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SEVERAL VARIATIONAL INCLUSIONS FOR A FRACTIONAL DIFFERENTIAL INCLUSION OF CAPUTO-FABRIZIO TYPE*

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Dedicated to Dr. Dan Tiba on the occasion of his 70^{th} anniversary

Abstract

We establish several fractional variational inclusions for solutions of a nonconvex fractional differential inclusion involving Caputo-Fabrizio fractional derivative.

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1 Introduction

Recently, a new fractional order derivative with regular kernel has been introduced by Caputo and Fabrizio [3]. This new definition is able to describe better heterogeneousness, systems with different scales with memory effects, the wave movement on surface of shallow water, the heat transfer model, mass-spring-damper model etc.. Another good property of this new

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definition is that using Laplace transform of the fractional derivative the fractional differential equation turns into a classical differential equation of integer order. Some properties of this definition have been studied in [1, 4, 8] etc.. Several papers are devoted to the development of this new fractonal derivative [6, 7, 8, 9, 10, 11] etc..

In Control Theory, mainly, if we want to obtain necessary optimality conditions, it is essential to have several "differentiability" properties of solutions with respect to initial conditions. One of the most powerful result in the theory of differential equations, the classical Bendixson-Picard-Lindelőf theorem states that the maximal flow of a differential equation is differentiable with respect to initial conditions and its derivatives verify the variational equation. This result has been generalized in various ways to differential inclusions by considering several variational inclusions and proving corresponding theorems that extend Bendixson-Picard-Lindelőf theorem. The present paper is concerned with fractional differential inclusions of the form

$$D^{\sigma}_{CF}x(t) \in F(t, x(t)) \quad a.e. \ ([0, T]), \quad x(0) = x_0, \quad x'(0) = x_1 \qquad (1.1)$$

where $\alpha \in (0, 1)$, $\sigma = \alpha+1$, D^{σ}_{CF} is the Caputo-Fabrizio fractional derivative, $F : [0, T] \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is a set-valued map and $x_0, x_1 \in \mathbf{R}$.

The aim of this note is to extend the results concerning the differentiability of solutions of differential inclusions with respect to initial conditions to the solutions of problem (1.1). The results we extend known as the contingent, the intermediate (quasitangent) and the circatangent variational inclusion are already obtained in the "classical case" of differential inclusions. For all these results and for a complete discussion on this topic we refer to [2]. The proofs of our results follows by an approach similar to the classical case of differential inclusions ([2]) and use a recent result ([5]) concerning the existence of solutions of problem (1.1).

The results in the present paper may be regarded as a continuation of the study in [7], where it is proved that the reachable set of the intermediate (quasitangent) Caputo-Fabrizio variational inclusion is a derived cone in the sense of Hestenes to the reachable set of a given Caputo-Fabrizio fractional differential inclusion.

The paper is organized as follows: in Section 2 we recall some preliminary results that we need in the sequel and in Section 3 we prove our results.

2 Preliminaries

Let Y be a normed space, $X \subset Y$ and $x \in \overline{X}$ (the closure of X). From the multitude of the tangent cones in the literature (e.g., [2]) we recall only the contingent, the quasitangent and Clarke's tangent cones, defined, respectively by:

$$\begin{array}{l} K_xX = \{v \in Y; \quad \exists s_m \to 0+,\ \exists v_m \to v:\ x + s_m v_m \in X\} \\ Q_xX = \{v \in Y; \quad \forall s_m \to 0+,\ \exists v_m \to v:\ x + s_m v_m \in X\}, \\ C_xX = \{v \in Y; \forall (x_m, s_m) \to (x, 0+),\ x_m \in X,\ \exists y_m \in X:\ \frac{y_m - x_m}{s_m} \to v\}. \end{array}$$

This cones are related as follows: $C_xX \subset Q_xX \subset K_xX$.

Corresponding to each type of tangent cone, say τ_xX, one may introduce a *set-valued directional derivative* of a multifunction $G(\cdot) : X \subset Y \to \mathcal{P}(Y)$ (in particular of a single-valued mapping) at a point $(x, y) \in Graph(G)$ as follows

$$\tau_yG(x; v) = \{w \in Y; \quad (v, w) \in \tau_{(x,y)}Graph(G)\}, \quad v \in \tau_xX.$$

Let $I := [0, T]$, denote by $C(I, \mathbf{R})$ the Banach space of all continuous functions from I to $\mathbf{R}$ endowed with the norm $|x|_C = \sup_{t\in I} |x(t)|dt$ and by $L^1(I, \mathbf{R})$ we denote the Banach space of Lebegue integrable functions $u(.) : I \to \mathbf{R}$ endowed with the norm $|u|_1 = \int_0^1 |u(t)|dt$.

The next definitions were introduced in [3].

a) The Caputo-Fabrizio integral of order $\alpha \in (0, 1)$ of a function $f \in AC_{loc}([0, \infty), \mathbf{R})$ (which means that $f'(.)$ is integrable on $[0, T]$ for any $T > 0$) is defined by

$$I^\alpha_{CF} f(t) = (1 - \alpha)f(t) + \alpha \int_0^t f(s)ds.$$

b) The Caputo-Fabrizio fractional derivative of order $\alpha \in (0, 1)$ of f is defined for $t \geq 0$ by

$$D^\alpha_{CF} f(t) = \frac{1}{1 - \alpha} \int_a^t e^{-\frac{\alpha}{1-\alpha}(t-s)} f'(s)ds.$$

c) The Caputo-Fabrizio fractional derivative of order $\sigma = \alpha + n$, $\alpha \in (0, 1)$ $n \in \mathbf{N}$ of f is defined by

$$D^\sigma_{CF} f(t) = D^\alpha_{CF}(D^n_{CF} f(t)).$$

In particular, if $\sigma = \alpha + 1$, $\alpha \in (0, 1)$ $D^\sigma_{CF} f(t) = \frac{1}{1-\alpha} \int_a^t e^{-\frac{\alpha}{1-\alpha}(t-s)} f''(s)ds$.

A function $x(.) \in C(I, \mathbf{R})$ is called a solution of problem (1.1) if there exists a function $f(.) \in L^1(I, \mathbf{R})$ with $f(t) \in F(t, x(t))$, *a.e.* (I) such that $D^{\sigma}_{CF}x(t) = f(t)$, *a.e.* (I) and $x(0) = x_0, x'(0) = x_1$. In this case we say that $(x(.), f(.))$ is a *trajectory-selection pair* of (1.1).

We shall use the following notations for the solution sets of (1.1).

$$\mathcal{S}(x_0, x_1) = \{(x(.), f(.)); \quad (x(.), f(.)) \text{ is a trajectory-selection pair of (1.1)}\}.$$

Hypothesis 2.1. i) $F(.,.) : I \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ has nonempty closed values and for every $x \in \mathbf{R}$, $F(., x)$ is measurable.

ii) There exist $L(.) \in L^1(I, (0, \infty))$ such that for almost all $t \in I, F(t, .)$ is $L(t)$-Lipschitz in the sense that

$$d_H(F(t, x_1), F(t, x_2)) \le L(t)|x_1 - x_2| \quad \forall \ x_1, x_2 \in \mathbf{R},$$

where $d_H(A, C)$ is Pompeiu-Hausdorff distance between closed sets $A, C \subset \mathbf{R}$

$$d_H(A, C) = \max\{d^*(A, C), d^*(C, A)\}, \quad d^*(A, C) = \sup\{d(a, C); a \in A\}.$$

On $C(I, \mathbf{R}) \times L^1(I, \mathbf{R})$ we consider the following norm

$$|(x, f)|_{C \times L} = |x|_C + |f|_1 \quad \forall \ (x, f) \in C(I, \mathbf{R}) \times L^1(I, \mathbf{R}).$$

The next result ([5]) is an extension of Filippov's theorem concerning the existence of solutions to a Lipschitzian differential inclusion to fractional differential inclusions of the form (1.1).

Consider $y_0, y_1 \in \mathbf{R}$, $g(.) \in L^1(I, \mathbf{R})$ and $y(.)$ is a solution of the problem

$$D^{\alpha}_{CF}y(t) = g(t), \quad y(0) = y_0, \quad y'(0) = y_1.$$

Theorem 2.2. *Assume that Hypothesis 2.1 is satisfied and there exists* $q(.) \in L^1(I, \mathbf{R})$ *such that* $d(D^{\alpha}_{CF}y(t), F(t, y(t))) \le q(t)$ *a.e.* (I). *Denote* $\eta(t) = (|x_0 - y_0| + t|x_1 - y_1| + |q|_1)e^{\int_0^t L(s)ds}$. *Then there exists* $(x(.), f(.)) \in C(I, \mathbf{R}) \times L^1(I, \mathbf{R})$ *a trajectory-selection pair of (1.1) satisfying for all* $t \in I$

$$|x(t) - y(t)| \le \eta(t) \quad \forall t \in I,$$

$$|f(t) - g(t)| \le L(t)\eta(t) + q(t) \quad a.e.\ (I).$$

3 Results

Let $(y(.), g(.))$ be a trajectory-selection pair of problem (1.1). We wish to "linearize" (1.1) along $(y(.), g(.))$ by replacing it by several fractional variational inclusions.

Consider, first, the contingent variational inclusion

$$\begin{cases} D^{\sigma}_{CF} w(t) \in \overline{co} K_{g(t)}(F(t,.))(y(t); w(t)) \quad a.e.\ (I) \\ w(0) = u, \quad w'(0) = v. \end{cases} \tag{3.1}$$

Theorem 3.1. *Consider the solution map $\mathcal{S}(.,.)$ as a set valued map from $\mathbf{R} \times \mathbf{R}$ into $C(I, \mathbf{R}) \times L^{\infty}(I, \mathbf{R})$, with $L^{\infty}(I, \mathbf{R})$ supplied with the weak-* topology and assume that Hypothesis 2.1 is satisfied.*

Then for any $u, v \in \mathbf{R}$ one has

$$K_{(y,g)}\mathcal{S}((y(0), y'(0); (u, v)) \subset$$
$$\{(w, \pi); (w, \pi) \text{ is a trajectory-selection pair of } (3.1)\}.$$

Proof. Let $u, v \in \mathbf{R}$ and let $(w, \pi) \in K_{(y,g)}\mathcal{S}((y(0), y'(0); (u, v))$. According to the definition of the contingent derivative there exist $h_n \to 0+, u_n \to u, v_n \to v$, $w_n(.) \to w(.)$ in $C(I, \mathbf{R})$, $\pi_n(.) \to \pi(.)$ in weak-* topology of $L^{\infty}(I, \mathbf{R})$ and $c > 0$ such that

$$\begin{aligned} &|\pi_n(t)| \leq c \quad a.e.\ (I), \\ &g(t) + h_n \pi_n(t) \in F(t, y(t) + h_n w_n(t)) \quad a.e.\ (I), \\ &w_n(0) = u_n, w'_n(0) = v_n. \end{aligned} \tag{3.2}$$

Therefore,

$$\begin{aligned} &w_n(.) \quad \text{converges pointwise to} \quad w(.) \\ &\pi_n(.) \quad \text{converges weak in } L^1(I, \mathbf{R}) \text{ to} \quad \pi(.) \end{aligned} \tag{3.3}$$

We apply Mazur's theorem and we find that there exists

$$v_m(t) = \sum_{p=m}^{\infty} a_m^p \pi_p(t)$$

$v_m(.) \to \pi(.)$ (strong) in $L^1(I, \mathbf{R})$, where $a_m^p \geq 0$, $\sum_{p=m}^{\infty} a_m^p = 1$ and for any m, $a_m^p \neq 0$ for a finite number of p.

Therefore, a subsequence (again denoted) $v_m(.)$ converges la $\pi(.)$ a.e.. From (3.2) for any p and for almost all $t \in I$

$$w'_p(t) \in \frac{1}{h_p}(F(t, y(t) + h_p w_p(t)) - g(t)) \cap cB$$

Let $t \in I$ be such that $v_m(t) \to \pi(t)$ and $g(t) \in F(t, y(t))$. Fix $n \geq 1$ and $\epsilon > 0$. From (3.2) there exists m such that $h_p \leq 1/n$ and $|w_p(t) - w(t)| \leq 1/n$ for any $p \geq m$.

If, we denote

$$\phi(z, h) := \frac{1}{h}(F(t, y(t) + hz) - g(t)) \cap cB$$

then

$$v_m(t) \in co(\cup_{h \in (0, \frac{1}{n}], z \in B(w(t), \frac{1}{n})} \phi(z, h))$$

and if $m \to \infty$, we get

$$\pi(t) \in \overline{co}(\cup_{h \in (0, \frac{1}{n}], z \in B(w(t), \frac{1}{n})} \phi(z, h)).$$

Since, $\phi(z, h) \subset cB$, we infer that

$$\pi(t) \in \overline{co} \cap_{\epsilon > 0, n \geq 1} (\cup_{h \in (0, \frac{1}{n}], z \in B(w(t), \frac{1}{n})} \phi(z, h) + \epsilon B).$$

On the other hand,

$$\cap_{\epsilon > 0, n \geq 1}(\cup_{h \in (0, \frac{1}{n}], z \in B(w(t), \frac{1}{n})} \phi(z, h) + \epsilon B) \subset K_{g(t)} F(t, .)(y(t); w(t))$$

and the proof is complete.

Next, we study the intermediate (quaitangent) variational inclusion

$$\begin{cases} D^{\sigma}_{CF} w(t) \in Q_{g(t)}(F(t, .))(y(t); w(t)) \quad a.e.\ (I) \\ w(0) = u, \quad w'(0) = v, \end{cases} \tag{3.4}$$

where $u, v \in \mathbf{R}$.

Theorem 3.2. *Consider the solution map $\mathcal{S}(., .)$ as a set valued map from $\mathbf{R} \times \mathbf{R}$ into $C(I, \mathbf{R}) \times L^1(I, \mathbf{R})$ and assume that Hypothesis 2.1 is satisfied.*

Then, for any $u, v \in \mathbf{R}$ and any trajectory-selection pair (w, π) of the fractional differential inclusion (3.4) one has

$$(w, \pi) \in Q_{(y,g)} \mathcal{S}((y(0), y'(0); (u, v)).$$

Proof. Let $u, v \in \mathbf{R}$ and let $(w, \pi) \in C(I, \mathbf{R}) \times L^1(I, \mathbf{R})$ be a trajectory-selection pair of (3.4). By the definition of the quasitangent derivative and from the Lipschitzianity of $F(t, .)$, for almost all $t \in I$, we have

$$\begin{aligned} &\lim_{h \to 0+} d(D^{\alpha}_{CF} w(t), \tfrac{F(t, y(t) + hw(t)) - D^{\alpha}_{CF} y(t)}{h}) = \\ &\lim_{h \to 0+} d(\pi(t), \tfrac{F(t, y(t) + hw(t)) - g(t)}{h}) = 0. \end{aligned} \tag{3.5}$$

Moreover, since $g(t) \in F(t, y(t))$ a.e. (I), from Hypothesis 2.1, for all enough small $h > 0$ and for almost all $t \in I$, one has

$d(D^{\alpha}_{CF}(y(t) + hw(t)), F(t, y(t) + hw(t))) = d(g(t) + h\pi(t), F(t, y(t)+ hw(t))) \leq h(|\pi(t)| + L(t)|w(t)|)$.

By standard arguments the function $t \to d(g(t) + h\pi(t), F(t, y(t) + hw(t)))$ is measurable. Therefore, using the Lebesgue dominated convergence theorem we infer

$$\int_0^T e^{-\frac{\alpha}{1-\alpha}(T-s)} d(D^{\alpha}_{CF}(y(t) + hw(t)), F(t, y(t) + hw(t)))dt \leq o(h), \quad (3.6)$$

where $\lim_{h\to 0+} \frac{o(h)}{h} = 0$.

We apply Theorem 2.2 and by (3.6) we deduce the existence of $M \geq 0$ and of trajectory-selection pairs $(y_h(.), g_h(.))$ of the fractional differential inclusion (1.1) satisfying

$$|y_h - y - hw|_C + |g_h - g - h\pi|_1 \leq Mo(h), \ y_h(0) = y(0) + hu, \ y'_h(0) = y'(0) + hv,$$

which implies

$$\lim_{h\to 0+} \frac{y_h - y}{h} = w \quad \text{in} \quad C(I, \mathbf{R}), \qquad \lim_{h\to 0+} \frac{g_h - g}{h_n} = \pi \quad \text{in} \quad L^1(I, \mathbf{R}).$$

Therefore

$$\lim_{h\to 0+} d_{C\times L}((w, \pi), \frac{\mathcal{S}((y(0) + hu, y'(0) + hv)) - (y, g)}{h}) = 0$$

and the proof is complete.

Finally we deal with the variational inclusion defined by the Clarke directional derivative of the set-valued map $F(t, .)$, i.e., the so called circatangent variational inclusion

$$\begin{cases} D^{\sigma}_{CF} w(t) \in C_{g(t)}(F(t, .))(y(t); w(t)) \quad a.e.\ (I) \\ w(0) = u, \quad w'(0) = v. \end{cases} \quad (3.7)$$

Theorem 3.3. *Consider the solution map $\mathcal{S}(.,.)$ as a set valued map from $\mathbf{R} \times \mathbf{R}$ into $C(I, \mathbf{R}) \times L^1(I, \mathbf{R})$ and assume that Hypothesis 2.1 is satisfied.*

Then, for any $u, v \in \mathbf{R}$ and any trajectory-selection pair (w, π) of the fractional differential inclusion (3.7) one has

$$(w, \pi) \in C_{(y,g)}\mathcal{S}((y(0), y'(0); (u, v)).$$

Proof. Let $u, v \in \mathbf{R}$, let $(w, \pi) \in C(I, \mathbf{R}) \times L^1(I, \mathbf{R})$ be a trajectory-selection pair of (3.7), let (y_n, g_n) be a sequence of trajectory-selection pairs of (1.1) that converges to $(y, g) \in C(I, \mathbf{R}) \times L^1(I, \mathbf{R})$ and let $h_n \to 0+$. Then there exists a subsequence $g_j(.) := g_{n_j}(.)$ such that

$$\lim_{j\to\infty} g_j(t) = g(t) \quad a.e.\ (I) \tag{3.8}$$

Denote $\lambda_j := h_{n_j}$. From (3.7) and from the definition of the Clarke directional derivative, for almost all $t \in I$ we have

$$\begin{aligned} &\lim_{h\to 0+} d(D^{\alpha}_{CF}w(t), \tfrac{F(t,y_j(t)+\lambda_j w(t))-D^{\alpha}_{CF}y_j(t)}{\lambda_j}) = \\ &\lim_{h\to 0+} d(\pi(t), \tfrac{F(t,y_j(t)+\lambda_j w(t))-g_j(t)}{\lambda_j}) = 0. \end{aligned} \tag{3.9}$$

Since $g_j(t) \in F(t, y_j(t))$ a.e. (I), for almost all $t \in I$, we get

$$d(D^{\alpha}_{CF}(y_j(t) + \lambda_j w(t)), F(t, y_j(t) + \lambda_j w(t))) = d(g_j(t) + \lambda_j \pi(t), F(t, y_j(t) + \lambda_j w(t))) \leq \lambda_j(|\pi(t)| + L(t)|w(t)|).$$

The last inequality together with Lebesgue's dominated convergence theorem implies

$$\int_0^T e^{-\frac{\alpha}{1-\alpha}(T-s)} d((D^{\alpha}_{CF}(y_j(t) + \lambda_j w(t)), F(t, y_j(t) + \lambda_j w(t)))dt \leq o(\lambda_j), \tag{3.10}$$

where $\lim_{j\to\infty} \frac{o(\lambda_j)}{\lambda_j} = 0$.

We apply Theorem 2.2 and by (3.10) we deduce the existence of $M \geq 0$ and of trajectory-selections pairs $(\overline{y}_j(.), \overline{g}_j(.))$ of the fractional differential inclusion in (1.1) satisfying

$$|\overline{y}_j - y_j - \lambda_j w|_C + |\overline{g}_j - g_j - \lambda_j \pi|_1 \leq M o(\lambda_j),$$

$$\overline{y}_j(0) = y(0) + \lambda_j u, \quad \overline{y}_j'(0) = y'(0) + \lambda_j v.$$

It follows that

$$\lim_{j\to\infty} \frac{\overline{y}_j - y}{\lambda_j} = w \quad \text{in} \quad C(I, \mathbf{R}), \quad \lim_{j\to\infty} \frac{\overline{g}_j - g}{\lambda_j} = \pi \quad \text{in} \quad L^1(I, \mathbf{R}),$$

which completes the proof.

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