q_i(\xi_i,\tau)=r_i^+(\xi_i,\tau)-r_i^-(\xi_i,\tau)=w_i^+(\tau-\delta_i\theta_{wi})-w_i^-(\tau+\delta_i\theta_{wi}(\xi_i-1)) \\
&\chi_i(\xi_i,\tau)=\frac{1}{\delta_i}[r_i^+(\xi_i,\tau)+r_i^-(\xi_i,\tau)]=\frac{1}{\delta_i}[w_i^+(\tau-\delta_i\theta_{wi})+ \\
&\qquad +w_i^-(\tau+\delta_i\theta_{wi}(\xi_i-1))]
\end{aligned}
\tag{53}
$$

Based on (53) and on the difference equations incorporated in (52)

$$
w_i^+(\tau)=-w_i^-(\tau-\delta_i\theta_{wi})\ ,\ i=1,2\ ;\ w_p^-(\tau)=w_p^+(\tau-\delta_p\theta_{wp})
$$

the Lyapunov functional is represented as

$$
\begin{aligned}
\mathcal{V}(z,w_1^-(\cdot),w_2^-(\cdot),w_p^+(\cdot))&=\frac{1}{2}\theta_s z(\tau)^2+\sum_1^2\frac{1}{\delta_i}\int_{-2\delta_i\theta_{wi}}^0 w_i^-(\tau+\vartheta)^2\mathrm{d}\vartheta+ \\
&+\frac{1}{\delta_p}\int_{-2\delta_p\theta_{wp}}^0 w_p^+(\tau+\vartheta)^2\mathrm{d}\vartheta,
\end{aligned}
\tag{54}
$$

the derivative functional $\mathcal{W}$ remaining the same as in (38). On the set where $\mathcal{W}$ vanishes we shall have

$$z(\tau) = \frac{-1}{\delta_1 + \delta_2 + \delta_p}[w_1^-(\tau - 2\delta_1\theta_{w1}) + w_2^-(\tau - 2\delta_2\theta_{w2}) - w_p^-(\tau - 2\delta_p\theta_{wp})] \tag{55}$$

while the difference system on this set becomes, by substituting $z(\tau)$ from (55)

$$(\delta_1 + \delta_2 + \delta_p)\begin{pmatrix} w_1^-(\tau) \\ w_2^-(\tau) \\ w_p^+(\tau) \end{pmatrix} = \begin{pmatrix} \delta_2 + \delta_p - \delta_1 & -2\delta_1 & 2\delta_1 \\ -2\delta_2 & \delta_1 + \delta_p - \delta_2 & 2\delta_2 \\ -2\delta_p & -2\delta_p & \delta_1 + \delta_2 - \delta_p \end{pmatrix} \times \\ \times \begin{pmatrix} w_1^-(\tau - 2\delta_1\theta_{w1}) \\ w_2^-(\tau - 2\delta_2\theta_{w2}) \\ w_p^-(\tau - 2\delta_p\theta_{wp}) \end{pmatrix} \tag{56}$$

Since $z(\tau) \equiv$ const on the set where $\mathcal{W}$ vanishes, we look for the constant solutions of (56). They are subject to

$$\begin{aligned} &\bar{w}_1^- + \bar{w}_2^- - \bar{w}_p^+ = 0 \\ &\bar{w}_1^- + \bar{w}_2^- - \bar{w}_p^+ = 0 \\ &\delta_p(\bar{w}_1^- + \bar{w}_2^-) + (\delta_1 + \delta_2)\bar{w}_p^+ = 0 \end{aligned} \tag{57}$$

and this system is under-determined and has a set of solutions given by $\bar{w}_p^+ = 0$, $\bar{w}_1^- = -\bar{w}_2^-$, which sends to the set of solutions of (47). The Barbashin Krasovskii LaSalle invariance principle ensures in this case that all bounded trajectories will approach asymptotically this invariant set of constant solutions.

Application of the invariance principle is however conditioned by the stability of the difference operator, as in the previous case. This aspect is still *under research* but the approach is much alike to the previous Subsection 4.1. The challenging aspect is the fact that now the difference operator displays *three delays*. We end this subsection by mentioning that besides the three-delay feature, there are other open problems: the aforementioned stability of a stationary set and of the equilibria inside it and the inclusion of the local hydraulic losses in the model, having as consequence the uniqueness of the equilibrium point under water hammer.

5 Huygens synchronization and complex behavior

According to a well established convention [39], Huygens synchronization means synchronization of lumped parameter oscillators through distributed parameter media, with reference to the classical Huygens experiment. Several cases for Huygens synchronization of mechanical and electrical oscillators have been discussed under a unitary framework in [40]. For the purpose of this study we choose the mechanical oscillator synchronization problem pointed out in [41], also considered in [40].

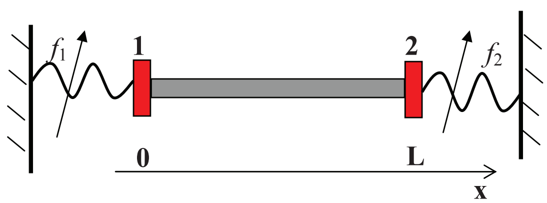


Figure 2: Two Van der Pol oscillators on the elastic rod

The basic equations of this mechanical structure are as follows [41, 40]

$$\begin{aligned} &y_{tt} - c^2 y_{xx} = 0 \ ; \ 0 < x < L \ , \ t > 0 \ ; \ c^2 = E/\rho \\ &m_1 \ddot{z}_1 + f_1(z_1, \dot{z}_1) = ES y_x(0,t) \\ &m_2 \ddot{z}_2 + f_2(z_2, \dot{z}_2) = -ES y_x(L,t) \end{aligned} \tag{58}$$

where the restoring terms $f_i(z_i, \dot{z}_i)$ account for the type of the oscillating system: for instance, if $f_i(z_i, \dot{z}_i) = f_i(z_i)$, the oscillator is a conservative system, displaying only closed (cyclic) trajectories in the phase plane. The Liénard oscillator, in particular, the Van der Pol oscillator, can be also obtained from (58) provided $f_i(z_i, \dot{z}_i) = -\varepsilon(1 - z_i^2)\dot{z}_i$ for the Van der Pol oscillator and $f_i(z_i, \dot{z}_i) = h_i(z_i)\dot{z}_i + g_i(z_i)$ for the Liénard equation. Applying the standard approach of Section 2 we associate to (58) the following nonlinear system of functional differential equations

$$\begin{aligned} &m_1 \ddot{z}_1 + f_1(z_1, \dot{z}_1) + \frac{ES}{c}\dot{z}_1 = 2\frac{ES}{c}\eta^-(t - L/c) \\ &m_2 \ddot{z}_2 + f_2(z_2, \dot{z}_2) + \frac{ES}{c}\dot{z}_2 = 2\frac{ES}{c}\eta^+(t - L/c) \\ &\eta^+(t) = \dot{z}_1(t) - \eta^-(t - L/c) \\ &\eta^-(t) = \dot{z}_2(t) - \eta^+(t - L/c) \end{aligned} \tag{59}$$

together with the representation formulae

$$
\begin{aligned}
v(x,t) &= \eta^-(t+(x-L)/c) + \eta^+(t-x/c) \\
w(x,t) &= \frac{1}{c}[\eta^-(t+(x-L)/c) - \eta^+(t-x/c)]
\end{aligned}
\tag{60}
$$

where $v(x,t) := y_t(x,t)$, $w(x,t) := y_x(x,t)$. Observe the terms $(ES/c)\dot{z}_i$ - additional damping factors in the equations of the lumped oscillators. Physicists call this aspect "dissipation radiation" but the approach described in Section 2 offers a more rigorous modeling of the phenomenon. The effect on the local oscillators is *oscillation quenching*: the interaction with the distributed medium (rod, string) quenches the local self-sustained oscillators – *synchronization to zero*. This phenomenon was described in [40] in the case of two electronic Van der Pol oscillators coupled through a lossless LC transmission line. With the mechanical oscillators this is no longer the case. System (59) belongs to the more general class described by

$$
\begin{aligned}
&\dot{x} = A_0 x(t) + A_1 y(t-\tau) - \sum_1^r b_m \phi_m(\sigma_m(t)) \\
&y(t) = A_2 x(t) + A_3 y(t-\tau)\ ,\ \sigma_m = c_m^* x
\end{aligned}
\tag{61}
$$

with A_3 having its eigenvalues on the unit circle. The second, difference equation can be represented in "integral" form as follows

$$
y(t) = A_3^k y_{\mathrm{o}}(t-k\tau) + \sum_0^{k-1} A_3^i A_2 x(t-i\tau)\ ,\ (k-1)\tau \le t < k\tau \tag{62}
$$

to obtain the following "integro-differential" system

$$
\begin{aligned}
\dot{x} = A_0 x(t) &+ \sum_0^{k-1} A_3^i A_2 x(t-i\tau) - \sum_1^r b_m \phi_m(\sigma_m(t)) + \\
&+ A_3^k y_{\mathrm{o}}(t-k\tau)\ ,\ k\tau \le t < (k+1)\tau
\end{aligned}
\tag{63}
$$

In (61) and (62) the integer k takes the values $\pm 1, \pm 2, \ldots$

Observe that the forcing term of (63) is almost periodic and it might be "responsible" for what was called in [41] - *complex behavior*, see also [42]. The explanation comes again from the fact that A_3 has, in the case of (59), its eigenvalues ± 1, thus on the unit circle.

6 Conclusions. Old problems remaining open

In this paper we started from the methodology associating to propagation systems (i.e described by initial boundary value problems for lossless/distortionless $1D$ hyperbolic partial differential equations) a system of functional differential equations with deviated argument, with particular reference to neutral functional differential equations. From the broad list of applications described by such systems and equations we chose several applications from Mechanical and Hydraulic Engineering, having the property that the difference operator of the associated neutral functional differential equations is only marginally stable (in a critical case), while the basic assumption occurring from the seminal memoir [26] was asymptotic stability of the difference operator. Our paper focused on the fact that marginal stability of the difference operator forbids application of the Barbashin Krasovskii LaSalle invariance principle (which is conditioned by the asymptotic stability of the difference operator). As a consequence, the aforementioned systems have only Lyapunov non-asymptotic stability obtained from energy-like Lyapunov function(al)s (which are "weak" Lyapunov function(al)s, as it is stated by N. G. Četaev). These aspects are quite well known since the 1977 edition of [23], which integrates the earlier results of of M. A. Cruz and J. K. Hale [27], see also the general list of references in [23].

While the strong stability assumption on the difference operator turned very fruitful, it however obscured those cases where it was not valid. In this way, a series of mathematical contributions to the critical case analysis [43, 44, 45, 46, 47] were forgotten.

Here we feel it would be useful to state *a beginning of a conjecture* - dealing with the linear case and associated characteristic equations. *Larger is the modulus of the eigenvalues approaching* $\imath\mathbb{R}$ *(in the line of Lemma 1), smaller is the modulus of the associated mode of the system.* In this way the weakly damped oscillations are "filtered" by system's dynamics. Here we consider helpful those studies and monographs dealing with the roots of entire functions, in particular quasi-polynomials [48, 49, 50].

Another line of research would be to establish a connection between the various difference operators and the dissipative/conservative boundary conditions of the corresponding boundary value problems for hyperbolic partial differential equations - see [51], pp. 161-163.

Meanwhile, the problem of applying the Barbashin Krasovskii LaSalle invariance principle without the aforementioned assumption on asymptotic stability of the difference operator has been also discussed [52]. More precisely, it is stated in Chapter VI, Section 8 dedicated to neutral functional differential equations, that the point is to obtain precompact positive orbits from boundedness, without the Ascoli theorem. One approach is to have an asymptotically stable difference operator, thus ensuring some smoothing of the solutions of neutral functional differential equations. If this is not possible (the difference operator is e.g. marginally stable - in critical case!), then it is suggested to embed the semi-dynamical system in a space wherein the positive orbits are precompact. This approach is illustrated in Chapter V, Section 4 of [52]. Worth mentioning that the application chosen in Chapter V originated from an initial boundary value problem for $1D$ hyperbolic partial differential equations!

Consequently this approach seems correct and fruitful to tackle the neutral functional differential equations with marginally stable difference operator. Additionally we have to refer to [53] but also to [54]; in this last reference it is given an example of linear neutral functional differential equation with marginally stable difference operator which is *asymptotically stable but not exponentially stable.*

It is felt that all aforementioned aspects can stimulate a research outside the standard paradigm which might be rewarding.

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