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THE EXISTENCE OF POSITIVE SOLUTIONS OF SINGULAR STURM-LIOUVILLE BOUNDARY VALUE PROBLEMS ON A MEASURE CHAIN *

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Abstract

The authors study the existence of positive solutions of singular Sturm-Liouville boundary value problem

$$(p(t)y^{\Delta}(t))^{\Delta} + \lambda q(t)f(t, y^{\sigma}(t)) = 0, \; \rho(a) < t < \sigma(b),$$

with boundary conditions

$$\alpha y(\rho(a)) - \beta p(\rho(a))y^{\Delta}(\rho(a)) = 0,$$
$$\gamma y(\sigma(b)) + \delta p(\sigma(b))y^{\Delta}(\sigma(b)) = 0,$$

on a measure chain, where $\lambda > 0$ and q is allowed to be singular at both end points $t = \rho(a)$ and $t = \sigma(b)$. We shall use a fixed point theorem on a cone in a Banach space to obtain the existence of positive solutions for λ in a suitable interval of a measure chain.

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keywords: Positive solution, Sturm-Liouville boundary value problems, singular, fixed point theorem, cone.

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1 Introduction

Consider the singular Sturm-Liouville boundary value problem consisting of the dynamic equation

$$(p(t)y^{\Delta}(t))^{\Delta} + \lambda q(t)f(t,\, y^{\sigma}(t)) = 0,\ t \in (\rho(a),\, \sigma(b))_{\mathbb{T}}, \tag{1.1}$$

with homogeneous boundary conditions

$$\begin{aligned} \alpha y(\rho(a)) - \beta p(\rho(a))y^{\Delta}(\rho(a)) = 0, \\ \gamma y(\sigma(b)) + \delta p(\sigma(b))y^{\Delta}(\sigma(b)) = 0, \end{aligned} \tag{1.2}$$

where $t \in (\rho(a),\, \sigma(b))_{\mathbb{T}} = (\rho(a),\, \sigma(b)) \cap \mathbb{T} = \{t \in \mathbb{T} : \rho(a) < t < \sigma(b)\}$, $\sigma : \mathbb{T} \to \mathbb{T}$, $y^{\sigma} : \mathbb{T} \to [0,\, \infty)$, $y : [\rho(a),\, \sigma(b)]_{\mathbb{T}} \to [0,\, \infty)$ and

(H_1) $p \in C_{rd}([\rho(a),\, \sigma(b)]_{\mathbb{T}}, (0,\, +\infty))$ and $0 < \int_{\rho(a)}^{\sigma(b)} \frac{\Delta t}{p(t)} < +\infty$;

(H_2) $\lambda > 0$, α, β, δ, and γ are non-negative, and $\beta\gamma + \alpha\gamma + \alpha\delta > 0$;

(H_3) $f(t,\, y^{\sigma}(t)) \in C_{rd}([\rho(a),\, \sigma(b)]_{\mathbb{T}} \times [0,\, +\infty), \mathbb{R}^+)$ and $q \in C_{rd}((\rho(a),\, \sigma(b))_{\mathbb{T}}, [0,\, +\infty))$;

(H_4) $0 < \int_{\rho(a)}^{\sigma(b)} G(\sigma(t),\, t)q(t)\Delta t < +\infty$;

where

$$G(\sigma(s),\, s) = \frac{1}{\eta}\left(\beta + \alpha \int_{\rho(a)}^{\sigma(s)} \frac{\Delta s}{p(s)}\right)\left(\delta + \gamma \int_{\sigma(s)}^{\sigma(b)} \frac{\Delta s}{p(s)}\right),\ \rho(a) \le \sigma(s) \le \sigma(b),$$

and

$$\eta = \alpha\delta + \alpha\gamma \int_{\rho(a)}^{\sigma(b)} \frac{\Delta s}{p(s)} + \beta\gamma.$$

For any $t \in [\rho(a),\, \sigma(b)]_{\mathbb{T}}$, let us define the limits

$$f_0(t) = \lim_{y^{\sigma} \to 0^+} \frac{f(t,\, y^{\sigma})}{y^{\sigma}},$$

and

$$f_{\infty}(t) = \lim_{y^{\sigma} \to +\infty} \frac{f(t,\, y^{\sigma})}{y^{\sigma}}.$$

Moreover both the limits exist uniformly in the extended reals. We consider the following cases:

(L_1) $f_0(t) = +\infty,\quad t \in [\rho(a),\, \sigma(b)]_{\mathbb{T}}$;

(L_2) $f_{\infty}(t) = +\infty,\quad t \in [\rho(a),\, \sigma(b)]_{\mathbb{T}}$;

(L_3) $f_0(t) = 0,\quad t \in [\rho(a),\, \sigma(b)]_{\mathbb{T}}$;

(L_4) $f_{\infty}(t) = 0,\quad t \in [\rho(a),\, \sigma(b)]_{\mathbb{T}}$;

(L_5) $f_0(t) = l_1 > 0, \quad t \in [\rho(a),\, \sigma(b)]_{\mathbb{T}}$;
(L_6) $f_\infty(t) = l_2 > 0, \quad t \in [\rho(a),\, \sigma(b)]_{\mathbb{T}}$.

The case $f_0 = 0$ and $f_\infty = \infty$ is called the *super linear case* and the case $f_0 = \infty$ and $f_\infty = 0$ is called the *sub linear case.*

For the special case $\mathbb{T} = \mathbb{R}$ and $q(t) = 1$, the existence of positive solution for the boundary value problem (1.1) and (1.2) has been investigated in [1]. In this paper, we would like to obtain some existence results of positive solution to the boundary value problem (1.1) and (1.2) for λ in a suitable interval of a measure chain. Our results improve and generalise some results in [2, 17, 19].

Remark 1.1 *In our theorems and corollaries of main results we are considering simultaneously two cases for establishing the existence of positive solution of the boundary value problem. We have total six number of cases. Out of six, we are selecting two cases simultaneously. So the total number of possible pairs are* ${}^6C_2 = 15$. *These pairs are* L_1L_2, L_1L_3, L_1L_4, L_1L_5, L_1L_6, L_2L_3, L_2L_4, L_2L_5, L_2L_6, L_3L_4, L_3L_5, L_3L_6, L_4L_5, L_4L_6 *and* L_5L_6. *We are considering only* L_1L_2, L_1L_4, L_1L_6, L_2L_3, L_2L_5,L_3L_4, L_3L_6 *and* L_4L_5 *in our theorems and corollaries. Rest of these* L_1L_3, L_1L_5, L_2L_4, L_2L_6, L_3L_5, *and* L_4L_6 *are invalid pairs. For example, let us consider*

$$(L_1)\, f_0(t) = \lim_{y^\sigma \to 0^+} \frac{f(t,\, y^\sigma)}{y^\sigma} = +\infty,$$

and

$$(L_3)\, f_0(t) = \lim_{y^\sigma \to 0^+} \frac{f(t,\, y^\sigma)}{y^\sigma} = 0.$$

In both the cases y^σ *approches to zero but their limiting value approches to two different limits. So the limit does not exist. Thus, we cannot take these pairs simultaneously.*

By a positive solution of the boundary value problem (1.1) and (1.2), we mean a function $y \in C_{rd}[\rho(a),\, \sigma(b)]_{\mathbb{T}} \cap C^1_{rd}[\rho(a),\, \sigma(b)]_{\mathbb{T}}$, $p(t)y^\Delta \in C^1_{rd}[\rho(a),\, \sigma(b)]_{\mathbb{T}}$ such that $y(t)$ satisfies equation (1.1) and boundary conditions (1.2), with $y(t) > 0$ on $[\rho(a),\, \sigma(b)]_{\mathbb{T}}$. Let a and b be such that $0 \le \rho(a) \le a < b \le \sigma(b) < \infty$ and $(\rho(a),\, \sigma(b))_{\mathbb{T}}$ has at least two points.

We are concerned with the calculus on a measure chain which is the unification of continuous and discrete calculus. An excellent introduction to this subject is given by S. Hilger [13] and monograph by B. Kayamakcalan [14]. For the basic knowledge of time-scale calculus readers are advised to refer the monographs of Bohner and Peterson [3, 4]. In order to understand and familiarize the notations of the time scale calculus, we need some preliminary definitions.

Definition 1.1 *Let $\mathbb{T}$ be a time scale, that is, $\mathbb{T}$ is an arbitrary non-empty closed subset of the real numbers $\mathbb{R}$. For each interval $\mathbb{I} = \mathbb{I} \cap \mathbb{T}$. We assume throught that $\mathbb{T}$ has the topology that it inherits from standard topology on the real numbers $\mathbb{R}$. For $t < sup\mathbb{T}$, define the forward jump operator by*

$$\sigma(t) := \inf\{s > t : s \in \mathbb{T}\} \in \mathbb{T}$$

and for $t > \inf(t)$ define the backward jump operator by

$$\rho(t) := \sup\{s < t : s \in \mathbb{T}\} \in \mathbb{T}$$

for all $t \in \mathbb{T}$. If $\sigma(t) > t$, we say t is right scattered, while if $\rho(t) < t$ we say t is left scattered. If $\sigma(t) = t$, we say t right dense, while $\rho(t) = t$, we say t is left dense.

Definition 1.2 *Define the interval in $\mathbb{T}$; $[a, b] = \{t \in \mathbb{T}$ such that $a \leq t \leq b\}$. Other types of intervals are defined similarly.*

Definition 1.3 *Assume $x : \mathbb{T} \to \mathbb{R}$ and fix $t \in \mathbb{T}$ (if $t = sup\mathbb{T}$ assume t is not left scattered), then define $x^{\Delta}(t)$ to be the number (provided it exists) with the property that given ane $\epsilon > 0$, there is a neighbourhood $\mathbb{U}$ of t such that*

$$|[x(\sigma(t) - x(s)] - x^{\Delta}(t)[\sigma(t) - s]| \leq \epsilon|\sigma(t) - s|$$

for all $s \in \mathbb{U}$. We call x^{Δ} the delta derivative of $x(t)$.

It can be shown that if $x : \mathbb{T} \to \mathbb{R}$ is continuous at $t \in \mathbb{T}$ and t is right scattered, then

$$x^{\Delta}(t) = \frac{x(\sigma(t)) - x(t)}{\sigma(t) - t}.$$

If it is right dense, then

$$x^{\Delta}(t) = \lim_{s \to t} \frac{x(\sigma(t)) - x(s)}{\sigma(t) - s}.$$

We say that $x : \mathbb{T} \to \mathbb{R}$ is right dense continuous on $\mathbb{T}$ provided it is continuous at all right dense points and at points that are left dense and right scattered we just assume that left hand limit exists (and finite). We denote this by $x \in C_{rd}(\mathbb{T})$. If $\mathbb{T} = \mathbb{Z}$, the set of integers, then

$$x^{\Delta}(t) = \Delta x(t) = x(t+1) - x(t).$$

Furthermore, the equation (1.1) reduce to the self-adjoint difference equation

$$\Delta(p(t)\Delta y(t)) + \lambda q(t) f(t,\, y(t+1)) = 0,\; a-1 \le t \le b+1.$$

If $\mathbb{T} = \mathbb{R}$, the set of reals, then the equation (1.1) reduce to the self-adjoint differential equation

$$(p(t)y'(t))' + \lambda q(t) f(t,\, y(t)) = 0,\; a \le t \le b.$$

Definition 1.4 *If*

$$F^{\Delta}(t) = f(t),$$

then we define an integral by

$$\int_a^t f(s)\Delta s = F(t) - F(a).$$

In this paper, we will use elementary properties of this integral which are available in [3].

Definition 1.5 *If* $\mathbb{T}$ *has a left-scattered maximum* m*, then* $\mathbb{T}^k = \mathbb{T} - \{m\}$*. Otherwise,* $\mathbb{T}^k = \mathbb{T}$*. In summary,*

$$\mathbb{T}^k = \begin{cases} \mathbb{T}\backslash(\rho(\sup \mathbb{T}),\, \sup \mathbb{T})) & if \quad \sup \mathbb{T} < \infty \\ \mathbb{T} & if \quad \sup \mathbb{T} = \infty. \end{cases}$$

Definition 1.6 *We say that* $x : \mathbb{T} \times \mathbb{T}^{k^2} \to \mathbb{R}$ *is a Cauchy function for*

$$Lx(t) = (px^{\Delta})^{\Delta}(t) + q(t)x^{\sigma}(t) = 0 \tag{1.3}$$

provided for each fixed $s \in \mathbb{T}^{k^2}$*,* $x(.,s)$ *is the solution of the initial value problem*

$$Lx(.,s) = 0,\; x(\sigma(s),\, s) = 0,\; x^{\Delta}(\sigma(s),\, s) = -\frac{1}{p(\sigma(s))}.$$

It can be easily varified that if $q = 0$, then Cauchy function for $(px^{\Delta})^{\Delta} = 0$ is given by

$$x(t,\, s) = \int_{\sigma(s)}^{t} \frac{1}{p(\tau)} \Delta\tau.$$

We will use the following results which has been proved in [3] with slightly modification of operator.

Theorem 1.1 *(Green's function for general two point boundary value problem). Assume that the boundary value problem:*

$$Lx(t) = (p(t)x^{\Delta}(t))^{\Delta} + q(t)x^{\sigma}(t) = 0, \tag{1.4}$$

$$\begin{aligned} \alpha x(a) - \beta x^{\Delta}(a) &= 0, \\ \gamma x(\sigma(b)) + \delta x^{\Delta}(\sigma(b)) &= 0, \end{aligned} \tag{1.5}$$

has only the trivial solution. For each fixed $s \in [a,\, b]$, let $u(t,\, s)$ be the unique solution of the boundary value problem:

$$\begin{aligned} Lu(t,\, s) &= 0, \\ \alpha u(a,\, s) - \beta u^{\Delta}(a,\, s) &= 0, \\ \gamma u(\sigma(b),\, s) + \delta u^{\Delta}(\sigma(b),\, s) &= -\gamma x(\sigma(b),\, s) - \delta x^{\Delta}(\sigma(b),\, s), \end{aligned}$$

where $x(t,\, s)$ is the Cauchy function for (1.3). Then we define Green's function $G : [a,\, \sigma(b)]_{\mathbb{T}} \times [a,\, b]_{\mathbb{T}} \to \mathbb{R}$ for the boundary value problem (1.4) and (1.5) by

$$G(t,\, s) = \begin{cases} u(t,\, s), & t \le s, \\ v(t,\, s), & t \ge \sigma(s), \end{cases}$$

where $v(t,\, s) := u(t,\, s) + x(t,\, s)$ for $t \in [a,\, \sigma(b)]$, $s \in [a,\, b]$. Then for each fixed $s \in [a,\, b]$, $v(.,\, s)$ is a solution of (1.4) and satisfies the second boundary condition in (1.5). If $h \in C_{rd}$, then

$$u(t) = \int_{a}^{\sigma(b)} G(t,\, s)h(s)\Delta s,$$

is the solution of the non-homogeneous boundary value problem:

$$\begin{aligned} Lu &= h(t), \\ \alpha u(a) - \beta u^{\Delta}(a) &= A, \\ \gamma u(\sigma(b)) + \delta u^{\Delta}(\sigma(b)) &= B, \end{aligned}$$

with $A = B = 0$ (where A and B are constants).

Lemma 1.1 *[8] The boundary value problem:*

$$\begin{aligned} -(p(t)y^\Delta(t))^\Delta &= 0, \\ \alpha y(\sigma(a)) - \beta y^\Delta(\sigma(a)) &= 0, \\ \gamma y(\sigma(b)) + \delta y^\Delta(\sigma(b)) &= 0, \end{aligned}$$

has only the trivial solution if and only if $\frac{\gamma\beta}{p(a)} + \frac{\alpha\beta}{p(\sigma(b))} + \alpha\gamma \int_a^{\sigma(b)} \frac{1}{p(s)} \Delta s \neq 0.$

2 Green's Function

Let us consider the homogeneous boundary value problem:

$$-(p(t)y^\Delta(t))^\Delta = 0, \; t \in (\rho(a), \sigma(b))_{\mathbb{T}}, \tag{2.1}$$

$$\begin{aligned} \alpha y(\rho(a)) - \beta p(\rho(a)) y^\Delta(\rho(a)) &= 0, \\ \gamma y(\sigma(b)) + \delta p(\sigma(b)) y^\Delta(\sigma(b)) &= 0. \end{aligned} \tag{2.2}$$

The linearly independent solutions of (2.1) are

$$y_1(t) = 1$$

and

$$y_2(t) = \int_{\rho(a)}^t \frac{1}{p(\tau)} \Delta\tau.$$

Let $y(t, s)$ be as in the statement of Theorem 1.1. Since for each fixed $s \in [a, b]_{\mathbb{T}}$, $y(t, s)$ is a solution of

$$-(p(t)y^\Delta(t))^\Delta = 0, \; \rho(a) < t < \sigma(b),$$

then, the general solution of (2.1) is given by

$$y(t, s) = c_1(s) + c_2(s) \int_{\rho(a)}^t \frac{1}{p(\tau)} \Delta\tau.$$

Note that the Cauchy function for (2.1) is given by

$$x(t, s) = -\int_{\sigma(s)}^t \frac{1}{p(\tau)} \Delta\tau.$$

Since

$$\alpha y(\rho(a)) - \beta p(\rho(a)) y^\Delta(\rho(a)) = 0,$$

then

$$\alpha c_1(s) - \beta p(\rho(a))\frac{c_2(s)}{p(\rho(a))} = 0,$$

that is,

$$c_1(s) = \frac{\beta c_2(s)}{\alpha}.$$

From the boundary condition

$$\gamma y(\sigma(b)) + \delta p(\sigma(b))y^{\Delta}(\sigma(b)) = -\gamma x(\sigma(b)) - \delta p(\sigma(b))x^{\Delta}(\sigma(b)),$$

we obtain

$$\gamma y(\sigma(b)) + \delta p(\sigma(b))y^{\Delta}(\sigma(b)) = \gamma\int_{\sigma(s)}^{\sigma(b)}\frac{1}{p(\tau)}\Delta\tau + \delta p(\sigma(b))\frac{1}{p(\sigma(b))}\ ,$$

implies that

$$\gamma\left(c_1(s) + c_2(s)\int_{\rho(a)}^{\sigma(b)}\frac{1}{p(\tau)}\Delta\tau\right) + \delta p(\sigma(b))\frac{c_2(s)}{p(\sigma(b))} = \gamma\int_{\sigma(s)}^{\sigma(b)}\frac{1}{p(\tau)}\Delta\tau + \delta$$

that is,

$$\gamma c_1(s) + \left(\gamma\int_{\rho(a)}^{\sigma(b)}\frac{1}{p(\tau)}\Delta\tau + \delta\right)c_2(s) = \gamma\int_{\sigma(s)}^{\sigma(b)}\frac{1}{p(\tau)}\Delta\tau + \delta.$$

Using $c_1(s) = \frac{\beta c_2(s)}{\alpha}$, we obtain

$$c_2(s) = \frac{\alpha}{\eta}\left(\gamma\int_{\sigma(s)}^{\sigma(b)}\frac{1}{p(\tau)}\Delta\tau + \delta\right).$$

The value of $c_1(s)$ is given by

$$\begin{aligned} c_1(s) &= \frac{\beta c_2(s)}{\alpha} \\ &= \frac{\beta}{\eta}\left(\gamma\int_{\sigma(s)}^{\sigma(b)}\frac{1}{p(\tau)}\Delta\tau + \delta\right), \end{aligned}$$

where

$$\eta = \alpha\delta + \beta\gamma + \alpha\gamma\int_{\rho(a)}^{\sigma(b)}\frac{1}{p(\tau)}\Delta\tau.$$

Hence,

$$\begin{aligned}
y(t,\,s) &= c_1(s) + c_2(s)\int_{\rho(a)}^{t}\frac{1}{p(\tau)}\Delta\tau \\
&= \frac{\beta}{\eta}\left(\gamma\int_{\sigma(s)}^{\sigma(b)}\frac{1}{p(\tau)}\Delta\tau+\delta\right)+\frac{\alpha}{\eta}\left(\gamma\int_{\sigma(s)}^{\sigma(b)}\frac{1}{p(\tau)}\Delta\tau+\delta\right)\left(\int_{\rho(a)}^{t}\frac{1}{p(\tau)}\Delta\tau\right) \\
&= \frac{1}{\eta}\left(\alpha\int_{\rho(a)}^{t}\frac{1}{p(\tau)}\Delta\tau+\beta\right)\left(\gamma\int_{\sigma(s)}^{\sigma(b)}\frac{1}{p(\tau)}\Delta\tau+\delta\right).
\end{aligned}$$

Now,

$$\begin{aligned}
v(t,\,s) &= y(t,\,s) + x(t,\,s) \\
&= \frac{1}{\eta}\left(\gamma\int_{\sigma(s)}^{\sigma(b)}\frac{1}{p(\tau)}\Delta\tau+\delta\right)\left(\alpha\int_{\rho(a)}^{t}\frac{1}{p(\tau)}\Delta\tau+\beta\right)-\int_{\sigma(s)}^{t}\frac{1}{p(\tau)}\Delta\tau \\
&= \frac{1}{\eta}\left(\alpha\int_{\rho(a)}^{\sigma(s)}\frac{1}{p(\tau)}\Delta\tau+\beta\right)\left(\gamma\int_{t}^{\sigma(b)}\frac{1}{p(\tau)}\Delta\tau+\delta\right).
\end{aligned}$$

Where,

$$\begin{aligned}
H &= \int_{\sigma(s)}^{\sigma(b)}\frac{1}{p(\tau)}\Delta\tau\int_{\rho(a)}^{t}\frac{1}{p(\tau)}\Delta\tau-\int_{\sigma(s)}^{t}\frac{1}{p(\tau)}\Delta\tau\int_{\rho(a)}^{\sigma(b)}\frac{1}{p(\tau)}\Delta\tau \\
&= \int_{t}^{\sigma(b)}\frac{\Delta\tau}{p(\tau)}\int_{\rho(a)}^{\sigma(s)}\frac{\Delta\tau}{p(\tau)}.
\end{aligned}$$

Hence, the Green's function $G(t,\,s)$ is given by

$$G(t,\,s) = \begin{cases} y(t,\,s), & t\le s, \\ v(t,\,s), & t\ge\sigma(s). \end{cases}$$

That is,

$$G(t,\,s) = \frac{1}{\eta}\begin{cases}\left(\alpha\int_{\rho(a)}^{t}\frac{1}{p(\tau)}\Delta\tau+\beta\right)\left(\gamma\int_{\sigma(s)}^{\sigma(b)}\frac{1}{p(\tau)}\Delta\tau+\delta\right), & t\le s, \\ \left(\alpha\int_{\rho(a)}^{\sigma(s)}\frac{1}{p(\tau)}\Delta\tau+\beta\right)\left(\gamma\int_{t}^{\sigma(b)}\frac{1}{p(\tau)}\Delta\tau+\delta\right), & t\ge\sigma(s),\end{cases} \tag{2.3}$$

and

$$G(\sigma(s),\,s) = \frac{1}{\eta}\left(\alpha\int_{\rho(a)}^{\sigma(s)}\frac{1}{p(\tau)}\Delta\tau+\beta\right)\left(\gamma\int_{\sigma(s)}^{\sigma(b)}\frac{1}{p(\tau)}\Delta\tau+\delta\right),\ \rho(a)\le t,\ \sigma(s)\le\sigma(b).$$

Clearly,

$$G(t,\, s) \le G(\sigma(s),\, s), \quad \rho(a) \le t,\, \sigma(s) \le \sigma(b). \tag{2.4}$$

By (H_4), there exits $t_0 \in (\rho(a),\, \sigma(b))_{\mathbb{T}}$ such that $q(t_0) \ge 0$. We may choose $\theta \in \left(\rho(a),\, \frac{\rho(a)+\sigma(b)}{2}\right)_{\mathbb{T}}$ such that $t_0 \in (\theta,\, \sigma(b)-\theta)_{\mathbb{T}}$. Define a cone K_θ as follows:

$$K_\theta = \{y \in C([\rho(a),\, \sigma(b)]_{\mathbb{T}}) : y(t) \ge 0,\, \min_{\theta \le t \le \sigma(b)-\theta} y(t) \ge M_\theta \|y\|\}, \tag{2.5}$$

where

$$M_\theta = \min\left\{ \frac{\delta + \gamma \int_{\sigma(b)-\theta}^{\sigma(b)} \frac{\Delta\tau}{p(\tau)}}{\delta + \gamma \int_{\rho(a)}^{\sigma(b)} \frac{\Delta\tau}{p(\tau)}},\, \frac{\beta + \alpha \int_{\rho(a)}^{\theta} \frac{\Delta\tau}{p(\tau)}}{\beta + \alpha \int_{\rho(a)}^{\sigma(b)} \frac{\Delta\tau}{p(\tau)}} \right\},$$

and

$$\|y\| = \sup_{t \in [\rho(a),\, \sigma(b)]} |y(t)|.$$

Let us denote

$$\phi(t) = \delta + \gamma \int_t^{\sigma(b)} \frac{\Delta\tau}{p(\tau)}, \quad \rho(a) \le t \le \sigma(b),$$
$$\psi(t) = \beta + \alpha \int_{\rho(a)}^{t} \frac{\Delta\tau}{p(\tau)}, \quad \rho(a) \le t \le \sigma(b).$$

Then

$$G(t,\, s) = \frac{1}{\eta}\begin{cases} \psi(t)\phi(\sigma(s)), & \rho(a) \le t \le s \le \sigma(b), \\ \psi(\sigma(s))\phi(t), & \rho(a) \le \sigma(s) \le t \le \sigma(b). \end{cases}$$

For $\theta \le t \le \sigma(b) - \theta$, we have

$$\frac{G(t,\, s)}{G(\sigma(s),\, s)} = \begin{cases} \frac{\psi(t)}{\psi(\sigma(s))} \ge \frac{\beta+\alpha \int_{\rho(a)}^{\theta} \frac{\Delta\tau}{p(\tau)}}{\beta+\alpha \int_{\rho(a)}^{\sigma(b)} \frac{\Delta\tau}{p(\tau)}}, & t \le s, \\ \frac{\phi(t)}{\phi(\sigma(s))} \ge \frac{\delta+\gamma \int_{\sigma(b)-\theta}^{\sigma(b)} \frac{\Delta\tau}{p(\tau)}}{\delta+\gamma \int_{\rho(a)}^{\sigma(b)} \frac{\Delta\tau}{p(\tau)}}, & t \ge \sigma(s). \end{cases}$$

Hence,

$$G(t,\, s) \ge M_\theta G(\sigma(s),\, s), \quad \theta \le t \le \sigma(b) - \theta. \tag{2.6}$$

Now define an operator $T_\lambda : K_\theta \to K_\theta$ as follows:

$$T_\lambda y(t) = \lambda \int_{\rho(a)}^{\sigma(b)} G(t,s)q(s)f(s,y^\sigma(s))\Delta s, \lambda \ge 0, y \in K_\theta. \tag{2.7}$$

Note that if $y(t)$ is a solution of the boundary value problem (1.1) and (1.2), then y(t) satisfies the integral equation (2.7).

Lemma 2.1 $T_\lambda(K_\theta) \subset K_\theta$.

Proof. By (2.4), we have for any $t \in [\rho(a),\, \sigma(b)]_{\mathbb{T}}$ and $y \in K_\theta$,

$$\begin{aligned} T_\lambda y(t) &= \lambda \int_{\rho(a)}^{\sigma(b)} G(t,\, s)q(s)f(s,\, y^\sigma(s))\Delta s \\ &\le \lambda \int_{\rho(a)}^{\sigma(b)} G(\sigma(s),\, s)q(s)f(s,\, y^\sigma(s))\Delta s. \end{aligned}$$

Hence,

$$\|T_\lambda\| \le \lambda \int_{\rho(a)}^{\sigma(b)} G(\sigma(s),\, s)q(s)f(s,\, y^\sigma(s))\Delta s. \tag{2.8}$$

By (2.6), we have

$$\begin{aligned} \min_{\theta \le t \le \sigma(b)-\theta} T_\lambda y(t) &= \min_{\theta \le t \le \sigma(b)-\theta} \lambda \int_{\rho(a)}^{\sigma(b)} G(t,\, s)q(s)f(s,\, y^\sigma(s))\Delta s \\ &\ge M_\theta \lambda \int_{\rho(a)}^{\sigma(b)} G(\sigma(s),\, s)q(s)f(s,\, y^\sigma(s))\Delta s. \end{aligned} \tag{2.9}$$

From (2.8) and (2.9), we have

$$\min_{\theta \le t \le \sigma(b)-\theta} T_\lambda y(t) \ge M_\theta \|Ty\|, \quad y \in K_\theta.$$

Hence, $T_\lambda(K_\theta) \subset K_\theta$. The proof of the Lemma is complete.

Lemma 2.2 $T_\lambda : K_\theta \to K_\theta$ *is a completely continuous operator.*

The proof is similar to that of Lemma 2 in [17].

Lemma 2.3 *([15]) Let K be a cone in a Banach space E and Ω_1, Ω_2 be two bounded open sets in E such that $0 \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$. Let $T : K \cap (\overline{\Omega_2} \backslash \Omega_1) \to K$ be a completely continuous operator. If*
$\|Ty\| \le \|y\|$, for all $y \in K \cap \partial\Omega_1$, and $\|Ty\| \ge \|y\|$, for all $y \in K \cap \partial\Omega_2$,
or
$\|Ty\| \ge \|y\|$, for all $y \in K \cap \partial\Omega_1$, and $\|Ty\| \le \|y\|$, for all $y \in K \cap \partial\Omega_2$;

then T has at least one fixed point in $K \cap (\overline{\Omega_2} \backslash \Omega_1)$.

Let

$$A = \max_{\rho(a)\le t\le\sigma(b)} \int_{\rho(a)}^{\sigma(b)} G(t,\, s)q(s)\Delta s\ ,$$

and

$$B_\theta = \min_{\theta\le t\le\sigma(b)-\theta} \int_{\theta}^{\sigma(b)-\theta} G(t,\, s)q(s)\Delta s.$$

Lemma 2.4 *Assume that* $(H_1)-(H_4)$ *hold and there exist two different positive numbers* c *and* d *such that*

$$\max_{\rho(a)\le t\le\sigma(b),\, 0\le y^\sigma\le c} f(t,\, y^\sigma) \le \frac{c}{\lambda A} \tag{2.10}$$

$$\min_{\rho(a)\le t\le\sigma(b)-\theta,\, M_\theta d\le y^\sigma\le d} f(t,\, y^\sigma) \ge \frac{d}{\lambda B_\theta} \tag{2.11}$$

Then the boundary value problem (1.1) and (1.2) has at least one positive solution $\overline{y}\in K_\theta$ *and* $\min\{c,\, d\}\le\|\overline{y}\|\le\max\{c,\, d\}$.

Proof. Without loss of generality, we may assume that $c\le d$. Let

$$\Omega_c = \{y\in C([\rho(a),\, \sigma(b)]_{\mathbb{T}}):\ \|y\|\le c\},$$

and

$$\Omega_d = \{y\in C([\rho(a),\, \sigma(b)]_{\mathbb{T}}):\ \|y\|\le d\}.$$

By (2.10), for any $t\in[\rho(a),\, \sigma(b)]_{\mathbb{T}}$ and $y\in K_\theta\cap\partial\Omega_c$, we have

$$f(t,\, y^\sigma)\le\frac{c}{\lambda A}.$$

Hence,

$$\begin{aligned} T_\lambda y(t) &= \lambda\int_{\rho(a)}^{\sigma(b)} G(t,\, s)q(s)f(s,\, y^\sigma(s))\Delta s \\ &\le \lambda\int_{\rho(a)}^{\sigma(b)} G(\sigma(s),\, s)q(s)f(s,\, y^\sigma(s))\Delta s \\ &\le c. \end{aligned}$$

Hence,

$$\|Ty\|\le\|y\|,\ y\in K_\theta\cap\partial\Omega_c.$$

By (2.11), for any $t\in[\theta,\, \sigma(b)-\theta]_{\mathbb{T}}$ and $y\in K_\theta\cap\partial\Omega_d$, we have

$$f(t,\, y^\sigma)\ge\frac{d}{\lambda B_\theta}.$$

Hence,

$$\begin{aligned}T_\lambda y(t_0) &= \lambda \int_{\rho(a)}^{\sigma(b)} G(t_0,\, s)q(s)f(s,\, y^\sigma(s))\Delta s \\ &\geq \lambda \int_{\theta}^{\sigma(b)-\theta} G(t_0,\, s)q(s)f(s,\, y^\sigma(s))\Delta s \\ &\geq d.\end{aligned}$$

Hence,

$$\|Ty\| \geq \|y\|,\ y \in K_\theta \cap \partial\Omega_d.$$

It follows from Lemmaa 2.3 that there exists a $\overline{y} \in K_\theta \cap (\overline{\Omega_d}\backslash\Omega_c)$ such that $T_\lambda \overline{y}(t) = \overline{y}(t)$ and $\overline{y}(t)$ lies in between c and d. This means that $\overline{y}(t)$ is a solution of boundary value problem (1.1) and (1.2) and $\min\{c,\, d\} \leq \|\overline{y}\| \leq \max\{c,\, d\}$. This completes the proof of the lemma.

3 Main Results

Let us denote

$$\begin{aligned}\lambda_1 &= \frac{1}{A}\sup_{r>0} \frac{r}{\displaystyle\max_{\rho(a)\leq t\leq\sigma(b),\, 0\leq y^\sigma\leq r} f(t,\, y^\sigma)}\ , \\ \lambda_2 &= \frac{1}{B_\theta}\inf_{r>0} \frac{r}{\displaystyle\min_{\rho(a)\leq t\leq\sigma(b),\, M_\theta r\leq y^\sigma\leq r} f(t,\, y^\sigma)}\ .\end{aligned}$$

By our hypothesis it is clear that $0 \leq \lambda_1 \leq +\infty$ and $0 \leq \lambda_2 < +\infty$.

Theorem 3.1 *Assume that* $(H_1) - (H_4)$*,* (L_1) *and* (L_2) *hold. Then there exists* $\lambda_1 > 0$ *such that the boundary value problem (1.1) and (1.2) has at least two positive solutions for* $0 < \lambda < \lambda_1$.

Proof. Define

$$s(r) = \frac{r}{A \displaystyle\max_{\rho(a)\leq t\leq\sigma(b),\ 0\leq y^\sigma\leq r} f(t,\, y^\sigma)}$$

By (H_3), we know that $s : (0,\, +\infty) \to (0,\, +\infty)$ is continuous. In view of (L_1) and (L_2) we have

$$\lim_{r\to 0^+} s(r) = \lim_{r\to+\infty} s(r) = 0.$$

Hence, there exists $r_0 \in (0, +\infty)$ such that $s(r_0) = \max_{r>0} s(r) = \lambda_1$. For $0 < \lambda < \lambda_1$, there exist two positive numbers c_1 and c_2 such that $0 < c_1 < r_0 < c_2 < +\infty$ and

$$s(c_1) = s(c_2) = \lambda.$$

Hence,

$$\max_{\rho(a) \le t \le \sigma(b),\ 0 \le y^\sigma \le c_1} f(t, y^\sigma) = \frac{c_1}{A\lambda} ,$$

and

$$\max_{\rho(a) \le t \le \sigma(b),\ 0 \le y^\sigma \le c_2} f(t, y^\sigma) = \frac{c_2}{A\lambda}.$$

On the other hand by (L_1) and (L_2), there exist d_1 and d_2 such that $0 < d_1 < c_1 < r_0 < c_2 < d_2 < +\infty$ and

$$\frac{f(t, y^\sigma)}{y^\sigma} \ge \frac{1}{\lambda M_\theta B_\theta}, \quad t \in [\rho(a), \sigma(b)]_{\mathbb{T}}, \quad y^\sigma \in [0, d_1) \cup [M_\theta d_2, +\infty),$$

that is,

$$f(t, y^\sigma) \ge \frac{y^\sigma}{\lambda M_\theta B_\theta} \ge \frac{d_1}{\lambda B_\theta}, \quad t \times y^\sigma \in [\rho(a), \sigma(b)]_{\mathbb{T}} \times [M_\theta d_1, d_1].$$

Hence,

$$f(t, y^\sigma) \ge \frac{y^\sigma}{\lambda M_\theta B_\theta} \ge \frac{d_2}{\lambda B_\theta}, \quad t \times y^\sigma \in [\rho(a), \sigma(b)]_{\mathbb{T}} \times [M_\theta d_2, d_2].$$

Thus,

$$\min_{\theta \le t \le \sigma(b)-\theta,\, M_\theta d_1 \le y^\sigma \le d_1} f(t, y^\sigma) \ge \min_{\rho(a) \le t \le \sigma(b),\, M_\theta d_1 \le y^\sigma \le d_1} f(t, y^\sigma) \ge \frac{d_1}{\lambda B_\theta} ,$$

and

$$\min_{\theta \le t \le \sigma(b)-\theta,\, M_\theta d_2 \le y^\sigma \le d_2} f(t, y^\sigma) \ge \min_{\rho(a) \le t \le \sigma(b),\, M_\theta d_2 \le y^\sigma \le d_2} f(t, y^\sigma) \ge \frac{d_2}{\lambda B_\theta} .$$

By Lemma 2.4, the boundary value problem (1.1) and (1.2) has at least two positive solutions. The proof of the theorem is complete.

Theorem 3.2 *Assume that $(H_1) - (H_4)$ and one of (L_1) and (L_2) hold. Then the boundary value problem (1.1) and (1.2) has at least one positive solutions for $0 < \lambda < \lambda_1$.*

The statement of the Theorem 3.2 is follows from the proof of the Theorem 3.1.

Theorem 3.3 *Assume that* $(H_1)-(H_4)$, (L_3) *and* (L_4) *hold. Then there exists* $\lambda_2 \geq 0$ *such that the boundary value problem (1.1) and (1.2) has at least two positive solutions for* $\lambda_2 < \lambda < +\infty$.

Proof. Define

$$w(r) = \frac{r}{B_\theta \min\limits_{\rho(a)\leq t\leq\sigma(b),\, M_\theta r\leq y^\sigma\leq r} f(t,\, y^\sigma)}\,.$$

Clearly, $w(r)$ is continuous in $(0,\, +\infty)$. From (L_3) and (L_4), we have

$$\lim_{r\to 0^+} w(r) = \lim_{r\to+\infty} w(r) = +\infty.$$

Hence, there exists $r_0 \in (0,\, +\infty)$ such that

$$w(r_0) = \min_{r>0} w(r) = \lambda_2 \geq 0.$$

Since $\lambda_2 < \lambda < +\infty$, we can find two positive numbers d_1 and d_2 such that $0 < d_1 < r_0 < d_2 < +\infty$ and

$$w(d_1) = w(d_2) = \lambda.$$

Hence,

$$\min_{\rho(a)\leq t\leq\sigma(b),\, M_\theta d_1\leq y^\sigma\leq d_1} f(t,\, y^\sigma) = \frac{d_1}{\lambda B_\theta},$$

and

$$\min_{\rho(a)\leq t\leq\sigma(b),\, M_\theta d_2\leq y^\sigma\leq d_2} f(t,\, y^\sigma) = \frac{d_2}{\lambda B_\theta}.$$

On the other hand, (L_3) implies that there exists $c_1 \in (0,\, d_1)$ such that

$$\frac{f(t,\, y^\sigma)}{y^\sigma} \leq \frac{1}{\lambda A}\,,\quad t\times y^\sigma \in [\rho(a),\, \sigma(b)]_{\mathbb{T}} \times [0,\, c_1]\,,$$

implies that,

$$f(t,\, y^\sigma) \leq \frac{c_1}{\lambda A}\,,\quad t\times y^\sigma \in [\rho(a),\, \sigma(b)]_{\mathbb{T}} \times [0,\, c_1].$$

From (L_4), there exists $c \in (d_2,\, +\infty)$ such that

$$\frac{f(t,\, y^\sigma)}{y^\sigma} \leq \frac{1}{\lambda A}\,,\quad t\times y^\sigma \in [\rho(a),\, \sigma(b)]_{\mathbb{T}} \times [c,\, +\infty].$$

Let

$$M = \max_{\rho(a) \le t \le \sigma(b),\, 0 \le y^\sigma \le c} f(t,\, y^\sigma).$$

We can choose $c_2 > c$ such that $c_2 \ge \lambda M A$. Hence,

$$f(t,\, y^\sigma) \le \frac{c_2}{\lambda A}, \quad t \in [\rho(a),\, \sigma(b)]_{\mathbb{T}},\, y^\sigma \in [0,\, c_2]$$

By Lemma 2.4, the boundary value problem (1.1) and (1.2) has at least two positive solutions for $\lambda_2 < \lambda < +\infty$. The proof of the theorem is complete.

From the proof of the Theorem 3.3, we have the following result.

Theorem 3.4 *Assume that* $(H_1) - (H_4)$ *and one of* (L_3) *and* (L_4) *hold. Then the boundary value problem (1.1) and (1.2) has at least one positive solutions for* $\lambda_2 < \lambda < +\infty$.

Corollary 3.1 *Assume that* $(H_1) - (H_4)$ *hold. Moreover, one of the following conditions is true:*
(i) (L_1) *and* (L_4) *hold;*
(ii) (L_2) *and* (L_3) *hold.*
Then the boundary value problem (1.1) and (1.2) has at least one positive solution for $\lambda > 0$.

Proof. Let us assume (i) hold. By Theorem 3.2, we only need to prove $\lambda_1 = +\infty$. If

$$\sup_{\rho(a) \le t \le \sigma(b),\, 0 \le y^\sigma \le +\infty} f(t,\, y^\sigma) = M < +\infty,$$

then

$$\lambda_1 \ge \frac{1}{A} \sup_{r > 0} \frac{r}{M} = +\infty.$$

If f is unbounded, then there exist $t_n \in [\rho(a),\, \sigma(b)]_{\mathbb{T}}$ and $r_n \to +\infty$ such that

$$f(t_n,\, r_n) = \max_{\rho(a) \le t \le \sigma(b),\, 0 \le y^\sigma \le r_n} f(t,\, y^\sigma)$$

By (L_4),

$$\lim_{r_n \to +\infty} \frac{f(t_n,\, r_n)}{r_n} = 0,$$

and

$$\begin{aligned}
\lambda_1 &= \max_{r>0} s(r) \\
&\geq \max_{r_n>0} s(r_n) \\
&= \max_{r_n>0} \frac{r_n}{A \max\limits_{\rho(a)\leq t\leq\sigma(b),\, 0\leq y^\sigma\leq r_n} f(t,\, y^\sigma)} \\
&= \max_{r_n>0} \frac{r_n}{Af(t_n,\, r_n)} = +\infty.
\end{aligned}$$

Hence, $\lambda_1 = +\infty$. Now, let us assume that (ii) holds. By Theorem 3.4, it is sufficient to prove $\lambda_2 = 0$. By (L_2), for $t \in [\rho(a),\, \sigma(b)]_{\mathbb{T}}$, $f(t,\, y^\sigma) \to +\infty$ as $y^\sigma \to +\infty$. There exist $t_n \in [\rho(a),\, \sigma(b)]_{\mathbb{T}}$ and $r_n \to +\infty$ such that

$$f(t_n,\, M_\theta r_n) = \min_{\rho(a)\leq t\leq\sigma(b),\, M_\theta r_n\leq y^\sigma\leq r_n} f(t,\, y^\sigma).$$

Again by (L_2),

$$\lim_{r_n\to+\infty} \frac{f(t_n,\, M_\theta r_n)}{r_n} = +\infty,$$

and

$$\begin{aligned}
\lambda_2 &= \min_{r>0} w(r) \\
&\leq \min_{r_n>0} w(r_n) \\
&= \min_{r_n>0} \frac{r_n}{B_\theta \min\limits_{\rho(a)\leq t\leq\sigma(b),\, M_\theta r_n\leq y^\sigma\leq r_n} f(t,\, y^\sigma)} \\
&= \min_{r_n>0} \frac{r_n}{f(t_n,\, M_\theta r_n)} = 0
\end{aligned}$$

Hence, $\lambda_2 = 0$. So the boundary value problem (1.1) and (1.2) has at least one positive solution for $\lambda > 0$. The proof of the corollary is complete.

Remark 3.1 *Corollary 3.1 improves and generalizes the results in [9, 17, 19].*

Corollary 3.2 *Assume that* $(H_1) - (H_4)$ *hold. Moreover, one of the following conditions is true:*
(i) (L_1) *and* (L_6) *hold;*
(ii) (L_2) *and* (L_5) *hold.*
Then the boundary value problem (1.1) and (1.2) has at least one positive solution for $0 < \lambda < \frac{1}{Al_1}$.

Proof. Let us assume (i) hold. Then, By (L_1) for $t \in [\rho(a), \sigma(b)]_{\mathbb{T}}$, $\frac{f(t, y^\sigma)}{y^\sigma} \to +\infty$ as $y^\sigma \to 0^+$. There exist $t_n \in [\rho(a), \sigma(b)]_{\mathbb{T}}$ and $r_n \to 0^+$ such that

$$f(t_n, M_\theta r_n) = \min_{\rho(a) \le t \le \sigma(b),\, M_\theta r_n \le y^\sigma \le r_n} f(t, y^\sigma)$$

Again by (L_1),

$$\lim_{r_n \to 0^+} \frac{f(t_n, M_\theta r_n)}{r_n} = +\infty,$$

and

$$\begin{aligned}
\lambda_2 &= \min_{r>0} w(r) \\
&\le \min_{r_n>0} w(r_n) \\
&= \min_{r_n>0} \frac{r_n}{B_\theta \min\limits_{\rho(a) \le t \le \sigma(b),\, M_\theta r_n \le y^\sigma \le r_n} f(t, y^\sigma)} \\
&= \min_{r_n>0} \frac{r_n}{f(t_n, M_\theta r_n)} \\
&= 0.
\end{aligned}$$

Hence $\lambda_2 = 0$.
Let us assume (ii) holds. For $t_n \in [\rho(a), \sigma(b)]_{\mathbb{T}}$, and $r_n \to 0^+$ such that

$$f(t_n, r_n) = \max_{\rho(a) \le t \le \sigma(b),\, 0 \le y^\sigma \le r_n} f(t, y^\sigma).$$

By (L_5),

$$\lim_{r_n \to 0} \frac{f(t_n, r_n)}{r_n} = l_1 > 0,$$

and

$$\begin{aligned}
\lambda_1 &= \max_{r>0} s(r) \\
&\ge \max_{r_n>0} s(r_n) \\
&= \max_{r_n>0} \frac{r_n}{A \max\limits_{\rho(a) \le t \le \sigma(b),\, 0 \le y^\sigma \le r_n} f(t, y^\sigma)} \\
&= \frac{1}{A l_1}.
\end{aligned}$$

Hence, $\lambda_1 \ge \frac{1}{A l_1}$. By Theorem 3.2, the boundary value problem (1.1) and (1.2) has at least one positive solution for $0 < \lambda < \frac{1}{A l_1}$.

Corollary 3.3 *Assume that $(H_1)-(H_4)$ hold. Moreover, one of the following conditions is true:*
(i) (L_3) and (L_6) hold;
(ii) (L_4) and (L_5) hold.
Then the boundary value problem (1.1) and (1.2) has at least one positive solution for $\frac{1}{M_\theta B_\theta l_2} < \lambda < +\infty$.

Proof. We only need to prove that $\lambda_2 \leq \frac{1}{M_\theta B_\theta l_2}$. Let us assume (i) hold. By (L_6), for $t \in [\rho(a), \sigma(b)]_{\mathbb{T}}$, $\frac{f(t, y^\sigma)}{y^\sigma} \to l_2 > 0$ as $y^\sigma \to +\infty$. There exist $t_n \in [\rho(a), \sigma(b)]_{\mathbb{T}}$ and $r_n \to +\infty$ such that

$$f(t_n, M_\theta r_n) = \min_{\rho(a)\leq t\leq\sigma(b),\, M_\theta r_n \leq y^\sigma \leq r_n} f(t, y^\sigma).$$

Again by (L_6),

$$\lim_{r_n \to +\infty} \frac{f(t_n, M_\theta r_n)}{r_n} = M_\theta l_2 > 0$$

and

$$\begin{aligned}
\lambda_2 &= \min_{r>0} w(r) \\
&\leq \min_{r_n>0} w(r_n) \\
&= \min_{r_n>0} \frac{r_n}{B_\theta \displaystyle\min_{\rho(a)\leq t\leq\sigma(b),\, M_\theta r_n \leq y^\sigma \leq r_n} f(t, y^\sigma)} \\
&= \min_{r_n>0} \frac{r_n}{f(t_n, M_\theta r_n)} \\
&= \frac{1}{M_\theta B_\theta l_2}.
\end{aligned}$$

Hence, $\lambda_2 \leq \frac{1}{M_\theta B_\theta l_2}$. Now, we assume (ii) holds. For $t_n \in [\rho(a), \sigma(b)]_{\mathbb{T}}$ and $r_n \to +\infty$ such that

$$f(t_n, r_n) = \max_{\rho(a)\leq t\leq\sigma(b),\, 0 \leq y^\sigma \leq r_n} f(t, y^\sigma)$$

By (L_4),

$$\lim_{r_n \to +\infty} \frac{f(t_n, r_n)}{r_n} = 0,$$

and

$$\begin{aligned}\lambda_1 &= \max_{r>0} s(r)\\ &\geq \max_{r_n>0} s(r_n)\\ &= \max_{r_n>0} \frac{r_n}{A \max\limits_{\rho(a)\leq t\leq\sigma(b),\, 0\leq y^\sigma\leq r_n} f(t,\, y^\sigma)}\\ &= +\infty.\end{aligned}$$

Hence, $\lambda_1 = +\infty$. So by Theorem 3.4, the boundary value problem (1.1) and (1.2) has at least one positive solution for $\frac{1}{M_\theta B_\theta l_2} < \lambda < +\infty$.

Remark 3.2 *Corollary 3.2 and Corollary 3.3 generalizes the results of Corollary 3.2 and Corollary 3.3 respectively in [19].*

4 Examples

In this section, several examples has been illustrated to validates the results obtained in the earlier section.

Example 4.1 *Consider the singular boundary value problem*

$$\begin{cases}((t^2+1)y^\Delta)^\Delta + \lambda\frac{1}{t\sqrt{\sigma(1)-t}}(y^\sigma)^2 = 0, \quad \rho(0) < t < \sigma(1),\\ \alpha y(\rho(0)) - \beta((\rho(0))^2+1)y^\Delta(\rho(0)) = 0,\\ \gamma y(\sigma(1)) + \delta((\sigma(1))^2+1)y^\Delta(\sigma(1)) = 0.\end{cases} \tag{4.1}$$

Here, $p(t) = (t^2+1)$, $q(t) = \frac{1}{t\sqrt{\sigma(1)-t}}$ *and* $f(t, y^\sigma) = (y^\sigma)^2$ *and* $\alpha,\ \beta,\ \gamma,\ \delta \geq 0$*.*

When $\mathbb{T} = \mathbb{R}$*, then the above boundary value problem becomes*

$$\begin{cases}((t^2+1)y')' + \lambda\frac{1}{t\sqrt{1-t}}y^2 = 0, \quad 0 < t < 1,\\ \alpha y(0) - \beta y^\Delta(0) = 0,\\ \gamma y(1) + 2\delta y^\Delta(1) = 0.\end{cases} \tag{4.2}$$

Here, $p(t) = (t^2+1)$, $q(t) = \frac{1}{t\sqrt{(1-t)}}$ *and* $f(t,\, y(t)) = (y(t))^2$*,* $p(0) = 1$*,* $p(1) = 2$ *and* $\alpha,\ \beta,\ \gamma,\ \delta \geq 0$*. Clearly* (H_1) *-* (H_4) *are satisfied. Note that*

$$G(s,\, s) = \frac{1}{\eta}\left(\gamma\left(\frac{\pi}{4} - tan^{-1}(s)\right) + \delta\right)\left(\alpha\ tan^{-1}(s) + \beta\right) \geq 0,$$

and

$$\lim_{y\to+\infty}\frac{f(t,\,y)}{y}=\lim_{y\to+\infty}\frac{y^2}{y}=+\infty,$$
$$\lim_{y\to 0}\frac{f(t,\,y)}{y}=\lim_{y\to 0}\frac{y^2}{y}=0.$$

Thus (L_2) *and* (L_3) *hold. By Corollary 3.1, the boundary value problem (4.2) has at least one positive solution for* $\lambda > 0$.

Example 4.2 *When* $\mathbb{T}=\mathbb{R}$, *let us consider the boundary value problem*

$$\begin{cases} ((t^2+1)y')'+\lambda\frac{1}{t\sqrt{(1-t)}}y^2 siny=0, & 0<t<1,\\ \alpha y(0)-\beta y'(0)=0, \\ \gamma y(1)+2\delta y'(1)=0. \end{cases} \tag{4.3}$$

Here, $p(t)=(t^2+1)$, $q(t)=\frac{1}{t\sqrt{(1-t)}}$ *and* $f(t,\,y)=y^2$, $p(0)=1$, $p(1)=2$ *and* $\alpha,\,\beta,\,\gamma,\,\delta\geq 0$. *Clearly* (H_1) - (H_4) *are satisfied. Note that*

$$G(s,\,s)=\frac{1}{\eta}\left(\gamma\left(\frac{\pi}{4}-tan^{-1}(s)\right)+\delta\right)\left(\alpha\, tan^{-1}(s)+\beta\right)\geq 0,$$

and

$$\lim_{y\to+\infty}\frac{f(t,\,y)}{y}=\lim_{y\to+\infty}\frac{y^2 siny}{y}=+\infty,$$
$$\lim_{y\to 0^+}\frac{f(t,\,y)}{y}=\lim_{y\to 0^+}\frac{y^2 siny}{y}=0.$$

Thus (L_2) *and* (L_3) *hold. By Corollary 3.1, the boundary value problem (4.3) has at least one positive solution for* $\lambda > 0$.

Example 4.3 *When* $\mathbb{T}=\mathbb{Z}$, *let us consider the boundary value problem*

$$\begin{cases} \Delta^2 y(t-1)+\lambda\frac{1}{t\sqrt{(2-t)}}\frac{y^2}{siny}=0, & 0<t<6,\\ \alpha y(0)=0, \\ \gamma y(6)=0. \end{cases} \tag{4.4}$$

Here, $p(t)=1$, $q(t)=\frac{1}{t\sqrt{(2-t)}}$ *and* $f(t,\,y)=\frac{y^2}{siny}$, $p(0)=1$, $p(6)=1$ *and* $\alpha,\,\beta,\,\gamma,\,\delta\geq 0$. *Clearly* (H_1) - (H_4) *are satisfied. Note that*

$$G(s,\,s)=(6-s)s\geq 0$$

and

$$\lim_{y\to+\infty}\frac{f(t,\,y)}{y}=\lim_{y\to+\infty}\frac{y^2}{ysiny}=+\infty,$$
$$\lim_{y\to0^+}\frac{f(t,\,y)}{y}=\lim_{y\to0^+}\frac{y^2}{ysiny}=1.$$

Thus (L_2) *and* (L_5) *hold. By Corollary 3.2, the boundary value problem (4.4) has at least one positive solution for* $0<\lambda<\frac{1}{Al_1}$.

Example 4.4 *When* $\mathbb{T}=h\mathbb{Z}$, *let us consider the boundary value problem*

$$\begin{cases}\Delta_h((t+1)\Delta_h y)+\lambda\frac{1}{t\sqrt{(2+h-t)}}sin(y(t+h))=0, & \rho(1)<t<\sigma(2),\\ \alpha y(\rho(1))-\beta p(\rho(1))y^{\Delta}(\rho(1))=0,\\ \gamma y(\sigma(2))+\delta p(\sigma(2))y^{\Delta}(\sigma(2))=0.\end{cases}\tag{4.5}$$

Here, $p(t)=(t+1)$, $q(t)=\frac{1}{t\sqrt{(2+h-t)}}$ *and* $f(t,\,y(t+h))=sin(y(t+h))$, *and* $\alpha,\,\beta,\,\gamma,\,\delta\geq0$. *Clearly* (H_1) - (H_4) *are satisfied. Note that*

$$G(s+h,\,s)=\frac{1}{\eta}\left(\gamma\left(\sum_{\frac{s}{h}}^{\frac{2}{h}}\frac{h}{1+kh}\right)+\delta\right)\left(\alpha\left(\sum_{\frac{1}{h}-1}^{\frac{s}{h}}\frac{h}{1+kh}\right)+\beta\right)\geq0,$$

and

$$\lim_{y(t+h)\to+\infty}\frac{f(t,\,y(t+h))}{y(t+h)}=\lim_{y\to+\infty}\frac{siny}{y}=0,$$
$$\lim_{y(t+h)\to0^+}\frac{f(t,\,y(t+h))}{y(t+h)}=\lim_{y\to0^+}\frac{siny}{y}=1.$$

Thus (L_4) *and* (L_5) *hold. By Corollary 3.3, the boundary value problem (4.5) has at least one positive solution for* $\frac{1}{M_\theta B_\theta l_2}<\lambda<+\infty$.

Example 4.5 *Let us consider the boundary value problem*

$$\begin{cases}(y^{\Delta})^{\Delta}+\lambda\frac{1}{t\sqrt{1-t}}\sqrt{y}\,siny=0, & 0<t<1,\\ \alpha y(0)-\beta y^{\Delta}(0)=0,\\ \gamma y(1)+\delta y^{\Delta}(1)=0.\end{cases}\tag{4.6}$$

When $\mathbb{T} = q^{\overline{\mathbb{Z}}}$, *where* $q^{\mathbb{Z}} = \{q^k : k \in \mathbb{Z}\}$. *The above boundary value problem becomes*

$$\begin{cases} D_q^2 y(t) + \lambda \frac{1}{t\sqrt{1-t}} \sqrt{y}\ siny = 0, \quad 0 < t < 1 \\ \alpha y(0) - \beta D_q y(0) = 0, \\ \gamma y(1) + \delta D_q y(1) = 0. \end{cases} \tag{4.7}$$

where

$$D_q y(t) = \frac{y(qt) - y(t)}{(q-1)t}, \ t \neq 0.$$

Here, $p(t) = 1$, $q(t) = \frac{1}{t\sqrt{1-t}}$ *and* $f(t, y) = \sqrt{y}\, siny$, $p(0) = 1$, $p(1) = 1$ *and* $\alpha, \beta, \gamma, \delta \geq 0$. *Clearly* (H_1) - (H_4) *are satisfied. Note that*

$$G(qs, qs) = \frac{1}{\eta}(\gamma + \delta - \gamma qs)(\alpha qs + \beta) \geq 0,$$

and

$$\lim_{y \to +\infty} \frac{f(t, y)}{y} = \lim_{y \to +\infty} \frac{\sqrt{y}\, siny}{y} = 0,$$
$$\lim_{y \to 0^+} \frac{f(t, y)}{y} = \lim_{y \to 0^+} \frac{\sqrt{y}\, siny}{y} = 0.$$

Thus (L_3) *and* (L_4) *hold. By Theorem 3.3, the boundary value problem (4.7) has at least two positive solution for* $\lambda_2 < \lambda < +\infty$.

Example 4.6 *Consider the singular boundary value problem*

$$\begin{cases} ((t^2+1)y^{\Delta})^{\Delta} + \lambda \frac{1}{t\sqrt{\sigma(1)-t}} f(t, y^{\sigma}(t)) = 0, \quad \rho(1) < t < \sigma(2), \\ \alpha y(\rho(1)) - \beta p(\rho(1)) y^{\Delta}(\rho(1)) = 0, \\ \gamma y(\sigma(2)) + \delta((\sigma(2))^2 + 1) y^{\Delta}(\sigma(2)) = 0, \end{cases} \tag{4.8}$$

where

$$f(t, y^{\sigma}(t)) = \begin{cases} 1, & y^{\sigma} < 1, \\ (y^{\sigma})^2, & y^{\sigma} \geq 1. \end{cases}$$

If $\mathbb{T} = \mathbb{P}_{a,b} = \cup_{k=0}^{\infty}[k(a+b), k(a+b)+a]$, $\sigma(t) = t$ *for* $t \in \cup_{k=0}^{\infty}[k(a+b), k(a+b)+a)$, *so the above boundary value problem becomes*

$$\begin{cases} ((t^2+1)y^{\Delta})^{\Delta} + \lambda \frac{1}{t\sqrt{1-t}} f(t, y(t)) = 0, \quad 0 < t < 3, \\ \alpha y(0) - \beta y^{\Delta}(0) = 0, \\ \gamma y(1) + 10\delta y^{\Delta}(1) = 0, \end{cases} \tag{4.9}$$

where

$$f(t,\,y(t)) = \begin{cases} 1, & y(t) < 1, \\ (y(t))^2, & y(t) \geq 1. \end{cases}$$

Here $p(t) = t^2 + 1$, $q(t) = \frac{1}{t\sqrt{1-t}}$, $p(\rho(1)) = 1$ *and* $p(\sigma(2)) = 10$ *and* α, β, γ, $\delta \geq 0$. *Clearly* (H_1) - (H_4) *are satisfied. Note that*

$$G(s,\,s) = \frac{1}{\eta}\left(\gamma\, tan^{-1}\left(\frac{3-s}{3+s}\right) + \delta\right)\left(\alpha\, tan^{-1}(s) + \beta\right) \geq 0,$$

and

$$\lim_{y\to+\infty} \frac{f(t,\,y)}{y} = \lim_{y\to+\infty} \frac{y^2}{y} = +\infty,$$
$$\lim_{y\to 0} \frac{f(t,\,y)}{y} = \lim_{y\to 0} \frac{1}{y} = +\infty.$$

Thus (L_1) *and* (L_2) *hold. By Theorem 3.1, the boundary value problem (4.9) has at least two positive solution for* $0 < \lambda < \lambda_1$.

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