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ALGORITHMS FOR THE RICCATI EQUATION WITH A NONSINGULAR M-MATRIX*

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Dedicated to Dr. Dan Tiba on the occasion of his 70^{th} anniversary

Abstract

We consider different algorithms with linear rate of convergence for computing the minimal nonnegative solution of M-matrix algebraic Riccati equation. The performance of the considered algorithms are illustrated on numerical examples.

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1 Introduction

The nonsymmetric algebraic Riccati equations have been the topic of extensive research. These equations arise in many important applications,

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including the total least squares problems, in the Markov chains [7], in the transport theory [3] and many others.

The equation has the form

$$\mathcal{R}(X) := \; XCX - XD - AX + B = 0\,, \tag{1}$$

where D, B, C and A are real matrices of dimensions $m\times m, m\times n, n\times m$ and $n \times n$, respectively. The block matrix $K = \begin{pmatrix} A & -C \\ -B & D \end{pmatrix}$ is an M-matrix. It is investigated in Bai and coauthors [2], Ma and Lu [10], Guan and Lu [4], Guan [5] and many other authors (see the reference there in).

In addition, nonsymmetric Riccati equation (1) arises in the game theory and more specially from the investigation of the open-loop Nash linear quadratic differential game [1, 12]. A more general problem on connected to the properties of the stabilising solution of the game theoretic algebraic Riccati equation is investigated in [8, 9]. The solution of practical interest is the minimal nonnegative solution of (1).

There are many numerical methods up to now proposed for the minimal nonnegative solution of (1) with a nonsingular M-matrix. An effective method called alternately linearized implicit iteration method (ALI) was proposed and investigated in [2, 10, 4]. This approach constructs two matrix sequences of nonnegative matrices which converges to the minimal solution. A new alternately linearized implicit iteration method (NALI) for computing the minimal nonnegative solution of (1) is introduced in [4].

Guan [5] has proposed the MALI iterative method: $A = (a_{ij}), D = (d_{ij}), X^{(0)} = 0 \in R^{n\times n}$. The matrix A is transformed $A = L_A - U_A$, where L_A is the lower triangular part of A and U_A is the strictly upper triangular part of A. The matrix D is received the type $D = L_D - U_D$ in the same way.

The iteration scheme is:

$$\begin{aligned} & Y^{(k)}(\alpha I + L_D) = (\alpha I - A + X^{(k)}C)X^{(k)} + X^{(k)}U_D + B\,, \\ & \;\alpha \geq max_i\; (a_{ii}) \\ & (\delta I + L_A)X^{(k+1)} = Y^{(k)}(\delta I - D + CY^{(k)}) + U_A Y^{(k)} + B\,, \\ & \;\delta \geq max_i\; (d_{ii})\,. \end{aligned} \tag{2}$$

The convergence analysis of the MALI iteration method and numerical experiments are executed by Guan. Moreover, Ivanov and Yang [11] have proposed a modification of (2):

$$\begin{aligned} Y^{(k)}(\gamma I + L_D) &= (\gamma I - A + X^{(k)}C)X^{(k)} + X^{(k)}U_D + B\,, \\ (\gamma I + A)X^{(k+1)} &= Y^{(k)}(\gamma I - D + CY^{(k)}) + B\,, \quad . \end{aligned} \tag{3}$$

The iteration (3) is an alternative to iteration (2).

Our investigation follows ideas presented of iterative methods by Bai and coauthors [2], Ma and Lu [10], Guan and Lu [4], Guan [5], Ivanov and Yang [11]. We propose a different iterative method to compute the minimal nonnegative solution and derive convergence properties of the new iteration.

The notation $\mathbf{R}^{r\times s}$ stands for $r \times s$ real matrices and I means an unit $n \times n$ matrix. A matrix $A = (a_{ij}) \in \mathbf{R}^{m\times n}$ is a nonnegative matrix if the inequalities $a_{ij} \geq 0$ are satisfied for all $1 \leq i \leq m$ and $1 \leq j \leq n$. We use an elementwise order relation. The inequality $P \geq Q (P > Q)$ for $P = (p_{ij}), Q = (q_{ij})$ means that $p_{ij} \geq q_{ij} (p_{ij} > q_{ij})$ for all indexes i and j. Define Z-matrices and M-matrices. A matrix $A = (a_{ij}) \in \mathbf{R}^{n\times n}$ is said to be a Z-matrix if it has nonpositive off-diagonal entries. Any Z-matrix A can be written in the form $A = \alpha I - N$ with N being a nonnegative matrix. Each M-matrix is a Z-matrix with if $\alpha \geq \rho(N)$, where $\rho(N)$ is the spectral radius of N. It is called a nonsingular M-matrix if $\alpha > \rho(N)$ and a singular M-matrix if $\alpha = \rho(N)$.

2 Iterative methods and properties

Here, we consider the another modification of the above iteration (2). Compute

$$\begin{aligned} &\alpha \geq max_i\ (a_{ii}), \{A = (a_{ij})\}\,, \quad \beta \geq max_i\ (d_{ii}), \{D = (d_{ij})\}\ , \\ &\gamma = max\{\alpha, \beta\}\,. \end{aligned}$$

The iteration has the form

$$\begin{aligned} Y^{(k)}(\gamma I + D) &= (\gamma I - A + X^{(k)}C)X^{(k)} + B\,, \\ (\gamma I + L_A)X^{(k+1)} &= Y^{(k)}(\gamma I - D + CY^{(k)}) + U_A Y^{(k)} + B \quad . \end{aligned} \tag{4}$$

The iteration start with $X^{(0)} = 0 \in R^{n\times n}$. We transform $A = L_A - U_A$, where L_A is the lower triangular part of A and U_A is the strictly upper triangular part of A. The iteration (4) is an alternative to iteration (2).

Lemma 1 *The matrix sequences* $\{X^{(k)}, Y^{(k)}\}_{k=0}^{\infty}$ *are obtained applying iteration ((4)) with initial values* $X^{(0)} = 0$*. Then for any positive* k *, the following equalities hold:*

$$\text{(i)} \quad \mathcal{R}(X^{(k)}) \quad = (Y^{(k)} - X^{(k)})(\gamma I + D)\,,$$

$$\text{(ii)} \quad \mathcal{R}(Y^{(k)}) = (\gamma I - A + X^{(k)}C)(Y^{(k)} - X^{(k)}) + (Y^{(k)} - X^{(k)})CY^{(k)}\,,$$

$$\text{(iii)} \ \mathcal{R}(Y^{(k)}) = (\gamma I + L_A)(X^{(k+1)} - Y^{(k)})\,,$$

$$\text{(iv)} \ \mathcal{R}(X^{(k+1)}) = (X^{(k+1)} - Y^{(k)})(\gamma I - D + CY^{(k)}) + (U_A + X^{(k+1)}C)(X^{(k+1)} - Y^{(k)})\,,$$

$$\text{(v)} \ \mathcal{R}(\hat{X}) \quad = (Y^{(k)} - \hat{X})(\gamma I + D) + (\gamma I - A + \hat{X}C)(\hat{X} - X^{(k)}) + (\hat{X} - X^{(k)})CX^{(k)}\,.$$

$$\text{(vi)} \ \mathcal{R}(\hat{X}) = (\gamma I + L_A)(X^{(k+1)} - \hat{X}) + (\hat{X} - Y^{(k)})(\gamma I - D + CY^{(k)}) + (U_A + \hat{X}C)(\hat{X} - Y^{(k)})\,.$$

Proof. The proof is completed by a direct calculation.

Theorem 1 *Assume the matrix* A *is an M-matrix and* $B \geq 0, C \geq 0$ *, and there exists* $\mu > 0$*, such that* $(\mu I + A)$ *is an M-matrix and* $\mu I - A$ *is nonpositive. Assume there exist a nonnegative matrix* $\hat{X}$*, such that* $\mathcal{R}(\hat{X}) \leq 0$ *.*

The matrix sequences $\{X^{(k)}, Y^{(k)}\}_{k=0}^{\infty}$ *defined by ((4)) satisfy the following properties:*

(i) $\hat{X} \geq X^{(k+1)} \geq Y^{(k)} \geq X^{(k)}$ *for* $k = 0, 1, \ldots;$

(ii) $\mathcal{R}(X^{(k)}) \geq 0\,, \quad \mathcal{R}(Y^{(k)}) \geq 0\,, \quad \mathcal{R}(X^{(k+1)}) \geq 0\,, \quad k = 0, 1, \ldots\,.$

(iii) *The matrix sequence* $\{X^{(k)}\}_{k=0}^{\infty}$ *converges to the nonnegative minimal solution* $\tilde{X}$ *to the Riccati equation* $\mathcal{R}(X) = 0$ *with* $\tilde{X} \leq \hat{X}$*.*

(iv) *The matrix sequence* $\{X^{(k)}\}_{k=0}^{\infty}$ *converges to the nonnegative minimal solution* $\tilde{X}$ *to the Riccati equation* $\mathcal{R}(X) = 0$ *with the property* $\tilde{X} \leq \hat{X}$*.*

Proof. We apply the decomposition of the matrix coefficient $A = L_A - U_A$, where L_A is the lower triangular part of A and U_A is the strictly upper triangular part of A. We remark $U_A \geq 0$. We begin with the facts that $(\gamma I + D)^{-1} \geq 0$, and $(\gamma I + L_A)^{-1} \geq 0$. We construct the matrix sequences $\{X^{(k)}, Y^{(k)}\}_{k=0}^{\infty}$ applying recursive equations (4) with $X^{(0)} = 0$ and $\gamma > 0$.

For $k = 0$ we obtain $Y^{(0)}(\gamma I + D) = B \geq 0$ and thus $Y^{(0)} = B(\gamma I + D)^{-1} \geq 0$. And $Y^{(0)} \geq X^{(0)} = 0$. In addition, $\mathcal{R}(X^{(0)}) = B \geq 0$.

Applying Lemma 1 (ii), we get $(\gamma I - A \geq 0)$

$$\mathcal{R}(Y^{(0)}) = (\gamma I - A)Y^{(0)} + Y_i^{(0)} C Y^{(0)} \geq 0\,.$$

We compute $X^{(1)}$ applying the recursive equation (4). We have

$$(\gamma I + L_A)X^{(1)} = W^{(0)} \geq 0\,,$$

where

$$W^{(0)} := Y^{(0)}(\gamma I - D + CY^{(0)}) + U_A Y^{(0)} + B\,.$$

Since $(\gamma I + L_A)^{-1} \geq 0$, we obtain $X^{(1)}$ is nonnegative.

Applying Lemma 1 (iii), we get

$$(X^{(1)} - Y^{(0)}) = (\gamma I + L_A)^{-1}\,\mathcal{R}(Y^{(0)}) \geq 0\,.$$

According to Lemma 1 (iv) we induce

$$\begin{aligned}\mathcal{R}(X^{(1)}) = {} & (X^{(1)} - Y^{(0)})(\gamma I - D + CY^{(0)}) \\ & +(U_A + X^{(1)}C)(X^{(1)} - Y^{(0)}) \ \geq 0\,,\end{aligned}$$

because $U_A \geq 0, \gamma\, I - D \geq 0, X^{(1)} \geq Y^{(0)} \geq X^{(0)}$.

In order to prove $\hat{X} \geq X^{(1)}$ we consider equality Lemma 1 (v)

$$\mathcal{R}(\hat{X}) \ \ = (Y^{(0)} - \hat{X})(\gamma I + D) + (\gamma I - A + \hat{X}C)\hat{X} \geq 0\,.$$

Note that $\gamma I - A \geq 0$ and then:

$$(Y^{(0)} - \hat{X}) = H^{(0)}\,(\gamma I + D)^{-1} \leq 0\,,$$

because

$$H^{(0)} := \mathcal{R}(\hat{X}) - (\gamma I - A + \hat{X}C)\hat{X} \ \leq 0\,.$$

Thus $\hat{X} \geq Y^{(0)}$. Moreover, applying equality Lemma 1 (vi) we obtain

$$\begin{aligned}(\gamma I + L_A)(X^{(1)} - \hat{X}) = {} & \mathcal{R}(\hat{X}) - (\hat{X} - Y^{(0)})(\gamma I - D + CY^{(k)}) \\ & -(U_A + \hat{X}C)(\hat{X} - Y^{(0)})\,.\end{aligned}$$

We infer $\hat{X} \geq X^{(1)}$.

So, we have proved inequalities (i) - (ii) for $k = 0$.

We assume that the inequalities (i) - (ii) hold for $k = 0, 1, \ldots, r$. We know matrices $X^{(r+1)}$ with the properties:

$$\hat{X} \geq X^{(r+1)} \geq Y^{(r)} \geq X^{(r)} \,,$$

and

$$\mathcal{R}(X^{(r)}) \geq 0 \,, \quad \mathcal{R}(Y^{(r)}) \geq 0 \,.\mathcal{R}(X^{(r+1)}) \geq 0 \;.$$

We will prove the inequalities (i) - (ii) for $k = r + 1$.

We compute $Y^{(r+1)}$ via (4), i.e.

$$Y^{(r+1)} = [(\gamma I - A + X^{(r+1)}C)X^{(r+1)} + B]\,(\gamma I + D)^{-1} \geq 0 \,.$$

According to Lemma 1 (i) we extract

$$Y^{(r+1)} - X^{(r+1)} = \mathcal{R}(X^{(r+1)})(\gamma I + D)^{-1} \geq 0 \,.$$

From Lemma 1 (ii), we conclude

$$\begin{aligned}\mathcal{R}(Y^{(r+1)}) = &\ (\gamma I - A + X^{(r+1)}C)(Y^{(r+1)} - X^{(r+1)}) \\ &+(Y^{(r+1)} - X^{(r+1)})CY^{(r+1)} \ \geq 0 \,,\end{aligned}$$

We compute $X^{(r+2)}$ via the second equation of (4). Consider the equality (iii) of Lemma 1 for $k = r + 1$. We write down:

$$X^{(r+2)} - Y^{(r+1)} = (\gamma I + L_A)^{-1}\,\mathcal{R}(Y^{(r+1)}) \geq 0 \,.$$

Next, we apply of Lemma 1 (iv) for

$$\begin{aligned}\mathcal{R}(X^{(r+2)}) &= (X^{(r+2)} - Y^{(r+1)})(\gamma I - D + CY^{(r+1)}) \\ &\quad +(U_A + X^{(r+2)}C_i)(X^{(r+2)} - Y^{(r+1)}) \geq 0 \,.\end{aligned}$$

Thus $\mathcal{R}(X^{(r+2)}) \geq 0$.

In order to prove $\hat{X} \geq X^{(r+2)}$ we consider equality Lemma 1 (v)

$$\begin{aligned}\mathcal{R}(\hat{X}) \ &= (Y^{(r+1)} - \hat{X})(\gamma I + L_D) \\ &\quad +(\gamma I - A + \hat{X}C)(\hat{X} - X^{(r+1)}) - (\hat{X} - X^{(r+1)})(U_D + CX^{(r+1)}) \,.\end{aligned}$$

Note that $\gamma I - A \geq 0$, $U_D \geq 0$, . Then

$$Y^{(r+1)} - \hat{X} = H^{(r+1)}\,(\gamma I + L_D)^{-1} \leq 0 \,,$$

because $H^{(r+1)} \leq 0$, and

$$H^{(r+1)} := \mathcal{R}(\hat{X}) - (\gamma I - A + \hat{X}C)(\hat{X} - X^{(r+1)})$$
$$-(\hat{X} - X^{(r+1)})(U_D + CX^{(r+1)})\,.$$

Thus $\hat{X} \geq Y^{(r+1)}$.

Further on, taking into account Lemma 1 (vi) we obtain

$$X_i^{(r+2)} - \hat{X}_i = (\gamma_i I_n + L_{A_i})^{-1}\, T_i^{(r+1)} \leq 0\,,$$

because $T_i^{(r+1)} \leq 0$, and

$$T_i^{(r+1)} := \mathcal{R}_i(\hat{X}_1, \ldots, \hat{X}_s) - (\hat{X}_i - Y_i^{(r+1)})(\gamma_i I_n - D_i + C_i Y_i^{(r+1)})$$
$$-(U_{A_i} + \hat{X}_i C_i)(\hat{X}_i - Y_i^{(r+1)}) - \textstyle\sum_{j\neq i} e_{ij}(\hat{X}_j - Y_j^{(r+1)})\,, \quad i = 1, \ldots, s\,.$$

We infer $\hat{X} \geq X^{(r+2)}$.

Hence, the induction process has been completed. Thus the matrix sequence $\{X^{(k)}\}_{k=0}^{\infty}$ are nonnegative, monotonically increasing and bounded from above by $(\hat{X}$ (in the elementwise ordering). We denote $\lim_{k\to\infty}(X^{(k)}) = (\tilde{X})$. By taking the limits in (4) it follows that $(\tilde{X})$ is a solution of $\mathcal{R}(X) = 0$ with the property $\tilde{X} \leq \hat{X}$.

Assume there is another solution $\tilde{Z}$ with $\tilde{Z} \leq \tilde{X}$. There exists sufficiently large index r such that $X^{(r+1)} \geq \tilde{Z} \geq Y^{(r)} \geq X^{(r)}$.

Applying Lemma 1 (vi) for $\hat{X} = \tilde{Z}$, we get

$$0 = (\gamma I + L_A)(X^{(r+1)} - \tilde{Z}) + (\tilde{Z} - Y^{(r)})(\gamma I - D + CY^{(r)})$$
$$+(U_A + \tilde{Z}C)(\tilde{Z} - Y^{(r)})\,.$$

We rewrite

$$(\gamma I + L_A)(X^{(r+1)} - \tilde{Z}) = Q^{(r)}\,.$$

The matrix $Q^{(r)}$ is nonnegative because $\tilde{Z} \geq Y^{(r)}$. Thus $X^{(r+1)} - \tilde{Z}$ is nonnegative, which is contradiction with the assumption $X^{(r+1)} \geq \tilde{Z}$. We infer the solution $\tilde{X}$ is the minimal one.

The theorem is proved.

3 Algorithms

The considered iterative methods have a linear convergence rate. We present algorithms which realize the iterations on Matlab.

Algorithm 1 presents iteration (2) using Matlab's commands.

Algorithm 1.

1. Input the coefficients A, B, C, D and compute $RESB = norm(B)$.

2. Choose tolerance $tol = 1.0e - 14$ and $n \times n$ initial matrix $X0 = 0$, normRE=1.

3. Compute $LA = tril(A)\,, UA = LA - A\,, LD = tril(D)\,, UD = LD - D$.

4. Compute $\alpha = max_i\ (a_{ii}),\ \delta = max_i\ (d_{ii})$.

5. Compute $aImA = \alpha I - A, dImD = \delta I - D\,.$

6. Compute $Z = inv(\alpha I + LD)\ \ T = inv(\delta I + LA)$.

7. Define the loop

while normRE > tol
Y0=((aImA + X0*C)*X0 + X0*UD + B)*Z
X0 = T*(Y0*(dImD + C*Y0) + UA*Y0 + B)
normRE = norm(X0*C*X0-A*10-X0*D+B)/RESB

END Algorithm 1

Algorithm 2 explains how to realize iteration (4).

Algorithm 2.

1. Input the coefficients A, B, C, D and compute $RESB = norm(B)$.

2. Choose tolerance $tol = 1.0e - 14$ and $n \times n$ initial matrix $X0 = 0$, normRE=1.

3. Compute $\alpha = max_i\ (a_{ii}),\ \delta = max_i\ (d_{ii})\,, \gamma = max(\alpha, \delta)$.

4. Compute $aImA = \gamma I - A, dImD = \gamma I - D\,.$

5. Compute $LA = tril(A)\,, UA = LA - A\,,\ Z = inv(\gamma I + D)\ \ T = inv(\gamma I + LA)$.

6. Define the loop

while normRE > tol
Y0=((aImA + X0*C)*X0 + B)*Z
X0 = T*(Y0*(dImD + C*Y0) + UA*Y0 + B)
normRE = norm(X0*C*X0-A*10-X0*D+B)/RESB

END Algorithm 2

We can introduce a mixed algorithm depending on the bigger value of α and δ.

We apply iterations (4) and (3) to construct next Algorithm 3.

Algorithm 3.

1. Input the coefficients A, B, C, D and compute $RESB = norm(B)$.

2. Choose the tolerance value $tol = 1.0e - 14$ and $n \times n$ initial matrix $X0 = 0$, normRE=1.
3. Compute $\alpha = max_i\ (a_{ii}),\ \delta = max_i\ (d_{ii})\,, \gamma = max(\alpha, \delta)$.
4. Compute $aImA = \gamma I - A, dImD = \gamma I - D$.
5. If $\gamma = \alpha$ then

Compute $LD = tril(D)\,, UD = LD - D\,,\ Z = inv(\alpha I + LD)\ \ T = inv(\delta I + A)$.

Apply iteration (3)

else

Compute $LA = tril(A)\,, UA = LA - A\,,\ Z = inv(\gamma I + D)\ \ T = inv(\gamma I + LA)$.

Apply iteration (4)

END Algorithm 3

Additional experiments with bigger size of matrix coefficients.

4 Numerical Experiments

We provide experiments wit different matrix coefficients with small dimension ($n = 2, 3$). Numerical experiments are executed on the computer with Intel(R) Core(TM) i7-1065G7 CPU @ 1.30GHz 1.50 GHz.

Example 1. We introduce an example with the 2×2 matrix coefficients. Using Matlab's notation we introduce the matrices:

A=[3 -1; -1 3];
D=[102 -100; -100 102];
C=[1.2 0.9; 1.0 0.65];
B=C;

Example 2. The 2×2 matrix coefficients are:

A=[3 -1; -1 3];
D=[102 -100; -100 102];
C=[0.97 0.6; 1.2 0.79];
A=[3 -1; -1 3];
B=ksi*C; (for different values of ksi).

Example 3. The 3×3 matrix coefficients are:

A=[65 -2 -0.18;-2 65 -0.9; -0.18 -0.9 65];
D=[202 -140 -98; -140 202 -101; -98 -101 202];
C=[15 2 1; 2 12 4; 1 4 15];
B=ksi*C; (for different values of ksi).

Example 4. The 3×3 matrix coefficients are:

D=[65 -2 -0.18;-2 65 -0.9; -0.18 -0.9 65];
A=[202 -140 -98; -140 202 -101; -98 -101 202];
C=[15 2 1; 2 12 4; 1 4 15];
B=ksi*C; (for different values of ksi).

Table 1. $tol = 1.0e - 14$, 1000 runs

	Algorithm 1		**Algorithm 2**	
	Example 1			
$\gamma =$	It	CPU	It	CPU
	1162	4.01s	1188	4.0s
	Example 2, ksi=1.0			
δ	845	2.8s	860	2.8s
	Example 2, ksi=1.27			
δ	4174	14.7s	4298	14.3s
	Example 3, ksi=1.0			
δ	483	2.4s	462	2.3s
	Example 3, ksi=1.03			
δ	1492	7.6s	1430	6.9s
	Example 4, ksi=1.03			
α	1495	7.7s	1936	9.4s

Applying Algorithm 3 to Example 4 we obtain 1429 iteration steps and CPU time of 7.1s. Algorithm 3 pays this computational price to obtain the minimal nonnegative solution of $R(X) = 0$.

Example 5. [2] The matrix coefficients A, B, C and D are $n \times n$ matrices and we compute them as follows:

$$A = D = \text{tridiag } (-I, T, -I),$$

are block tridiagonal matrices. The matrix C is

$$C = \frac{1}{50} \text{ tridiag } (1, 2, 1)$$

is a tridiagonal matrix and $B = SD + AS - SCS$, i.e. S is the minimal nonnegative solution of $R(X) = 0$. We take e the vector of units and compute S

$$S = \frac{1}{50} ee^T .$$

In addition, the $m \times m, (n = m^2)$ matrix T is

$$T = \text{tridiag}\left(-1, 4 + \frac{200}{(m+1)^2}, -1\right).$$

Computational experiment to compute the minimal nonnegative solution gives the results described in Table 2.

Table 2. $tol = 1.0e - 14$, 1000 runs

	Algorithm 1		**Algorithm 2**		**Algorithm 3**	
m	*It*	CPU	*It*	CPU	*It*	CPU
8	25	10.2s	22	9.8s	22	9.4s
12	52	159.7s	44	165.0s	44	143.0s

Results from experiments show the effectiveness of the considered approach.

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