Ann. Acad. Rom. Sci.
Ser. Math. Appl.
ISSN 2066-6594 Vol. 15, No. 1-2/2023

ON DICHOTOMY WITH DIFFERENTIABLE GROWTH RATES FOR SKEW-EVOLUTION COCYCLES IN BANACH SPACES*

M. Megan † A. Găină ‡ R. Boruga (Toma)§

DOI https://doi.org/10.56082/annalsarscimath.2023.1-2.473

Dedicated to Dr. Dan Tiba on the occasion of his 70^{th} anniversary

Abstract

The paper considers a general concept of dichotomy in Banach spaces for skew-evolution cocycles. As particular cases the properties of exponential dichotomy, polynomial dichotomy and logarithmic dichotomy are obtained. Characterizations of these concepts are presented.

MSC: 34D05; 34D09

keywords: Skew-evolution cocycle, h-dichotomy, exponential dichotomy, polynomial dichotomy, logarithmic dichotomy.

*Accepted for publication on July 3-rd, 2023

†mihail.megan@gmail.com Department of Mathematics, Faculty of Mathematics and Computer Science, West University of Timişoara, V. Pârvan Blv. No. 4, 300223 Timişoara, Romania

‡ariana.gaina@gmail.com Department of Mathematics, Faculty of Mathematics and Computer Science, West University of Timişoara, V. Pârvan Blv. No. 4, 300223 Timişoara, Romania

§rovanaboruga@gmail.com Department of Mathematics, Faculty of Mathematics and Computer Science, West University of Timişoara, V. Pârvan Blv. No. 4, 300223 Timişoara, Romania

1 Introduction

In the last years, an impressive development is represented by the study of asymptotic behaviors for skew-evolution cocycles in Banach spaces, which can be considered generalizations for evolution operators and skew-product semiflows.

Lately, significant progress has been made in the study of exponential dichotomy in Banach spaces. The importance of the role played by this concept in the theory of dynamical systems is illustrated by the appearance of various papers in this domain for the exponential case that started with the work of Perron [15] and continued by Massera and Schffer [10], Daleckii and Krein [7], Chicone and Latushkin [6], Megan, Sasu and Sasu [11], Stoica and Megan [20], Sasu and Sasu [18, 19], Megan and Stoica [12].

The polynomial case was introduced by Barreira and Valls in [1] and it was also studied by Boruga (Toma) and Megan [3, 4, 5], Megan and Stoica [12], Rămneanţu, Ceauşu and Megan [17].

Throughout the years an important extension of exponential dichotomy and polynomial dichotomy is introduced by Pinto [16] and it is called dichotomy with growth rates or h-dichotomy, where the growth rate is a non-decreasing and bijective function $h : \mathbb{R}_+ \to [1, \infty)$. For recent contributions we refer to the works of Bento, Lupa, Megan and Silva [2], Mihiţ, Borlea and Megan [14], Găină [9], Megan and Găină [13].

The main aim of this paper is to give some necessary and sufficient conditions for dichotomy with growth rates with the particular cases of exponential dichotomy, polynomial dichotomy and logarithmic dichotomy for skew-evolution cocycles in Banach spaces. More precisely, considering a skew-evolution cocycle with h-growth, strong h-growth, exponential growth, polynomial growth, respectively logarithmic growth and a family of projectors invariant to the skew-evolution cocycle, we obtain different characterizations of Datko [8] type for these concepts.

2 Skew-evolution cocycles

Let X be a metric space, V a Banach space and $\mathcal{B}(V)$ the Banach algebra of all bounded linear operators acting on V. Moreover, we consider the following sets

$$\Delta = \{(t, s) \in \mathbb{R}_+^2 : t \geq s\}$$

$$T = \{(t, s, t_0) \in \mathbb{R}_+^3 : t \geq s \geq t_0\}.$$

Definition 2.1. A mapping $\varphi : \Delta \times X \to X$ is said to be an *evolution semiflow* on X if:

(es_1) $\varphi(s, s, x) = x$, for all $(s, x) \in \mathbb{R}_+ \times X$;

(es_2) $\varphi(t, s, \varphi(s, t_0, x_0)) = \varphi(t, t_0, x_0)$, for all $(t, s, t_0, x_0) \in T \times X$.

Definition 2.2. A mapping $\Phi : \Delta \times X \to \mathcal{B}(V)$ is said to be a *skew-evolution semiflow* on $X \times V$ over the evolution semiflow φ if the following properties are satisfied:

(ses_1) $\Phi(s, s, x) = I$ (the identity operator on X), for all $(s, x) \in \mathbb{R}_+ \times X$;

(ses_2) $\Phi(t, s, \varphi(s, t_0, x_0))\Phi(s, t_0, x_0) = \Phi(t, t_0, x_0)$, for all $(t, s, t_0, x_0) \in T \times X$.

If $\varphi : \Delta \times X \to X$ is an evolution semiflow and $\Phi : \Delta \times X \to \mathcal{B}(V)$ a skew-evolution semiflow over the evolution semiflow φ, then the pair $C = (\Phi, \varphi)$ is said to be *a skew-evolution cocycle.*

Example 2.1. Let X be a metric space, V a Banach space, $\varphi : \Delta \times X \to X$ an evolution semiflow on X and $A : X \to \mathcal{B}(V)$ a continuous mapping. If $\Phi(t, s, x)$ is the solution of the Cauchy problem

$$\begin{cases} \dot{v}(t) = A(\varphi(t, s, x))v(t), t > s \\ v(s) = x \ , \end{cases}$$

then $C = (\Phi, \varphi)$ is a skew-evolution cocycle.

Example 2.2. Let us consider $V = \mathbb{R}^2, X = \mathbb{R}_+$ and $f : \mathbb{R} \to \mathbb{R}_+^*$ a nondecreasing function. The mapping $\varphi : \Delta \times \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\varphi(t, s, x) = t - s + x$$

is an evolution semiflow on $\mathbb{R}_+$ and the mapping $\Phi : \Delta \times \mathbb{R}_+ \to \mathcal{B}(\mathbb{R}^2)$ defined by

$$\Phi(t, s, x)(v_1, v_2) = \left(\frac{f(x)}{f(t - s + x)} v_1, \frac{f(t - s + x)}{f(x)} v_2 \right)$$

is a skew-evolution semiflow over the evolution semiflow φ.

So $C = (\Phi, \varphi)$ is a skew-evolution cocycle.

Example 2.3. We consider $X = \mathbb{R}_+$. The mapping $\varphi : \Delta \times \mathbb{R}_+ \to \mathbb{R}_+$ defined by $\varphi(t, s, x) = t - s + x$ is an evolution semiflow on $\mathbb{R}_+$. For every evolution operator $E : \Delta \to \mathcal{B}(V)$, we have

$$\Phi_E : \Delta \times X \to \mathcal{B}(V), \Phi_E(t, s, x) = E(t - s + x, x)$$

is a skew-evolution cocycle on $X \times V$ over the evolution semiflow φ.

3 Invariant family of projectors for skew-evolution cocycles

Definition 3.1. A mapping $P : \mathbb{R}_+ \times X \to \mathcal{B}(V)$ is said to be a *family of projectors* if

$$P^2(t,x) = P(t,x), \text{ for all } (t,x) \in \mathbb{R}_+ \times X.$$

Definition 3.2. If $P : \mathbb{R}_+ \times X \to \mathcal{B}(V)$ is a family of projectors, then $Q : \mathbb{R}_+ \times X \to \mathcal{B}(V)$, $Q(s,x) = I - P(s,x)$ is also a family of projectors which is called *the complementary family of projectors of P.*

Definition 3.3. A family of projectors $P : \mathbb{R}_+ \times X \to \mathcal{B}(V)$ is said to be *invariant* for the skew-evolution cocycle C if

$$\Phi(t,s,x)P(s,x) = P(t,\varphi(t,s,x))\Phi(t,s,x),$$

for all $(t,s,x) \in \Delta \times X$.

Remark 3.1. If the family of projectors $P : \mathbb{R}_+ \times X \to \mathcal{B}(V)$ is invariant for the skew-evolution cocycle C, then its complementary family of projectors $Q : \mathbb{R}_+ \times X \to \mathcal{B}(V)$ is also invariant for the skew-evolution cocycle C.

In what follows, if P is invariant for the skew-evolution cocycle C, then we denote by $\Phi_P : \Delta \times X \to \mathcal{B}(V)$ the application defined by

$$\Phi_P(t,s,x) = \Phi(t,s,x)P(s,x).$$

Proposition 3.1. *The application Φ_P has the following properties:*

(i) $\Phi_P(t,s,x) = P(t,\varphi(t,s,x))\Phi_P(t,s,x)$, *for all* $(t,s,x) \in \Delta \times X$;

(ii) $\Phi_P(t,t,x) = P(t,x)$, *for all* $(t,x) \in \mathbb{R}_+ \times X$;

(iii) $\Phi_P(t,t_0,x_0) = \Phi_P(t,s,\varphi(s,t_0,x_0))\Phi_P(s,t_0,x_0)$, *for all* $(t,s,t_0,x_0) \in T \times X$.

Proof. The properties (i) and (ii) are immediate from definition of Φ_P, *Definition 3.3* and *Definition 2.2.*

For (iii) we observe that

$$\begin{aligned} \Phi_P(t,t_0,x_0) &= \Phi(t,t_0,x_0)P(t_0,x_0) = \\ &= \Phi(t,s,\varphi(s,t_0,x_0))P(s,\varphi(s,t_0,x_0))\Phi_P(s,t_0,x_0) = \\ &= = \Phi_P(t,s,\varphi(s,t_0,x_0))\Phi_P(s,t_0,x_0), \end{aligned}$$

for all $(t,s,t_0,x_0) \in T \times X$. □

4 h-dichotomy for skew-evolution cocycles

Definition 4.1. A nondecreasing map $h : \mathbb{R}_+ \to [1, \infty)$ with $\lim_{t\to\infty} h(t) = \infty$ is called a *growth rate.*

Let C be a skew-evolution cocycle and P a family of projectors invariant to C.

Definition 4.2. The pair (C, P) is said to be *h-dichotomic* if there are $N \geq 1, \nu > 0$ and $\varepsilon \geq 0$ such that:

$$h(t)^{\nu}(||\Phi_P(t, t_0, x_0)v_0|| + ||\Phi_Q(s, t_0, x_0)v_0||) \leq$$

$$\leq Nh(s)^{\nu}(h(s)^{\varepsilon}||\Phi_P(s, t_0, x_0)v_0|| + h(t)^{\varepsilon}||\Phi_Q(t, t_0, x_0)v_0||),$$

for all $(t, s, t_0, x_0, v_0) \in T \times X \times V$.

Remark 4.1. *In the previous definition we can consider* $\nu \in (0, 1)$.

If we consider the particular cases $h(t) = e^t$, $h(t) = t + 1$ and $h(t) = \ln(t + e)$, then it results the concepts of *exponential dichotomy*, *polynomial dichotomy* and respectively *logarithmic dichotomy.*

If $\varepsilon = 0$, then we obtain the concepts of *uniform h-dichotomy*, *uniform exponential dichotomy*, *uniform polynomial dichotomy* and respectively *uniform logarithmic dichotomy.*

Remark 4.2. The concept of uniform h-dichotomy implies the concept of h-dichotomy. The converse implication is not true, as we can see in:

Example 4.1. We consider X a metric space, V a Banach space, P an invariant family of projectors with $P(t, x)P(s, x) = P(s, x)$ and $C = (\Phi, \varphi)$ a skew-evolution cocycle where

$$\varphi : \Delta \times X \to X, \varphi(t, s, x)(\tau) = x(t - s + \tau)$$

is an evolution semiflow on X and $\Phi : \Delta \times X \to \mathcal{B}(V)$ defined by

$$\Phi(t, s, x) = \frac{f(s)}{f(t)}P(s, x) + \frac{g(t)}{g(s)}Q(s, x)$$

where $Q(s, x) = I - P(s, x)$, $f(t) = h(t)^{2-\cos \ln h(t)}$ and $g(t) = h(t)^{-\cos 2t}$ is a skew-evolution semiflow.

It is easy to see that (C, P) is h-dichotomic for $\varepsilon = 2$ and $\nu = 1$.

If we suppose that (C, P) is uniformly h-dichotomic, then it means that there exist $N \geq 1$ and $\nu > 0$ with

$$Nh(t)^{-\nu}||\Phi_Q(t,s,x)v|| \geq h(s)^{-\nu}||Q(s,x)v||,$$

which is equivalent to

$$Nh(t)^{-\nu}\frac{h(t)^{-\cos 2t}}{h(s)^{-\cos 2s}}||Q(s,x)|| \geq h(s)^{-\nu}||Q(s,x)v||,$$

for all $(t,s,x,v) \in \Delta \times X \times V$.
For $s = 0$ we have

$$Nh(t)^{\nu+\cos 2t} \leq h(0)^{\nu-1},$$

In particular, for $t = n\pi + \frac{\pi}{4}$ with $n \in \mathbb{N}$ and $n \to \infty$ we obtain a contradiction.

Theorem 4.1. The following statements are equivalent:

(i) (C, P) is h-dichotomic;

(ii) there are $N \geq 1, \nu > 0$ and $\varepsilon \geq 0$ with:

$$(hd') \qquad h(t)^{\nu}(||\Phi_P(t,s,x_0)v_0|| + ||Q(s,x_0)v_0||) \leq$$

$$\leq Nh(s)^{\nu}(h(s)^{\varepsilon}||P(s,x_0)v_0|| + h(t)^{\varepsilon}||\Phi_Q(t,s,x_0)v_0||),$$

for all $(t,s,x_0,v_0) \in \Delta \times X \times V$;

(iii) there are $N \geq 1, \nu > 0$ and $\varepsilon \geq 0$ such that:

(hd'_1) $h(t)^{\nu}||\Phi_P(t,s,x)v|| \leq Nh(s)^{\nu+\varepsilon}||P(s,x)v||$;
(hd'_2) $h(t)^{\nu}||Q(s,x)v||) \leq Nh(s)^{\nu}h(t)^{\varepsilon}||\Phi_Q(t,s,x)v||$,

for all $(t,s,x,v) \in \Delta \times X \times V$;

(iv) there exist $N \geq 1, \nu > 0$ and $\varepsilon \geq 0$ with:

(hd_1) $h(t)^{\nu}||\Phi_P(t,t_0,x_0)v_0|| \leq Nh(s)^{\nu+\varepsilon}||\Phi_P(s,t_0,x_0)v_0||$;
(hd_2) $h(t)^{\nu}||\Phi_Q(s,t_0,x_0)v_0|| \leq Nh(s)^{\nu}h(t)^{\varepsilon}||\Phi_Q(t,t_0,x_0)v_0|$,

for all $(t,s,t_0,x_0,v_0) \in T \times X \times V$.

Proof. $(i) \implies (ii)$ It results from (i) for $s = t_0$.

$(ii) \implies (iii)$ For $(hd') \implies (hd'_1)$ we take $v_0 = P(s, x_0)v$ and we notice that $Q(s, x_0)v_0 = Q(s, x_0)P(s, x_0)v = 0$ and $\Phi_Q(t, s, x_0)P(s, x_0)v = \Phi(t, s, x_0)Q(s, x_0)P(s, x_0)v = 0$.

In a similar way, for $(hd') \implies (hd'_2)$ we take $v_0 = Q(s, x_0)v$ and we observe that $P(s, x_0)v_0 = P(s, x_0)Q(s, x_0)v = 0$ and $\Phi_P(t, s, x_0)Q(s, x_0)v = \Phi(t, s, x_0)P(s, x_0)Q(s, x_0)v = 0$.

$(iii) \implies (iv)$ For $(hd'_1) \implies (hd_1)$ we have $x = \varphi(s, t_0, x_0)$ and $v = \Phi_P(s, t_0, x_0)v_0$ in (hd'_1) and we obtain

$$\begin{aligned} h(t)^{\nu}||\Phi_P(t, t_0, x_0)v_0|| &= h(t)^{\nu}||\Phi_P(t, s, \varphi(s, t_0, x_0))\Phi_P(s, t_0, x_0)v_0|| \leq \\ &\leq Nh(s)^{\nu+\varepsilon}||P(s, \varphi(s, t_0, x_0))\Phi_P(s, t_0, x_0)v_0|| = \\ &= Nh(s)^{\nu+\varepsilon}||\Phi_P(s, t_0, x_0)v_0||, \end{aligned}$$

for all $(t, s, t_0, x_0, v_0) \in T \times X \times V$.

We use the same method for $(hd'_2) \implies (hd_2)$ we take $x = \varphi(s, t_0, x_0)$ and $v = \Phi_Q(s, t_0, x_0)v_0$ in (hd'_2) and we obtain

$$\begin{aligned} & Nh(s)^{\nu}h(t)^{\varepsilon}||\Phi_Q(t, t_0, x_0)v_0|| \\ & = Nh(s)^{\nu}h(t)^{\varepsilon}||\Phi_Q(t, s, \varphi(s, t_0, x_0))\Phi_Q(s, t_0, x_0)v_0|| \geq \\ & \geq h(t)^{\nu}||Q(s, \varphi(s, t_0, x_0))\Phi_Q(s, t_0, x_0)v_0|| = \\ & = h(t)^{\nu}||\Phi_Q(s, t_0, x_0)v_0||, \end{aligned}$$

for all $(t, s, t_0, x_0, v_0) \in T \times X \times V$.

$(iv) \implies (i)$ It results by adding the conditions (hd_1) and (hd_2).

□

5 h-growth concepts for skew-evolution cocycles

Definition 5.1. The pair (C, P) has *h-growth* if there are $M \geq 1, \omega > 0$ and $\delta \geq 0$ such that:

$$h(s)^{\omega}(||\Phi_P(t, t_0, x_0)v_0|| + ||\Phi_Q(s, t_0, x_0)v_0||) \leq$$

$$\leq Mh(t)^{\omega}(h(s)^{\delta}||\Phi_P(s, t_0, x_0)v_0|| + h(t)^{\delta}||\Phi_Q(t, t_0, x_0)v_0||),$$

for all $(t, s, t_0, x_0, v_0) \in T \times X \times V$.

Remark 5.1. The concept of h-dichotomy implies the h-growth property, but the converse implication is not true, as we can see in:

Example 5.1. Let $V = \mathbb{R}^2$, $\varphi : \Delta \times X \to X$ a evolution semiflow on X and $\Phi : \Delta \times X \to \mathcal{B}(V)$ defined by

$$\Phi(t, s, x)(v_1, v_2) = \left(\frac{h(t)}{h(s)} v_1, \frac{h(s)}{h(t)} v_2 \right)$$

a skew-evolution semiflow on $X \times V$ over φ. Then C is a skew-evolution cocycle.

We consider the families of projectors $P, Q : \mathbb{R}_+ \times X \to \mathcal{B}(V)$ defined by

$$P(t, x)(v_1, v_2) = (v_1, 0),$$

$$Q(t, x)(v_1, v_2) = (0, v_2),$$

for all $(t, x, v_1, v_2) \in \mathbb{R}_+ \times X \times V$.

It is easy to see that (C, P) has h-growth, but is not h-dichotomic.

6 Growth concepts with differentiable growth rates of skew-evolution cocycles

Let $h : \mathbb{R}_+ \to [1, \infty)$ be a differentiable growth rate and P a family of projectors invariant to the skew-evolution cocycle C.

Definition 6.1. The pair (C, P) has *strong h-growth* if there are $M \geq 1, \omega > 0$ and $\delta \geq 0$ with:

$$h(s)^\omega (||\Phi_P(t, t_0, x_0)v_0|| + ||\Phi_Q(s, t_0, x_0)v_0||) \leq$$

$$\leq M h(t)^\omega \left(\frac{h'(s)}{h(s)} h(s)^\delta ||\Phi_P(s, t_0, x_0)v_0|| + \frac{h'(t)}{h(t)} h(t)^\delta ||\Phi_Q(t, t_0, x_0)v_0|| \right),$$

for all $(t, s, t_0, x_0, v_0) \in T \times X \times V$.

For the particular cases $h(t) = e^t$, $h(t) = t + 1$ and $h(t) = \ln(t + e)$ we obtain the concepts of *strong exponential growth*, *strong polynomial growth* and *strong logarithmic growth*.

Finally, if $\delta = 0$ it results the concepts of *uniform strong h-growth*, *uniform strong exponential growth*, *uniform strong polynomial growth* and respectively *uniform strong logarithmic growth*.

Remark 6.1. The pair (C, P) has strong h-growth if and only if there are $M \geq 1, \omega > 0$ and $\delta \geq 0$ with:

$$h(s)^{\omega}(||\Phi_P(t,s,x)v|| + ||Q(s,x)v||) \leq$$
$$\leq Mh(t)^{\omega}(\frac{h'(s)}{h(s)}h(s)^{\delta}||P(s,x)v|| + \frac{h'(t)}{h(t)}h(t)^{\delta}||\Phi_Q(t,s,x)v||),$$

for all $(t,s,x,v) \in \Delta \times X \times V$.

In what follows we will denote by

- $\mathcal{H}$ the set of differentiable growth rates $h : \mathbb{R}_+ \to [1, \infty)$ with the property that there exists $H > 1$ such that

$$h'(t) \leq Hh(t), \text{ for all } t \geq 0;$$

- $\mathcal{H}_1$ the set of differentiable growth rates $h : \mathbb{R}_+ \to [1, \infty)$ with the property that there exist $m > 0$ and $M > 0$ such that

$$mh(t) \leq h'(t) \leq Mh(t), \text{ for all } t \geq 0.$$

Example 6.1. If $e(t) = e^t$, $p(t) = t+1$, $l(t) = \ln(t+e)$ and $r(t) = \sqrt{t^2+1}$, then $e \in \mathcal{H}_1 \subset \mathcal{H}$ and $p, l, r \in \mathcal{H}$.

Proposition 6.1. If $h \in \mathcal{H}$, then:

(i) $h(t) \leq h(s)e^{H(t-s)}$, for all $(t,s) \in \Delta$;

(ii) $h(t+1) \leq e^H h(t)$, for all $t \geq 0$;

Proof. (i) From $h \in \mathcal{H}$ it results that

$$\int_s^t \frac{h'(t)}{h(t)} d\tau \leq H(t-s),$$

which implies that

$$h(t) \leq e^{H(t-s)}h(s),$$

for all $(t,s) \in \Delta$.

(ii) It is immediate from *(i)*.

□

7 Integral h-dichotomy

Let C be a strongly measurable skew-evolution cocycle (i.e. the mapping $t \mapsto ||\Phi(t,s,x)v||$ is measurable on $[s,\infty)$, for all $(s,x,v) \in \mathbb{R}_+ \times X \times V$), P a family of projectors invariant to C and $h : \mathbb{R}_+ \to [1,\infty)$ a differentiable growth rate.

Definition 7.1. *The pair (C,P) is integrally h-dichotomic if there are $D \geq 1$, $\varepsilon \geq 0$ and $d \in (0,1)$ such that*

$$(hD) \int_t^\infty \frac{h'(\tau)}{h(\tau)} h(\tau)^d ||\Phi_P(\tau,t_0,x_0)v_0||d\tau + \int_{t_0}^t \frac{h'(s)}{h(s)} h(s)^{-d} ||\Phi_Q(s,t_0,x_0)v_0||ds$$
$$\leq Dh(t)^\varepsilon (h(t)^d ||\Phi_P(t,t_0,x_0)v_0|| + h(t)^{-d}||\Phi_Q(t,t_0,x_0)v_0||),$$

for all $(t,t_0,x_0,v_0) \in \Delta \times X \times V$.

Remark 7.1. In particular, if $\varepsilon = 0$, then we obtain the concept of *uniform integral h-dichotomy.*

Proposition 7.1. *The pair (C,P) is integrally h-dichotomic if and only if there are $D \geq 1$, $\varepsilon \geq 0$ and $d \in (0,1)$ with*

$$(hD_1) \int_t^\infty \tfrac{h'(\tau)}{h(\tau)} h(\tau)^d ||\Phi_P(\tau,t_0,x_0)v||d\tau \leq Dh(t)^{\varepsilon+d}||\Phi_P(t,t_0,x_0)v||;$$

$$(hD_2) \int_{t_0}^t \tfrac{h'(s)}{h(s)} h(s)^{-d} ||\Phi_Q(s,t_0,x_0)v||ds \leq Dh(t)^{\varepsilon-d}||\Phi_Q(t,t_0,x_0)v||,$$

for all $(t,t_0,x_0,v_0) \in \Delta \times X \times V$.

Proof. Necessity. For $(hD) \implies (hD_1)$ we take $v_0 = P(t_0,x_0)v$ in (hD) and observe that $\Phi_P(\tau,t_0,x_0)v_0 = \Phi_P(\tau,t_0,x_0)v$ and $\Phi_Q(s,t_0,x_0)v_0 = 0$.

Similarly, for $(hD) \implies (hD_1)$ we take $v_0 = Q(t_0,x_0)v$ in (hD) and it follows that $\Phi_P(\tau,t_0,x_0)v_0 = 0$ and $\Phi_Q(s,t_0,x_0)v_0 = \Phi_Q(s,t_0,x_0)v$.

Sufficiency. It is immediate. □

Theorem 7.1. If (C,P) is h-dichotomic, then it is also integrally h-dichotomic.

Proof. If (C,P) is h-dichotomic, then there exist $N > 1, \nu \in (0,1)$ and $\varepsilon \geq 0$ such that for every $d \in (0,\nu)$ we have

$$\int_t^\infty \frac{h'(\tau)}{h(\tau)} h(\tau)^d ||\Phi_P(\tau,t_0,x_0)v_0|| d\tau + \int_{t_0}^t \frac{h'(s)}{h(s)} h(s)^{-d} ||\Phi_Q(s,t_0,x_0)v_0|| ds$$
$$\leq \int_t^\infty \frac{h'(\tau)}{h(\tau)} h(\tau)^d N h(t)^\varepsilon \left(\frac{h(\tau)}{h(t)}\right)^{-\nu} ||\Phi_P(t,t_0,x_0)v_0|| d\tau$$
$$+ \int_{t_0}^t \frac{h'(s)}{h(s)} N h(t)^\varepsilon h(s)^{-d} \left(\frac{h(t)}{h(s)}\right)^{-\nu} ||\Phi_Q(t,t_0,x_0)v_0|| ds$$
$$= N h(t)^\varepsilon (h(t)^\nu ||\Phi_P(t,t_0,x_0)v_0|| \int_t^\infty h'(\tau) h(\tau)^{d-\nu-1} d\tau$$
$$+ h(t)^{-\nu} ||\Phi_Q(t,t_0,x_0)v_0|| \int_{t_0}^t h'(s) h(s)^{\nu-d-1} ds)$$
$$\leq N h(t)^\varepsilon \left(h(t)^\nu ||\Phi_P(s,t_0,x_0)v_0|| \frac{h(t)^{d-\nu}}{\nu-d} + h(t)^{-\nu} ||\Phi_Q(t,t_0,x_0)v_0|| \frac{h(t)^{\nu-d}}{\nu-d}\right)$$
$$= \frac{N h(t)^\varepsilon}{\nu-d} \left(h(t)^d ||\Phi_P(t,t_0,x_0)v_0|| + h(t)^{-d} ||\Phi_Q(t,t_0,x_0)v_0||\right)$$
$$\leq D h(t)^\varepsilon (h(t)^d ||\Phi_P(t,t_0,x_0)v_0|| + h(t)^{-d} ||\Phi_Q(t,t_0,x_0)v_0||),$$

where $D = \frac{\nu-d+N}{\nu-d}$.

□

Theorem 7.2. We suppose that (C,P) has strong h-growth with $h \in \mathcal{H}$. If there exist $D \geq 1, \varepsilon \geq 0$ and $d > \delta$ with (hD), then the pair (C,P) is h-dichotomic.

Proof. Case I: $t \geq s+1$.

By Definition 6.1 and (hD) we obtain

$$
\begin{aligned}
&||\Phi_P(t,t_0,x_0)v_0|| + ||\Phi_Q(s,t_0,x_0)v_0|| = \\
&= \int\limits_{t-1}^{t} ||\Phi_P(t,t_0,x_0)v_0||d\tau + \int\limits_{s}^{s+1} ||\Phi_Q(s,t_0,x_0)v_0||d\tau \\
&\leq \int\limits_{t-1}^{t} Mh(\tau)^{\delta}\left(\frac{h(t)}{h(\tau)}\right)^{\omega}\frac{h'(\tau)}{h(\tau)}||\Phi_P(\tau,t_0,x_0)v_0||d\tau \\
&+ \int\limits_{s}^{s+1} Mh(\tau)^{\delta}\left(\frac{h(\tau)}{h(s)}\right)^{\omega}\frac{h'(\tau)}{h(\tau)}||\Phi_Q(\tau,t_0,x_0)v_0||d\tau \\
&\leq Mh(t)^{\delta}\left(\frac{h(t)}{h(s)}\right)^{-d}\int\limits_{t-1}^{t}\frac{h'(\tau)}{h(\tau)}\left(\frac{h(t)}{h(\tau)}\right)^{\omega+d}\left(\frac{h(\tau)}{h(s)}\right)^{d}||\Phi_P(\tau,t_0,x_0)v_0||d\tau \\
&+ Mh(s)^{\delta}e^{H\delta}\left(\frac{h(t)}{h(s)}\right)^{-d}\int\limits_{s}^{s+1}\frac{h'(\tau)}{h(\tau)}\left(\frac{h(\tau)}{h(s)}\right)^{\omega+d}\left(\frac{h(t)}{h(\tau)}\right)^{d}||\Phi_Q(\tau,t_0,x_0)v_0||d\tau \\
&\leq Mh(t)^{\delta}\left(\frac{h(t)}{h(s)}\right)^{-d}e^{H(\omega+d)}\int\limits_{s}^{\infty}\frac{h'(\tau)}{h(\tau)}\left(\frac{h(\tau)}{h(s)}\right)^{d}||\Phi_P(\tau,t_0,x_0)v_0||d\tau \\
&+ Mh(s)^{\delta}e^{H\delta}\left(\frac{h(t)}{h(s)}\right)^{-d}e^{H(\omega+d)}\int\limits_{s}^{s+1}\frac{h'(\tau)}{h(\tau)}\left(\frac{h(t)}{h(\tau)}\right)^{d}||\Phi_Q(\tau,t_0,x_0)v_0||d\tau \\
&\leq DMe^{H(\omega+d)}\Bigg[\left(\frac{h(t)}{h(s)}\right)^{\delta-d}h(s)^{\varepsilon+\delta}||\Phi_P(s,t_0,x_0)v_0|| + \\
&+ e^{H\delta}h(t)^{\varepsilon+\delta}\left(\frac{h(t)}{h(s)}\right)^{-(d+\delta)}||\Phi_Q(t,t_0,x_0)v_0||\Bigg],
\end{aligned}
$$

for all $(t,s,t_0,x_0,v_0) \in T \times X \times V$.

Case II: $t \in [s,s+1)$.

By Definition 6.1 we have

$$\begin{aligned}
& ||\Phi_P(t,t_0,x_0)v_0|| + ||\Phi_Q(s,t_0,x_0)v_0|| \leq \\
\leq \ & Mh(s)^\delta \left(\frac{h(t)}{h(s)}\right)^\omega \frac{h'(s)}{h(s)} ||\Phi_P(s,t_0,x_0)v_0|| + \\
+ \ & Mh(t)^\delta \left(\frac{h(t)}{h(s)}\right)^\omega \frac{h'(t)}{h(t)} ||\Phi_Q(t,t_0,x_0)v_0|| \leq \\
\leq \ & MH\Bigg[e^{H(\omega+d-\delta)} h(s)^{\varepsilon+\delta} \left(\frac{h(t)}{h(s)}\right)^{-(d-\delta)} ||\Phi_P(s,t_0,x_0)v_0|| + \\
+ \ & e^{H(\omega+d+\delta)} h(t)^{\varepsilon+\delta} \left(\frac{h(t)}{h(s)}\right)^{-(d+\delta)} ||\Phi_Q(t,t_0,x_0)v_0||\Bigg],
\end{aligned}$$

for all $(t,s,t_0,x_0,v_0) \in T \times X \times V$.

□

Corollary 7.1. If (C,P) has uniform strong h-growth with $h \in \mathcal{H}$, then (C,P) is uniformly h-dichotomic if and only if there exist $D \geq 1$ and $d \in (0,1)$ with

$$\begin{aligned}
& \int_t^\infty \frac{h'(\tau)}{h(\tau)} h(\tau)^d ||\Phi_P(\tau,t_0,x_0)v_0|| d\tau + \int_{t_0}^t \frac{h'(s)}{h(s)} h(s)^{-d} ||\Phi_Q(s,t_0,x_0)v_0|| ds \\
& \leq D(h(t)^d ||\Phi_P(t,t_0,x_0)v_0|| + h(t)^{-d} ||\Phi_Q(t,t_0,x_0)v_0||),
\end{aligned}$$

for all $(t,t_0,x_0,v_0) \in \Delta \times X \times V$.

Proof. It results from *Theorem 7.1* and *Theorem 7.2* for $\varepsilon = 0$.

□

Remark 7.2. For the particular cases $h(t) = e^t$, $h(t) = t+1$ and $h(t) = \ln(t+e)$ from *Theorem 7.1* and *Theorem 7.2* we obtain related properties for exponential dichotomy, polynomial dichotomy and respectively logarithmic dichotomy.

Similarly from *Corollary 7.1* it results characterizations for uniform exponential dichotomy, uniform polynomial dichotomy and respectively uniform logarithmic dichotomy.

For evolution operators these theorems are proved for uniform exponential dichotomy and uniform polynomial dichotomy by Boruga and Megan in [5].

Theorem 7.3. If the pair (C,P) is h-dichotomic with $\nu>\varepsilon$, then there exist $D\geq 1$ and $d\in(0,\nu-\varepsilon)$ such that

$$(hD')\int_{t_0}^{t}\frac{h'(s)}{h(s)^{d+1}}\frac{1}{||\Phi_P(s,t_0,x_0)v_0||}ds+\int_{t}^{\infty}\frac{h'(\tau)}{h(\tau)^{1-d}}\frac{1}{||\Phi_Q(\tau,t_0,x_0)v_0||}d\tau$$
$$\leq Dh(t)^{\varepsilon}\left(\frac{h(t)^{-d}}{||\Phi_P(t,t_0,x_0)v_0||}+\frac{h(t)^{d}}{||\Phi_Q(t,t_0,x_0)v_0||}\right),$$

for all $(t,t_0,x_0,v_0)\in\Delta\times X\times V$ with $\Phi_P(t,t_0,x_0)v_0\neq 0$ and $Q(t_0,x_0)v_0\neq 0$.

Proof. If the pair (C,P) is h-dichotomic with $\nu>\varepsilon$, then for $d\in(0,\nu-\varepsilon)$ we have

$$\int_{t_0}^{t}\frac{h'(s)}{h(s)^{d+1}}\frac{1}{||\Phi_P(s,t_0,x_0)v_0||}ds+\int_{t}^{\infty}\frac{h'(\tau)}{h(\tau)^{1-d}}\frac{1}{||\Phi_Q(\tau,t_0,x_0)v_0||}d\tau\leq$$
$$\leq\int_{t_0}^{t}\frac{h'(s)}{h(s)^{d+1}}N\left(\frac{h(t)}{h(s)}\right)^{-\nu}\frac{h(s)^{\varepsilon}}{||\Phi_P(t,t_0,x_0)v_0||}ds+$$
$$+\int_{t}^{\infty}\frac{h'(\tau)}{h(\tau)^{1-d}}Nh(\tau)^{\varepsilon}\left(\frac{h(\tau)}{h(t)}\right)^{-\nu}\frac{1}{||\Phi_Q(t,t_0,x_0)v_0||}d\tau\leq$$
$$\leq\frac{Nh(t)^{-\nu}}{||\Phi_P(t,t_0,x_0)v_0||}\frac{h(t)^{\nu+\varepsilon-d}}{\nu+\varepsilon-d}+\frac{Nh(t)^{\nu}}{||\Phi_Q(t,t_0,x_0)v_0||}\frac{h(t)^{d-\nu+\varepsilon}}{\nu-d-\varepsilon}\leq$$
$$\leq Dh(t)^{\varepsilon}\left(\frac{h(t)^{-d}}{||\Phi_P(t,t_0,x_0)v_0||}+\frac{h(t)^{d}}{||\Phi_Q(t,t_0,x_0)v_0||}\right),$$

where $D=1+N\left(\frac{1}{\nu+\varepsilon-d}+\frac{1}{\nu-d-\varepsilon}\right)$

□

Theorem 7.4. If (C,P) has strong h-growth with $h\in\mathcal{H}_1$ and there exist $D\geq 1,\varepsilon\geq 0$ and $d>\varepsilon$ such that (hD') takes place, then the pair (C,P) is h-dichotomic.

Proof. We suppose that (C,P) has strong h-growth with $h\in\mathcal{H}_1$ and (hD') holds for $D\geq 1,\varepsilon\geq 0$ and $d>\varepsilon$.

Case I: From Defintion 6.1 and inequality (hD'), *we have* $(t,s,t_0,x_0,v_0) \in T \times X \times V$ *with* $t \geq s+1$, $\Phi_P(t,t_0,x_0)v_0 \neq 0$ *and* $Q(t_0,x_0)v_0 \neq 0$.

$$\begin{aligned}
& \frac{h(s)^{-d}}{||\Phi_P(s,t_0,x_0)v_0||} + \frac{h(t)^d}{||\Phi_Q(t,t_0,x_0)v_0||} = \\
= \quad & \int\limits_s^{s+1} \frac{h(s)^{-d}}{||\Phi_P(s,t_0,x_0)v_0||} d\tau + \int\limits_{t-1}^{t} \frac{h(t)^d}{||\Phi_Q(t,t_0,x_0)v_0||} d\tau \leq \\
\leq \quad & \int\limits_s^{s+1} Mh(s)^\delta \left(\frac{h(\tau)}{h(s)}\right)^\omega \frac{h'(s)}{h(s)} \frac{h(s)^{-d}}{||\Phi_P(\tau,t_0,x_0)v_0||} d\tau + \\
+ \quad & \int\limits_{t-1}^{t} Mh(t)^\delta \left(\frac{h(t)}{h(\tau)}\right)^\omega \frac{h'(t)}{h(t)} \frac{h(t)^d}{||\Phi_Q(\tau,t_0,x_0)v_0||} d\tau \leq \\
\leq \quad & \frac{MH}{m} e^{H(\omega+d)} \left(h(s)^\delta \int\limits_{t_0}^{t} \frac{h'(\tau)}{h(\tau)^{1+d}} \frac{1}{||\Phi_P(\tau,t_0,x_0)v_0||} d\tau + \right. \\
+ \quad & \left. h(t)^\delta \int\limits_s^\infty \frac{h'(\tau)}{h(\tau)^{1-d}} \frac{1}{||\Phi_Q(\tau,t_0,x_0)v_0||} d\tau \right) \leq \\
\leq \quad & \frac{DMH}{m} e^{H(\omega+d)} \left(h(s)^{\delta+\varepsilon} \frac{h(t)^{\varepsilon-d}}{||\Phi_P(t,t_0,x_0)v_0||} + h(t)^\delta \frac{h(s)^{\varepsilon+d}}{||\Phi_Q(s,t_0,x_0)v_0||} \right).
\end{aligned}$$

So we have

$$\begin{aligned}
& ||\Phi_P(t,t_0,x_0)v_0|| + ||\Phi_Q(s,t_0,x_0)v_0|| \leq \\
\leq \quad & \frac{DMH}{m} e^{H(\omega+d)} \left[h(s)^{\delta+\varepsilon} \left(\frac{h(t)}{h(s)}\right)^{-(d-\varepsilon)} h(s)^{\delta+\varepsilon} ||\Phi_P(s,t_0,x_0)v_0|| + \right. \\
+ \quad & \left. \left(\frac{h(t)}{h(s)}\right)^{-(d-\varepsilon)} h(t)^{\delta+\varepsilon} ||\Phi_Q(t,t_0,x_0)v_0|| \right].
\end{aligned}$$

Case II: By Defintion 6.1 we obtain $(t,s,t_0,x_0,v_0) \in T \times X \times V$ *with* $t \in [s,s+1)$, $\Phi_P(t,t_0,x_0)v_0 \neq 0$ *and* $Q(t_0,x_0)v_0 \neq 0$.

$$
\begin{aligned}
& h(t)^d||\Phi_P(t,t_0,x_0)v_0|| + h(t)^d||\Phi_Q(s,t_0,x_0)v_0|| \leq \\
\leq \quad & Mh(s)^\delta h(t)^d \left(\frac{h(t)}{h(s)}\right)^\omega \frac{h'(s)}{h(s)}||\Phi_P(s,t_0,x_0)v_0|| + \\
+ \quad & Mh(t)^\delta h(t)^d \left(\frac{h(t)}{h(s)}\right)^\omega \frac{h'(t)}{h(t)}||\Phi_Q(t,t_0,x_0)v_0|| = \\
= \quad & Mh(s)^\delta \frac{h'(s)}{h(s)} \left(\frac{h(t)}{h(s)}\right)^{\omega+d} h(s)^d||\Phi_P(s,t_0,x_0)v_0|| + \\
+ \quad & Mh(t)^\delta \left(\frac{h(t)}{h(s)}\right)^{\omega+d} \frac{h'(t)}{h(t)} h(s)^d||\Phi_Q(t,t_0,x_0)v_0|| \leq \\
\leq \quad & MHe^{H(\omega+d)}h(s)^d(h(s)^\delta||\Phi_P(s,t_0,x_0)v_0|| + h(t)^\delta||\Phi_Q(t,t_0,x_0)v_0||).
\end{aligned}
$$

So

$$
\begin{aligned}
& ||\Phi_P(t,t_0,x_0)v_0|| + ||\Phi_Q(s,t_0,x_0)v_0|| \leq \\
\leq \quad & MHe^{H(\omega+d)}\left[\left(\frac{h(t)}{h(s)}\right)^{-(d-\varepsilon)} h(s)^{\delta+\varepsilon}||\Phi_P(s,t_0,x_0)v_0|| + \right. \\
+ \quad & \left. \left(\frac{h(t)}{h(s)}\right)^{-(d-\varepsilon)} h(t)^{\delta+\varepsilon}||\Phi_Q(t,t_0,x_0)v_0||\right],
\end{aligned}
$$

for all $(t,s,t_0,x_0,v_0) \in T \times X \times V$.

Finally, we obtain that the pair (C,P) is h-dichotomic.

□

Corollary 7.2. *If (C,P) has strong h-growth with $h \in \mathcal{H}_1$, then a necessary and sufficient condition for (C,P) to be h-dichotomic with $\nu > \varepsilon$ is that there are $D \geq 1$ and $d > \varepsilon$ such that (hD') is satisfied.*

Proof. It results from *Theorem 7.3* and *Theorem 7.4*. □

Corollary 7.3. *If (C,P) has an uniform strong h-growth, then (C,P) is uniformly h-dichotomic if and only if there are $D \geq 1$ and $d > 0$ with the property*

$$
\begin{aligned}
& \int_{t_0}^{t} \frac{h'(s)}{h(s)^{d+1}} \frac{1}{||\Phi_P(s,t_0,x_0)v_0||} ds + \int_{t}^{\infty} \frac{h'(\tau)}{h(\tau)^{1-d}} \frac{1}{||\Phi_Q(\tau,t_0,x_0)v_0||} d\tau \leq \\
\leq \quad & D\left(\frac{h(t)^{-d}}{||\Phi_P(t,t_0,x_0)v_0||} + \frac{h(t)^d}{||\Phi_Q(t,t_0,x_0)v_0||}\right),
\end{aligned}
$$

for all $(t, t_0, x_0, v_0) \in \Delta \times X \times V$ *with* $\Phi_P(t, t_0, x_0)v_0 \neq 0$ *and* $Q(t_0, x_0)v_0 \neq 0$.

Proof. It is a particular case ($\varepsilon = 0$) from the previous Corollary. □

Remark 7.3. In *Theorem 7.3* and *Theorem 7.4* when the growth rate is e^t we obtain the particular cases, such as exponential dichotomy, respectively uniform exponential dichotomy. The particular case of uniform exponential dichotomy of these two theorems was proved for evolution operators in [5].

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