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STATIC OUTPUT FEEDBACK CONTROL OF LINEAR PARAMETER VARYING SYSTEMS*

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Dedicated to Dr. Dan Tiba on the occasion of his 70^{th} anniversary

Abstract

The synthesis problem of static output feedback controllers within the anisotropic-norm setup for Linear Parameter Varying system is considered. A synthesis approach involving iterations over a convex optimisation problem is suggested, leading to a gain-scheduled controller. The results are formulated by a couple of Linear Matrix Inequalities and a coupling bilinear equality, using a parameter dependent Lyapunov function. Following LPVTOOL, the problem which is continuous in the gain-scheduling parameter, and hence infinite dimensional, is approximated by a finite grid leading to a tractable sequence of convex optimization problems. The design method is demonstrated on a simple example from the field of flight control.

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keywords: H_∞, LPV, Static Output Feedback, Anisotropic Norm

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1 Introduction

Flight control and other industrial control problems, often involve ad-hoc gain scheduling. The common practice starts with linearization of the nonlinear plant around equilibrium (trim) conditions over a grid of operating points (e.g. Mach number, altitude). The resulting collection of linear plants is then used to design corresponding controllers for each operating condition, which are tailored together to form a gain-scheduled controllers. The above practice works fine in many applications, given a slow enough variation of the gain-scheduling (GS) parameters. However, no stability or performance guarantee applies for rapid changes of the GS parameters [18, 17]. While the gain scheduled design under the Linear Parameters Varying (LPV) setup is completely covered by [18] and the tools described there, the synthesis of Static Output Feedback (SOF) controllers have received much less attention (see [21] for a somewhat conservative approach assuming a structural constraint on the Lyapunov function). Nevertheless, SOF synthesis is practically important, in many applications, since full order controllers are rather challenging for implementation even for Linear Time Invariant (LTI) systems (see [13, 14, 16] . The dependence on GS parameters, makes it even more difficult.

In [20] a static output feedback design have been considered aiming to minimize the a-anisotropic norm of the resulting closed-loop system. The aim of the present paper is to generalize those results to allow synthesis of gain scheduled controllers under a Linear Parameters Varying (LPV) setup by applying the methods and tools of [18] and [19].

The a-anisotropic norm setup, becomes a useful alternative to modeling exogenous signals either as white noise or of finite energy. When the external input signals are of white noise type, H_2-norm minimisation is applied, leading to the Kalman filter [5] and Linear Quadratic Gaussian (LQG) control. An alternative modeling of the exogenous inputs is based on deterministic bounded energy signals associated with the H_∞-norm based framework [11] applicable both to filtering and control e.g. [12]. Since H_2 is not entirely suitable when signals are strongly coloured and H_∞ may result in poor performance when these signals are weakly coloured (e.g. white noise), mixed H_2/H_∞ norm minimisation becomes useful (see, e.g. [8]). Another option to accomplish a compromise between the H_2 and the H_∞ norms is to use the so-called *a-anisotropic norm* [6] defined as follows: consider the discrete-time stable system denoted by F with the state-space equations

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t), t = 0, 1, ... \end{aligned} \tag{1}$$

where $A \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times m}$, $C_f \in \mathcal{R}^{p\times n}$, $D_f \in \mathcal{R}^{p\times m}$. By definition, the a-anisotropic norm of F is

$$|||F|||_a = \sup_{G\in\mathcal{G}_a} \frac{\|FG\|_2}{\|G\|_2}, \tag{2}$$

where G denotes a discrete-time stable filter of form

$$\begin{aligned} x_f(t+1) &= A_f x_f(t) + B_f v(t) \\ w(t) &= C_f x_f(t) + D_f v(t), t = 0, 1, ... \end{aligned} \tag{3}$$

with m inputs and m outputs, where the inputs $v \in \mathcal{R}^m$ are independent Gaussian white noises. In equation (2), $\mathcal{G}_a$ denotes the set of all systems (3) with the *mean anisotropy* $\bar{A}(G) \leq a$. The mean anisotropy of stationary Gaussian sequences has been introduced in [6] and it represents an entropy theoretic measure of the deviation of a probability distribution from Gaussian distributions[6, 9]. Based on the Szegö-Kolmogorov theorem, the mean anisotropy of a signal generated by an m-dimensional Gaussian white noise $v(t)$ with zero mean and identity covariance applied to a stable linear system G with m outputs has the form

$$\bar{A}(G) = -\frac{1}{2} \ln \det \left(\frac{mE\left[\tilde{w}(0)\tilde{w}(0)^T\right]}{Tr\left(E\left[w(0)w(0)^T\right]\right)} \right), \tag{4}$$

where $E[\tilde{w}(0)\tilde{w}(0)^T]$ is the covariance of the prediction error $\tilde{w}(0) := w(0) - E[w(0)|(w(k), k < 0]$. In the case when the output w of the filter G is a zero mean Gaussian white noise (i.e. its optimal estimate is just zero), $w(0)$ cannot be estimated from its past values and $\tilde{w}(0) = w(0)$ leading to $\bar{A}(G) = 0$. The relationship between the H_2, H_∞ and the a-anisotropic norms are given by the following inequalities :

$$\frac{1}{\sqrt{m}}||F||_2 = |||F|||_0 \leq |||F|||_a \leq ||F||_\infty = \lim_{a\to\infty} |||F|||_a \tag{5}$$

showing that the a-anisotropic norm may be regarded as a relaxation of the H_∞ norm. In a study case presented in [10] it is concluded that using the a-anisotropic norm instead of H_∞ norm one may reduce the controller gains and the controls effort. In [20] a static output feedback design have been considered aiming to minimise the a-anisotropic norm of the resulting closed-loop system. The main result proved there states that the optimal static output feedback gain may be obtained solving a non-convex optimisation problem.

The aim of the present paper is to derive solvability conditions for the static output feedback problem with respect to the anisotropic norm expressed in terms of convex optimisation conditions suitable for tractable numerical implementation in the case of LPV systems.

Notation. Throughout the paper the superscript 'T' stands for matrix transposition, $\mathcal{R}$ denotes the set of scalar real numbers whereas $\mathcal{Z}_+$ stands for the non-negative integers. Moreover, $\mathcal{R}^n$ denotes the n dimensional Euclidean space, $\mathcal{R}^{n\times m}$ is the set of all $n\times m$ real matrices, and the notation $P>0$ $(P\geq 0)$, for $P\in\mathcal{R}^{n\times n}$ means that P is symmetric and positive definite (positive semi-definite). The trace of a matrix Z is denoted by $Tr(Z)$, and $|v|$ denotes the Euclidian norm of an n-dimensional vector v. Finally note that the terms Lyapunov and Riccati equations in this paper, refer to generalised versions of the standard equations appearing in the H_2 and H_∞ control literature.

2 Static Output Feedback for LPV Systems

Consider the following plant

$$x(t+1) \quad = \quad A(\rho(t))x(t)+B_1(\rho(t))w(t)+B_2u(t) \tag{6}$$

where we seek for a stabilising static control matrix K, such that $u(t)=Ky(t)$, where $y(t)=C_2x(t)$ will minimize

$$z(t) \quad = \quad C_1(\rho(t))x(t)+D_{12}u(t)+D_{11}(\rho(t))w(t) \tag{7}$$

in the sense of bounded anisotropic norm. Here $\rho(t)\in R^{n_\rho}$ is a time varying parameters vector with a bounded rate $|d\rho_i/dt|<\nu_i, i=1,2,...,n_\rho$, defining the convex hull $\dot{\mathcal{P}}:=\{\dot\rho, |d\rho_i/dt|<\nu_i, i=1,2,...,n_\rho\}$ and A,B_1,C_1,D_{11} are of affine dependence on $\rho(t)$. The compact set of allowable ρ is denoted by $\mathcal{P}$ where we adopt the grid based approach, and where the system denoted by $S(\rho)$ of (7) is approximated as a state-space array of $S(\hat\rho_k)$ where $\hat\rho_k\in\mathcal{P}_g$ represent a parameters vector corresponding to the k'th point on the grid. The parameters space for the sake of control synthesis is, therefore, approximated by $\Omega:=\mathcal{P}_g\times\dot{\mathcal{P}}$ This approach is motivated by traditional gain scheduling framework in flight control. where linear models are derived around various flight conditions characterized by e.g. mach number and altitude (or dynamic pressure). When, however, the flight conditions around which linearization is performed includes also fast variables (e.g. angle of attack), the LPV framework can ensure stability and performance, in contrast to the general practice of designing gains to each point in the grid and then tailoring them together by either a polynomial or linear interpolation over a table, as a function of the gain scheduling parameters.

Note that, in the sequel, we may omit the dependence on ρ for the sake of simpler notations.

Define the cost function associated to the above problem

$$J(K) = |||\mathcal{F}_{c\ell}(K)|||_a \tag{8}$$

where $\mathcal{F}_{c\ell}(K)$ denotes the closed loop system obtained from (6) and (7) with the static output feedback $u_2(t) = Ky(t)$, having the realisation

$$\begin{array}{rcl} x(t+1) &=& (A + B_2KC_2)x(t) + B_1w(t) \\ z(t) &=& (C_1 + D_{12}KC_2)x(t) + D_{11}w(t)\,. \end{array}$$

Using the time-varying version (e.g.[22]) it follows that the above closed loop system $\mathcal{F}_{c\ell}(K)$ is stable and it has the a-anisotropic norm less than a given $\gamma > 0$ if and only if there exist a $q \in \left(0, \min(\gamma^{-2}, \|\mathcal{F}_{c\ell}\|_\infty^{-2})\right)$ and a symmetric matrix $X(\rho) > 0$ such that

$$\begin{bmatrix} \mathcal{E}_1(X,K) & \mathcal{E}_2(X,K) \\ \mathcal{E}_2(X,K)^T & -\frac{1}{q}I + B_1^TXB_1 + D_{11}^TD_{11} \end{bmatrix} < 0 \tag{9}$$

where one denoted

$$\begin{array}{rcl} \mathcal{E}_1(X,K) &:=& -\frac{\partial X}{\partial \rho}(\rho)\dot{\rho}h - X(\rho) + (A + B_2KC_2)^T X(\rho)\,(A + B_2KC_2) \\ && + (C_1 + D_{12}KC_2)^T (C_1 + D_{12}KC_2) \\ \mathcal{E}_2(X,K) &:=& (A + B_2KC_2)^T X(\rho)B_1 + (C_1 + D_{12}KC_2)^T D_{11} \end{array}$$

and

$$\frac{1}{q} - \gamma^2 < e^{-\frac{2a}{m}} \left(\det\left(\frac{1}{q}I - B_1^TX(\rho)B_1 - D_{11}^TD_{11}\right)\right)^{\frac{1}{m}}\,. \tag{10}$$

In the above expression of $\mathcal{E}_1(X,K)$ the first order approximation of the Taylor expansion

$$X(\rho(k+1)) = X(\rho(k)) + \frac{\partial X}{\partial \rho}(\rho_k)\frac{d\rho}{dt}h$$

was used, where h is the interval between consequent time steps and $\nu = d\rho_k/dt$.

Based on Schur complements arguments, it follows that the inequality (9) is equivalent with the condition

$$\mathcal{Z} + \mathcal{P}^T K \mathcal{Q} + \mathcal{Q}^T K^T \mathcal{P} < 0, \tag{11}$$

where by definition

$$\mathcal{Z} := \begin{bmatrix} -\bar{X} & 0 & A^T X & C_1^T \\ 0 & -\frac{1}{q} I & B_1^T X & D_{11}^T \\ XA & XB_1 & -X & 0 \\ C_1 & D_{11} & 0 & -I \end{bmatrix}, \mathcal{P}^T := \begin{bmatrix} 0 \\ 0 \\ XB_2 \\ D_{12} \end{bmatrix}, \mathcal{Q}^T := \begin{bmatrix} C_2^T \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{12}$$

where we denoted

$$\bar{X}(\rho, \dot{\rho}) := X(\rho) + \frac{\partial X}{\partial \rho}(\rho)\dot{\rho}h. \tag{13}$$

Further, using the so-called Projection lemma, one obtains that the inequality (11) is feasible with respect to K if and only if the following conditions are accomplished

$$W_{\mathcal{P}}^T Z W_{\mathcal{P}} < 0 \tag{14}$$

and

$$W_{\mathcal{Q}}^T Z W_{\mathcal{Q}} < 0, \tag{15}$$

where $W_{\mathcal{P}}$ and $W_{\mathcal{Q}}$ are any bases of the null spaces of $\mathcal{P}$ and $\mathcal{Q}$, respectively. Since a base of the null space of $\mathcal{P}$ is

$$W_{\mathcal{P}} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & X^{-1} W_1 \\ 0 & 0 & W_2 \end{bmatrix} \tag{16}$$

where $W := \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}$ is the orthogonal complement of $\begin{bmatrix} B_2^T & D_{12}^T \end{bmatrix}$. Similarly, a base of the null space of $\mathcal{Q}$ is

$$W_{\mathcal{Q}} = \begin{bmatrix} W_3 & 0 & 0 \\ W_4 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \tag{17}$$

where $V := \begin{bmatrix} W_3 \\ W_4 \end{bmatrix}$ is the orthogonal complement of $\begin{bmatrix} C_2 & 0 \end{bmatrix}$. In order to simplify the inequality of (14) we next express

$$W_{\mathcal{P}} = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & W_1^T & W_2^T \end{bmatrix} \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & X^{-1} & 0 \\ 0 & 0 & 0 & I \end{bmatrix},$$

namely

$$W_{\mathcal{P}} = \begin{bmatrix} I & 0 \\ 0 & W^T \end{bmatrix} \left[\begin{array}{cc|cc} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \hline 0 & 0 & X^{-1} & 0 \\ 0 & 0 & 0 & I \end{array} \right]$$

with the above definition of W. Therefore, (14) is simply expressed as

$$\begin{bmatrix} I & 0 \\ 0 & W^T \end{bmatrix} \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{12}^T & \mathcal{M}_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & W \end{bmatrix} < 0 \tag{18}$$

where $\mathcal{M}$ is given by

$$\mathcal{M} := \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{12}^T & \mathcal{M}_{22} \end{bmatrix}$$

with

$$\mathcal{M}_{11} = \begin{bmatrix} -\bar{X} & 0 \\ 0 & -\frac{1}{q}I \end{bmatrix}, \mathcal{M}_{12} = \begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix}^T, \mathcal{M}_{22} = \begin{bmatrix} -X^{-1} & 0 \\ 0 & -I \end{bmatrix}. \tag{19}$$

Then, from (18), using the Schur complement of $\mathcal{M}_{11}$ it follows that

$$W^T(\mathcal{M}_{22} - \mathcal{M}_{12}^T \mathcal{M}_{11}^{-1} \mathcal{M}_{12})W < 0\,. \tag{20}$$

Substituting the definition for $\mathcal{M}_{ij}, i, j = 1, 2$ and recalling the definition $\eta^2 = \frac{1}{q}$ we obtain the following convenient form of (20)

$$W^T \begin{bmatrix} -Y + A\bar{Y}A^T + B_1 B_1^T & A\bar{Y}C_1^T + B_1 D_{11}^T \\ C_1 \bar{Y} A^T + D_{11} B_1^T & -\Phi_Y \end{bmatrix} W < 0 \tag{21}$$

where

$$\Phi_Y := \eta^2 I - C_1 \bar{Y} C_1^T - D_{11} D_{11}^T$$

and where we have defined

$$\eta^{-2} Y = X^{-1} \text{ and }, \eta^{-2}\bar{Y} = \bar{X}^{-1} \tag{22}$$

We next repeat the same lines to simplify (15) as well. To this end, we partition

$$W_Q = \begin{bmatrix} V & 0 \\ 0 & I \end{bmatrix}$$

and readily obtain using Schur complements, that (15) is equivalent to

$$V^T(\mathcal{N}_{11} - \mathcal{N}_{12}\mathcal{N}_{22}^{-1}\mathcal{N}_{12}^T)V < 0$$

where

$$\begin{aligned} \mathcal{N}_{11} &= \begin{bmatrix} -\bar{X} & 0 \\ 0 & -\frac{1}{q}I \end{bmatrix}, \mathcal{N}_{12} = \begin{bmatrix} XA & XB_1 \\ C_1 & D_{11} \end{bmatrix}^T, \\ \mathcal{N}_{22} &= \begin{bmatrix} -X & 0 \\ 0 & -I \end{bmatrix}. \end{aligned} \tag{23}$$

We, therefore, obtain the following form of (15)

$$V^T \begin{bmatrix} -\bar{X} + A^T XA + C_1^T C_1 & A^T XB_1 + C_1^T D_{11} \\ B_1^T XA^+ D_{11}^T C_1 & -\Phi_X \end{bmatrix} V < 0 \tag{24}$$

where $\Phi_X := \eta^2 I - B_1^T XB_1 - D_{11}^T D_{11}$. We summarize the above derivations in the following result.

Theorem 1 *The closed loop system $\mathcal{F}_{c\ell}(K)$ is stable and it has the a-anisotropic norm less than a given $\gamma > 0$ if there exist for all $\{\rho, \dot{\rho}\} \in \Omega$ symmetric matrices $X(\rho) > 0$, $Y(\rho) > 0$, $\bar{X}(\rho, \dot{\rho})$, $\bar{Y}(\rho, \dot{\rho})$ and a scalar η satisfying the dual LMIs (21) and (24) together with the convex condition*

$$\eta^2 - det(\Phi_X)^{1/m} e^{-2a/m} < \gamma^2 \tag{25}$$

and the additional bilinear conditions

$$XY = \eta^2 I \ \ and\ \bar{X}\bar{Y} = \eta^2 I. \tag{26}$$

where $\bar{X}$ is defined in (13).

If the conditions of the above theorem are satisfied then the static output gain may be obtained solving the linear matrix inequality (11) with respect to $K(\rho)$.

However, the above requires a solution of a set of Bilinear Matrix Inequalities (BMI) due to the $XY = \eta^2 I$ equality. One way to tackle the BMI is to adopt a by first relaxing $XY = \eta^2 I$ and $\bar{X}\bar{Y} = \eta^2 I$ by

$$\begin{bmatrix} X & \eta I \\ \eta I & Y \end{bmatrix} > 0, \text{ and }, \begin{bmatrix} \bar{X} & \eta I \\ \eta I & \bar{Y} \end{bmatrix} > 0. \tag{27}$$

Then if one minimizes $Tr\{XY\}$ and $Tr\{\bar{X}\bar{Y}\}$, the bilinear constraints are satisfied. To this end, a sequential linearization algorithm (see e.g. [14]) can be used. In the initialization step, the convex problem comprised of the inequalities of Theorem 1 and (27) is solved for a given $\gamma > 0$, and $\ell = 0$, $X_\ell = 0$ and $Y_\ell = 0$ are set. Next step where ℓ is set to $\ell + 1$ and X, Y are found so as to minimize

$$f_\ell := Tr\{X_\ell Y + XY_\ell + \bar{X}_\ell \bar{Y} + \bar{X}\bar{Y}_\ell\}$$

subject to (21) and (24). Then $X_\ell = X$ and $Y_\ell = Y$ are set. This step is repeated until f_ℓ is small enough. A related algorithm requiring also line search but with improved convergence properties has been suggested in [7] and will be applied in the calculations below of the numerical example of the next section. To this end we note that one could choose also $X = Y^{-1}$ rather than $\eta^{-2}Y = X^{-1}$. In such a case, the inequality (21) is replaced by

$$W^T \begin{bmatrix} -Y + A\bar{Y}A^T + qB_1B_1^T & A\bar{Y}C_1^T + qB_1D_{11}^T \\ C_1\bar{Y}A^T + qD_{11}B_1^T & -\Phi_{\tilde{Y}} \end{bmatrix} W < 0$$

where $\Phi_{\tilde{Y}}$ is defined to be

$$\Phi_{\tilde{Y}} := I - C_1\bar{Y}C_1^T - qD_{11}D_{11}^T$$

and (27) becomes

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0, \begin{bmatrix} \bar{X} & I \\ I & \bar{Y} \end{bmatrix} > 0 \tag{28}$$

Although this different choice of Y reveals the duality between the control and filtering type inequalities in a less obvious manner, it is more convenient to deal with. Note that to apply [7] one needs to define a new variable $q = \eta^{-2}$ where in addition to $X\tilde{Y} = I$ also the scalar valued bilinear equality constraint $\eta^2 q = 1$ has to be satisfied. To this end, one needs also to consider the relaxed version

$$\begin{bmatrix} \eta^2 & 1 \\ 1 & q \end{bmatrix} > 0 \tag{29}$$

so that the minimization steps involve now searching for X, Y, η^2, q are found so as to minimise

$$g_\ell := Tr\{X_\ell Y + XY_\ell + \bar{X}_\ell \bar{Y} + \bar{X}\bar{Y}_\ell\} + \eta_\ell^2 q + \eta^2 q_\ell$$

subject to (21), (24), (28) and (29).

At this point, it should be noted, that the inequalities for $X(\rho) > 0$, $Y(\rho) > 0$, $\bar{X}(\rho, \dot{\rho})$, $\bar{Y}(\rho, \dot{\rho})$ over $\mathcal{P} \times \dot{\mathcal{P}}$ is, in fact, infinite dimensional, and to get a tractable optimization problem, a couple of simplifying assumptions are made in the next section.

3 Simplified Finite Dimensional Conditions

We next provide some simplifying assumptions to facilitate the solution process :

- As mentioned above, we adopt the grid based approach [18, 19] where the system denoted by $S(\rho)$ of (6) and (7) is approximated as a state-space array of $S(\hat{\rho}_j)$ where $\hat{\rho}_j \in \mathcal{P}_g$ represent a parameters vector corresponding to the k'th point on the grid. The parameters space for the sake of control synthesis is, therefore, approximated by $\Omega := \mathcal{P}_g \times \dot{\mathcal{P}}$

- Following [18, 19] we also pick n_b basis functions $f_j(\rho), j = 1, 2, ..., n_b$ so that

$$X(\rho) = \sum_{j=1}^{n_b} f_j(\rho) X^{(j)} \text{ and } Y(\rho) = \sum_{j=1}^{n_b} f_j(\rho) Y^{(j)},$$

  and $\bar{X}(\rho, \dot{\rho}) := \sum_{k=1}^{n_b} f_j(\rho) X^{(j)} + \sum_{j=1}^{n_b} \frac{\partial f_j}{\partial \rho}(\rho) X^{(j)} \dot{\rho} h$,

$$\bar{Y}(\rho, \dot{\rho}) := \sum_{k=1}^{n_b} f_j(\rho) Y^{(j)} + \sum_{j=1}^{n_b} \frac{\partial f_j}{\partial \rho}(\rho) Y^{(j)} \dot{\rho} h,$$

  where the superscripts $^{(j)}$ related to the basis function index, is not to be confused with the subscripts $_k$ of the iterations resolving the bilinear inequalities.

Therefore, the infinite dimensional conditions of Theorem 1 are now reduced to solutions to $X^{(j)}, Y^{(j)}, \bar{X}^{(j)}, \bar{Y}^{(j)}$, $j = 1, 2, ..., n_b$ of $n_r 2^{n_\rho}$ LMIs, and the $2n_b^2$ bilinear equalities, where n_b is the number of basis functions, n_r is the grid size for ρ and 2^{n_ρ} is the number of vertices of $\dot{\mathcal{P}}$. Note that the convenient version (28) of the bilinear equalities is used, along with sequential application of the convex inequalities of Theorem 1 and minimization of

$$g_\ell := Tr\{X_\ell(\hat{\rho}_j)Y(\hat{\rho}_j) + X(\hat{\rho}_j)Y_\ell(\hat{\rho}_j) + \bar{X}_\ell(\hat{\rho}_j)\bar{Y}(\hat{\rho}_j) + \bar{X}(\hat{\rho}_j)\bar{Y}_\ell(\hat{\rho}_j)\} + \eta_\ell^2 q + \eta^2 q_\ell$$

for each $\hat{\rho}_k \in \mathcal{P}_g$, where at completion of each step in the iteration, $X_\ell(\rho_j) = X(\rho_j)$ and $Y_\ell(\rho_j) = Y(\rho_j)$ are set.

The LPVTOOL [19] allows selection of arbitrary basis functions set, for a given problem, where examples in the tools' documentation show that some basis functions, are more successful than others.

One can also suggest seeking for basis functions, that minimize γ while just fixing the number of basis functions n_b, but it may turn out to be a non tractable problem. Another ad-hoc options is to select basis functions that are consistent with the dependence of the systems matrices in (3) where $n_b = n_r$ restriction is imposed. This may work for examples with small number of available grid points and will be consdered in the future.

4 Application to Flight Control

We next consider the numerical example of [15] with the synthesis of pitch control loop for the F4E aircraft. We consider here a slightly modified version, where a low-pass filter is added in cascade with the actuator, in order to get a vertex-independent B_2 matrix of the four-blocks plant representation. Consider

$$\frac{\mathrm{d}}{\mathrm{dt}}\begin{bmatrix} \mathrm{N_z} \\ \mathrm{q} \\ \delta_{\mathrm{e}} \\ \delta_1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \mathrm{a}_{13} & \mathrm{b}_1 \\ a_{21} & a_{22} & \mathrm{a}_{23} & 0 \\ 0 & 0 & -30 & 30 \\ 0 & 0 & 0 & -200 \end{bmatrix} \begin{bmatrix} \mathrm{N_z} \\ \mathrm{q} \\ \delta_{\mathrm{e}} \\ \delta_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 200 \end{bmatrix} \mathrm{u} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \omega$$

$$\mathrm{z} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathrm{x} + \begin{bmatrix} 0 \\ 0 \\ 0.001 \end{bmatrix} \mathrm{u}$$

$$\mathrm{y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \mathrm{x}.$$

The state-vector consists of the load-factor N_z, the pitch-rate q and elevon angle δ_e and a low-pass version of the elevon angle command of

Operating point	1	2	3	4
Mach number	.5	.9	.85	1.5
Altitude (ft)	5000	35000	5000	35000
a_{11}	-.9896	-.06607	-1.702	-.5162
a_{12}	17.41	18.11	50.72	29.96
a_{13}	96.15	84.34	263.5	178.9
a_{21}	.2648	.08201	.2201	-.6896
a_{22}	-.8512	-.6587	-1.418	-1.225
a_{23}	-11.39	-10.81	-31.99	-30.38
b_1	-97.78	-85.09	-272.2	-175.6

Table 1: The parameters of the four operating points

$200 rad/\sec$. The actuator is modeled as a first-order system with a bandwidth of $30 rad/\sec$. The parameters $a_{ij}, i = 1, 2; j = 1, 2, 3, b_1$ are given in [15] at the four operating points listed in Table 1.

Discrete-time representation of the above systems have been obtained with the sampling period $T_s = 0.001\,\text{sec}$. The static output controller $u = Ky$ with $K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$ were designed in [20] treating these points as distinct points. There, for each of the $n_r = 4$ operating points, a mean anisotropy level of a that tending to ∞ was firstly taken, and then one considered $a = 0.2$.

Here we will treat those points as representing a continuous envelope, by first looking for corresponding $n_b = 2$ simple basis functions, namely $f_1(\rho) = 1$ and $f_2(\rho) = \rho$ where $\rho = q_{dyn}$ is the dynamic pressure corresponding to each of the operating points in Table 1. We will then apply the LPV approach outlined in Section 2 using a mean anisotropy $a \to \infty$ with $\gamma = 3.6$ and a mean anisotropy of $a = 0.5$ with $\gamma = 3.6$ as well. Note that γ serves merely as an upper bound on the anisotropic norm of the closed-loop system.

The design results are compared in Figures 1-3. In Figures 1 and 2 the singular values of the closed-loop transfer function matrix, for the four operation points is depicted, along with the upper bound $\gamma = 3.6$ and the H_∞-norm γ_{LPV} of the closed-loop overall LPV system, obtained using the instruction *LPVNORM* from the LPV tools [19]. Note that In Fig. 3 it is shown that considerably lower gain values are obtained, with the anisotropic-norm design, comparing to the H_∞ design, virtualy at no cost in the sense of the closed-loop H_∞-norm of the closed-loop ($\gamma_{LPV} = 5.0889$ for the anisotropic design, while it is $\gamma_{LPV} = 5.0756$ for the H_∞ - design)

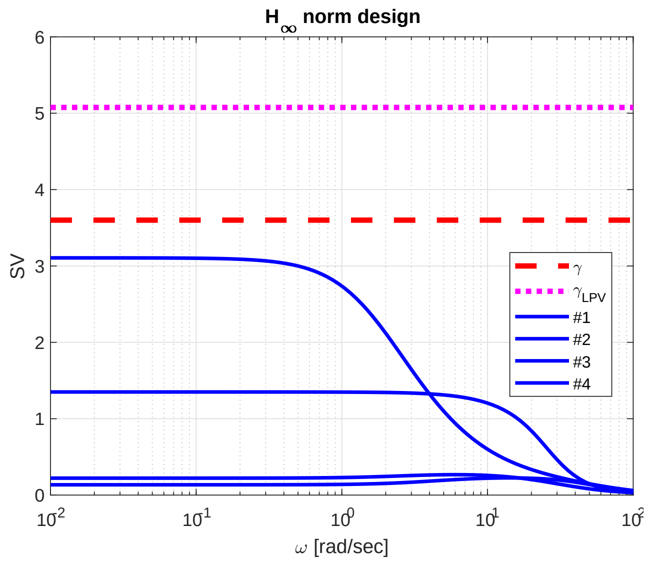

Figure 1: H_∞-Norm Design - 4 operating points - Singular Values

.Namely, the anisotropic-norm design is less conservative.

5 Conclusions

An LPV synthesis scheme for static output feedback controllers has been derived, under the setup of a-anisotropic norm which is based on an intermediate topology between H_2 and H_∞. Given a required norm-bound, the set of Linear Matrix Inequalities, along with a geometric-mean convex inequality, and an additional bilinear equality, characterize sub-optimal controllers revealing the duality in the style of [16] between the control and filtering type Linear Matrix Inequalities. Adopting a well known grid approach, the resulting infinite dimensional problem is relaxed to a finite-dimensional one, where bilinear equations are replaced with iterations on LMIs. A simple example of the field of flight control, demonstrates the design method and the advantage of the anisotropic-norm approach in the sense of smaller overdesign .This advantage is a consequence of the smaller class of exogenous signals of the anisotropic-norm approach with respect to the finite energy assumption associated with the H_∞-norm design.

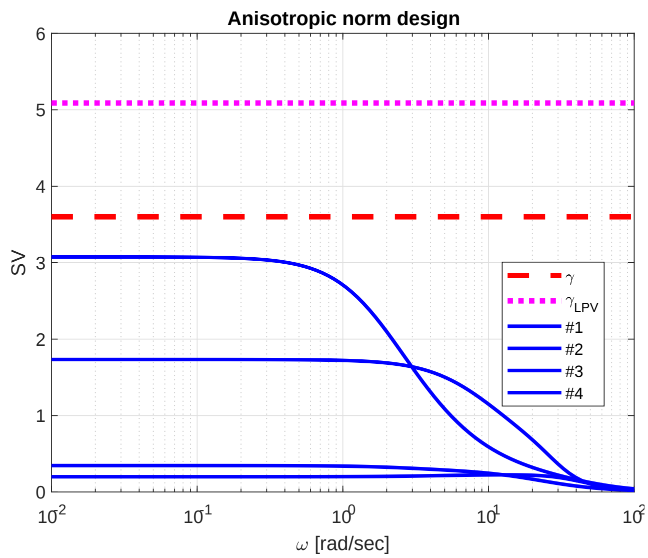

Figure 2: Anisotropic-Norm Design - 4 operating points - Singular Values

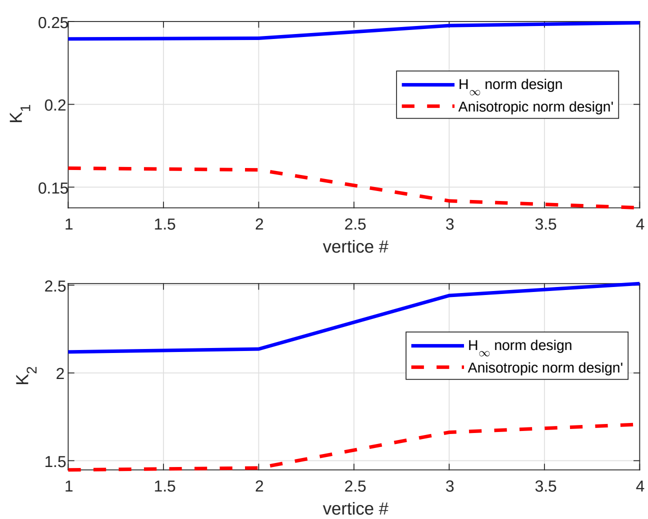

Figure 3: Comparison of H_∞ and anisotropic-norm design - Control Gains

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