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A FRICTIONAL CONTACT PROBLEM WITH NORMAL COMPLIANCE FOR VISCOELASTIC MATERIALS WITH LONG MEMORY*

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Abstract

This paper is devoted to study a quasistatic contact problem between a viscoelastic material with long memory and a foundation. The contact is modelled with a version of Coulomb's law of dry friction and a general normal compliance condition. We derive a variational formulation of the model and, under a smallness assumption, we establish the existence of a weak solution to the problem. The proof is based on the time-discretization method, the Banach fixed point theorem and arguments of compactness, lower semicontinuity and monotonicity.

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1 Introduction

It is well known that the Kelvin-Voigt model of viscoelasticity cannot predict the stress relaxation whereas the Maxwell model cannot adequately describe

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viscoelastic behaviour in creep. For this reason, we need to build up more sophisticated models. For example, the standard solid model which can be expressed in the form

$$\sigma\left(t\right)=\mathcal{A}\varepsilon(u\left(t\right))+\int_{0}^{t}\mathcal{B}\left(t-s\right)\varepsilon\left(u\left(s\right)\right)ds, \tag{1}$$

can describe both precedent phenomena (see, e.g., [3, 4, 19]). Here t is the time variable, u denotes the displacement field, σ represents the stress tensor, $\varepsilon\left(u\right)$ is the linearized strain tensor, $\mathcal{A}$ is the elasticity operator and $\mathcal{B}$ is the tensor of relaxation.

Analysis of various contact problems with constitutive laws of the form (1) also known as the viscoelastic law with long memory can be found in [8, 9, 13, 18, 19], for instance. In [8], the contact was assumed to be frictionless and was modelled with a version of normal compliance condition including unilateral constraint and an adhesive condition of a nonconvex function; the unique solvability of the problem was shown through existence and uniqueness results on abstract inclusions and abstract variational–hemivariational inequalities. In [9], the contact condition was modelled with Tresca's law involving slip dependent coefficient of friction; an existence result to the corresponding problem, for small enough friction coefficients, was established. Contact problems with Tresca's law and a number of frictional contact conditions were considered in [18]; an abstract existence and uniqueness result for a class of evolutionary variational inequalities were used to prove the unique solvability of the corresponding problems.

The novelty, in this paper, consists in dealing with a quasistatic contact problem for viscoelastic materials with a constitutive law of the form (1), such that the contact is modelled with normal compliance and the associated version of Coulomb's law of dry friction, which leads to a new and nonstardard mathematical problem.

We note that an early attempt to study the contact problem with normal compliance was done in [12]. Since then, the normal compliance contact condition has been extensively employed as an approximation of the Signorini contact condition, see for instance [1, 10, 11, 16, 17].

Our analysis is based on the time-discretization method. We construct a sequence of elliptic quasi-variational inequalities for which at each time step, under a smallness assumption, we prove the existence of a unique solution. Then, we construct approximate solutions and prove that the limit of a subsequence of the solutions of the approximate problems is a solution of the continuous problem.

The rest of this paper is organized as follows. In Section 2 we present the notation and some preliminary material. In Section 3 we describe the contact problem and state the assumptions on the data, we derive its variational formulation. Section 4 is dedicated to establish the existence of a weak solution to the model.

2 NOTATION AND PRELIMINARIES

Here we introduce the notation we shall use and some preliminary materials. For further details we refer the reader to [5, 7, 15, 19]. We use the notation $\mathbb{N}^*$ for the set of positive integers. We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ (d=2, 3). We define the inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ by

$$u \cdot v=\sum_{i=1}^{d} u_i v_i, \ \ |u| = \sqrt[2]{u \cdot u}, \forall u, \ v \ \in \mathbb{R}^d;$$

$$\sigma \cdot \xi = \sum_{1 \le i,j \le d} \sigma_{ij}\xi_{ij}, \ |\sigma| = \sqrt[2]{\sigma \cdot \sigma}, \forall \sigma, \ \xi \in \mathbb{S}^d.$$

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary Γ and let ν denote the unit outer normal on Γ. Let $T > 0$, let $[0, T]$ be the time interval of interest and let $x \in \overline{\Omega}$ be the spatial variable. We introduce the spaces

$$H = L^2(\Omega; \mathbb{R}^d), \ \mathcal{Q} = L^2(\Omega; \mathbb{S}^d),$$

$$H_1 = \{u \in H; \ \varepsilon(u) \in \mathcal{Q}\}, \ \mathcal{Q}_1 = \{\sigma \in \mathcal{Q}; \ Div\sigma \in H\},$$

where $\varepsilon : H_1 \to \mathcal{Q}$ is the deformation operator defined by

$$\varepsilon(u) = (\varepsilon_{ij}(u)), \ \varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), 1 \le i, j \le d, \forall u \in H_1,$$

$Div : \mathcal{Q}_1 \to H$ is the divergence operator for tensor functions defined by

$$Div\sigma = ((Div\sigma)_i)_{1 \le i \le d} = \left(\sum_{j=1}^{d} \frac{\partial \sigma_{ij}}{\partial x_j}\right)_{1 \le i \le d}, \forall \sigma \in \mathcal{Q}_1.$$

Note that H, $\mathcal{Q}$, H_1 and $\mathcal{Q}_1$ are Hilbert spaces equipped with the respective canonical inner products

$$(u, v)_H = \int_\Omega u \cdot v dx, \ (\sigma, \tau)_{\mathcal{Q}} = \int_\Omega \sigma \cdot \tau dx,$$

$$(u,v)_{H_1} = (u,v)_H + (\varepsilon(u), \varepsilon(v))_{\mathcal{Q}},$$
$$(\sigma,\tau)_{\mathcal{Q}_1} = (Div\sigma, Div\tau)_H + (\sigma,\tau)_{\mathcal{Q}},$$

where the associated norms are denoted by $\|.\|_H$, $\|.\|_{\mathcal{Q}}$, $\|.\|_{H_1}$ and $\|.\|_{\mathcal{Q}_1}$. Let $\mathbf{Q}_\infty$ be the space of fourth-order tensor fields defined by

$$\mathbf{Q}_\infty = \{\mathcal{E} = (\mathcal{E}_{ijkl}) \mid \ \mathcal{E}_{ijkl} = \mathcal{E}_{jikl} = \mathcal{E}_{klij} \in L^\infty(\Omega), \ \forall i,j,k,l \in \{1,...,d\}\}.$$

The space $\mathbf{Q}_\infty$ is a real Banach space with the norm

$$\|\mathcal{E}\|_{\mathbf{Q}_\infty} = \max_{1\le i,j,k,l\le d} \|\mathcal{E}_{jikl}\|_{L^\infty(\Omega)}. \tag{2}$$

Let $\tilde{\gamma} : H_1 \to L^2\left(\Gamma; \mathbb{R}^d\right)$ be the trace map. For every element $v \in H_1$ we use the notation v to denote the trace $\tilde{\gamma}(v)$ of v on Γ and for all $v \in H_1$ we denote by v_ν and v_τ, respectively, the normal and the tangential components of v on the boundary Γ

$$v_\nu = v \cdot \nu, \ v_\tau = v - v_\nu \nu \text{ on } \Gamma.$$

In a similar manner, the normal and the tangential components of a regular (say C^1) tensor field σ are defined by

$$\sigma_\nu = \sigma\nu \cdot \nu, \ \sigma_\tau = \sigma\nu - \sigma_\nu \nu \text{ on } \Gamma,$$

moreover the following Green formula holds

$$(Div\sigma, v)_H + (\sigma, \varepsilon(v))_{\mathcal{Q}} = \int_\Gamma \sigma\nu \cdot v da, \ \forall \ v \in H_1, \tag{3}$$

where da is the surface measure element. For every real Banach space $(X, \|.\|_X)$, we denote by $C([0,T]; X)$ the space of continuous functions from $[0,T]$ to X and we use the standard notation for the spaces $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$, $p \in [1,\infty]$ and $k \ge 1$.

Finally, we conclude this section with two Gronwall type inequalities. Other versions of Gronwall inequalities can be found for instance in [6] and references therein.

Lemma 1. *Assume that $\tilde{a}$ and $\tilde{b} : [0,T] \to \mathbb{R}$ are two functions in $L^1(0,T)$ satisfying*

$$\tilde{a}(t) \le \tilde{b}(t) + \alpha \int_0^t \tilde{a}(s)\, ds, \ \forall \ t \in [0,T], \tag{4}$$

where α is a nonnegative constant. Then, it follows

$$\tilde{a}(t) \le \tilde{b}(t) + \alpha \int_0^t e^{\alpha(t-s)} \tilde{b}(s)\, ds, \ \forall \ t \in [0,T]. \tag{5}$$

Proof. Use arguments similar to those in [6, proof of Proposition 2.1]. □

Lemma 2. *Let $T > 0$ be a constant. Let α_1 and α_2 be two nonnegative constants. For each $m \in \mathbb{N}^*$, let $\{w_i\}_{i=0}^{m} \subset \mathbb{R}$ be a nonnegative sequence, which satisfies*

$$w_{i+1} \leq \alpha_1 + \alpha_2 h \sum_{j=0}^{i} w_j,\ 0 \leq i \leq m-1, \tag{6}$$

where $h = \dfrac{T}{m}$. Then, it holds

$$w_{i+1} \leq (\alpha_1 + \alpha_2 T w_0)\, e^{\alpha_2 T},\ 0 \leq i \leq m-1. \tag{7}$$

The proof of Lemma 2 may be found in [10].

3 Problem statement and variational formulation

The physical setting is as follows. A deformable body occupies a bounded domain $\Omega \subset \mathbb{R}^d$ (with d=2, 3). We assume that the boundary Γ of the domain Ω is Lipschitz continuous, and it is divided into three disjoint measurable parts Γ_1, Γ_2, Γ_3, such that meas$(\Gamma_1) > 0$. The mechanical behaviour of the material is described with a viscoelastic law with long memory and the process is assumed to be quasistatic in the time interval $[0, T]$. The body is clamped on Γ_1 and therefore the displacement field vanishes there, while volume forces of density f_0 act in Ω and surface tractions of density f_2 act on Γ_2. The body is supposed to be in contact over Γ_3 with a foundation and, moreover, both normal compliance and a version of Coulomb's law of dry friction are included. To simplify the notation, we do not indicate explicitly the dependence of various functions on the spatial variable $x \in \Omega \cup \Gamma$. Under the above assumptions, the classical formulation of our problem is the following.

Problem 1. *Find a displacement field $u : \Omega \times [0, T] \to \mathbb{R}^d$ and stress field*

$\sigma \ : \Omega \times [0,T] \to \mathbb{S}^d$ *such that*

$$\sigma(t) = \mathcal{A}\varepsilon(u(t)) + \int_0^t \mathcal{B}(t-s)\,\varepsilon(u(s))\,ds, \ \ in \ \Omega \times (0,T), \tag{8}$$

$$Div\sigma + f_0 = 0, \ \ in \ \Omega \times (0,T), \tag{9}$$

$$u = 0, \ \ on \ \Gamma_1 \times (0,T), \tag{10}$$

$$\sigma\nu = f_2, \ \ on \ \Gamma_2 \times (0,T), \tag{11}$$

$$-\sigma_\nu = p_\nu(u_\nu - g), \ \ on \ \Gamma_3 \times (0,T), \tag{12}$$

$$\begin{cases} |\sigma_\tau| \le p_\tau(u_\nu - g), \\ |\sigma_\tau| < p_\tau(u_\nu - g) \Rightarrow \dot{u}_\tau = 0, \\ |\sigma_\tau| = p_\tau(u_\nu - g) \Rightarrow \exists\lambda \ge 0 \\ such\ that \ \ \sigma_\tau = -\lambda\dot{u}_\tau, \end{cases} \quad on \ \Gamma_3 \times (0,T), \tag{13}$$

$$u(0) = u_0 \ in \ \Omega. \tag{14}$$

Equation (8) represents the viscoelastic law with long memory. Equation (9) is the equilibrium equation posed on the domain Ω. Conditions (10)-(11) are the displacement-traction boundary conditions, where $\sigma\nu$ represents the Cauchy stress vector. (12)-(13) characterize the contact boundary conditions where u_ν denotes the normal displacement, $\dot{u}_\tau$ represents the tangential velocity, σ_ν is the normal stress, σ_τ represents the tangential traction and g is the gap, between Γ_3 and the foundation, measured along the direction of the outward normal ν. Here and below the dot above a variable represents its derivative with respect to the time variable. Equation (12) is a general expression of the normal compliance condition, where p_ν is a nonnegative prescribed function which vanishes for negative arguments, such that when $u_\nu < g$ there is no contact and the normal pressure vanishes; and when the contact takes place then $u_\nu - g \ge 0$ is a measure of the penetration of the surface asperities into those of the foundation. A possible choice of the function p_ν is

$$p_\nu(r) = c_\nu(r)_+,$$

where $(r)_+$ denotes the positive part of r, that is $(r)_+ = \max\{r, 0\}$, c_ν is the surface stiffness coefficient, such that Signorini's nonpenetration condition is obtained in the limit $c_\nu \to \infty$ and thus interpenetration is not allowed.

The relations (13) represent a version of Coulomb's law of dry friction where p_τ is a prescribed nonnegative function, the so-called friction bound.

This condition states that, if there is contact, the tangential traction σ_τ is bounded by the friction bound p_τ. If the strict inequality is satisfied, then sliding does not occur and when equality holds the friction stress is proportional and opposed to the slip rate (see, e.g., [17]). Finally, (14) is the initial condition.

In order to obtain the variational formulation of the mechanical problem (8)-(14), we introduce the space V defined by

$$V = \{v \in H_1,\ v = 0 \ on\ \Gamma_1\}.$$

Since $meas\,(\Gamma_1) > 0$, Korn's inequality holds

$$C_K \left\| v \right\|_{H_1} \leq \left\| \varepsilon\,(v) \right\|_{\mathcal{Q}},\ \forall v \in V, \tag{15}$$

where $C_K > 0$ is a positive constant depending only on Ω and Γ_1 (see, e.g., [14]). Over the space V, we consider the inner product given by

$$(w, v)_V = (\varepsilon(w), \varepsilon(v))_{\mathcal{Q}},\ \forall w, v \in V,$$

and let $\|.\|_V$ be the associated norm. It follows from Korn's inequality (15) that $\|.\|_{H_1}$ and $\|.\|_V$ are equivalent norms on V. Therefore $(V, (.,.)_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem, there exists a positive constant c_0 depending only on the domain Ω, Γ_1 and Γ_3 such that

$$\|v\|_{L^2(\Gamma_3;\mathbb{R}^d)} \leq c_0 \|v\|_V,\ \forall v \in V. \tag{16}$$

In the study of the mechanical Problem (8)-(14), we consider the following assumptions. We assume that the elasticity operator $\mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies

$$\begin{cases} \text{(i) There exists } m_{\mathcal{A}} > 0 \text{ such that} \\ (\mathcal{A}\,(x, \varepsilon_1) - \mathcal{A}\,(x, \varepsilon_2)) \cdot (\varepsilon_1 - \varepsilon_2) \geq m_{\mathcal{A}}\, |\varepsilon_1 - \varepsilon_2|^2 \\ \ a.e.\ x \in \Omega,\ \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d; \\ \\ \text{(ii) There exists } L_{\mathcal{A}} > 0 \text{ such that} \\ |\mathcal{A}\,(x, \varepsilon_1) - \mathcal{A}\,(x, \varepsilon_2)| \leq L_{\mathcal{A}}\, |\varepsilon_1 - \varepsilon_2| \\ \ a.e.\ x \in \Omega,\ \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d; \\ \\ \text{(iii) The mapping } x \longmapsto \mathcal{A}\,(x, \varepsilon) \text{ is Lebesgue measurable on } \Omega \\ \text{for any } \varepsilon \in \mathbb{S}^d; \\ \\ \text{(iv) The mapping } x \longmapsto \mathcal{A}\,(x, 0_{\mathbb{S}^d}) \text{ belongs to } \mathcal{Q}. \end{cases} \tag{17}$$

We assume that the relaxation tensor $\mathcal{B}$ satisfies

$$\mathcal{B} \in W^{1,\infty}(0,T;\mathbf{Q}_\infty). \tag{18}$$

We assume that the function $p_\alpha : \Gamma_3 \times \mathbb{R} \to \mathbb{R}^+$, $(\alpha = \nu,\, \tau)$, satisfies

$$\begin{cases} \text{(i) There exists } L_\alpha > 0 \text{ such that} \\ |p_\alpha(x,r_1) - p_\alpha(x,r_2)| \le L_\alpha |r_1 - r_2|, \\ \forall r_1,\ r_2 \in \mathbb{R},\ a.e.\ x \in \Gamma_3\ ; \\ \\ \text{(ii) } p_\alpha(x,r) = 0,\ \forall r \le 0,\ a.e.\ x \in \Gamma_3; \\ \\ \text{(iii) For each } r \in \mathbb{R},\ x \mapsto p_\alpha(x,r) \text{ is Lebesgue measurable on } \Gamma_3. \end{cases} \tag{19}$$

The densities of forces satisfy

$$(i)\ \ f_0 \in W^{1,\infty}(0,T;H),\ \ (ii)\ \ f_2 \in W^{1,\infty}(0,T;L^2(\Gamma_2;\mathbb{R}^d)). \tag{20}$$

The gap function satisfies

$$g \in L^2(\Gamma_3),\ \ g \ge 0\ a.e.\ x \in \Gamma_3\ . \tag{21}$$

Finally, we assume that the initial data satisfies

$$u_0 \in V. \tag{22}$$

In the sequel, we use the functional $\psi : V \times V \longrightarrow \mathbb{R}$ defined by

$$\psi(z,w) = \int_{\Gamma_3} p_\nu(z_\nu - g)\, w_\nu da + \int_{\Gamma_3} p_\tau(z_\nu - g)\, |w_\tau|\, da, \tag{23}$$

for all $z,\ w \in V$. Using (16), (19), (21) and (23), we deduce that the functional ψ satisfies the following

$$\psi(\eta,v) - \psi(\eta,w) + \psi(z,w) - \psi(z,v) \le c_0^2 (L_\tau + L_\nu) \|\eta - z\|_V \|v - w\|_V, \tag{24}$$

$$\psi(\eta,-z) - \psi(\eta,\eta - z) \le c_0 (L_\tau + L_\nu) \Big(c_0 \|\eta\|_V + \|g\|_{L^2(\Gamma_3)} \Big) \|\eta\|_V, \tag{25}$$

$$\psi(\eta,v) - \psi(\eta,w) \le \psi(\eta, v - w), \tag{26}$$

$$\psi(\eta,w) - \psi(z,w) \le c_0^2 (L_\tau + L_\nu) \|\eta - z\|_V \|w\|_V, \tag{27}$$

$$|\psi(\eta,v) - \psi(\eta,w)| \le c_0 (L_\tau + L_\nu) \Big(c_0 \|\eta\|_V + \|g\|_{L^2(\Gamma_3)} \Big) \|v - w\|_V, \tag{28}$$

$$\psi(\eta,w) \leq (L_\tau + L_\nu)\left(c_0 \|\eta\|_V + \|g\|_{L^2(\Gamma_3)}\right)\|w\|_{L^2(\Gamma_3;\mathbb{R}^d)}\,, \tag{29}$$

for all $\eta, v, w,\ z \in V$. It follows from (20) that the function $f : [0,T] \to V$ defined by

$$(f(t),w)_V = \int_\Omega f_0(t)\cdot w dx + \int_{\Gamma_2} f_2(t)\cdot w da,\ \forall w\ \in V,\ \forall t \in [0,T]\,, \tag{30}$$

has the following regularity

$$f \in W^{1,\infty}(0,T;V). \tag{31}$$

Also, using (18) and (2), we conclude that there exists $L_\mathcal{B} > 0$ such that

$$\|(\mathcal{B}(t) - \mathcal{B}(s))\,\varepsilon(w)\|_\mathcal{Q} \leq L_\mathcal{B}\,|t-s|\,\|w\|_V\,, \tag{32}$$

$$\|\mathcal{B}(t)\,\varepsilon(w)\|_\mathcal{Q} \leq \left(TL_\mathcal{B} + \|\mathcal{B}(0)\|_{\mathbf{Q}_\infty}\right)\|w\|_V\,, \tag{33}$$

for all $w \in V$ and for all $t,\ s \in [0,T]$. Now, assume u and σ are smooth functions satisfying (8)-(14) and use the Green formula (3) to obtain the following variational formulation of the mechanical Problem 1 in terms of the displacement field only.

Problem 2. *Find a displacement field* $u : [0,T] \to V$ *such that*

$$\begin{cases} (\mathcal{A}\varepsilon(u(t)), \varepsilon(w - \dot{u}(t)))_\mathcal{Q} + \left(\int_0^t \mathcal{B}(t-s)\,\varepsilon(u(s))\,ds, \varepsilon(w - \dot{u}(t))\right)_\mathcal{Q} \\ +\psi(u(t), w) - \psi(u(t), \dot{u}(t)) \\ \geq (f(t), w - \dot{u}(t))_V\,,\ \textit{for all}\ w \in V,\ \textit{for a.e.}\ t \in (0,T), \end{cases} \tag{34}$$

$$u(0) = u_0. \tag{35}$$

To study the problem (34)-(35), we need the following additional assumption on the initial data

$$(\mathcal{A}(\varepsilon(u_0)), \varepsilon(w))_\mathcal{Q} + \psi(u_0, w) \geq (f(0), w)_V\,,\ \forall\ w \in V, \tag{36}$$

and we make the following smallness assumption

$$L_\tau + L_\nu < \frac{m_\mathcal{A}}{c_0^2}. \tag{37}$$

where c_0, $m_\mathcal{A}$ and L_α, $(\alpha = \nu,\ \tau)$ are given in (16), (17) and (19), respectively.

4 Existence of a weak solution

The following theorem is the main result of this paper.

Theorem 1. *Assume that (17)-(22) and (36)-(37) hold. Then, Problem (34)-(35) has at least one solution u which satisfies*

$$u \in W^{1,\infty}(0,T;V). \tag{38}$$

We will split the proof into several steps.

First step. For each $m \in \mathbb{N}^*$, we introduce a uniform partition of the time interval $[0,T]$, denoted by $t_i^m = ih_m$, $h_m = \dfrac{T}{m}$, $i = 0,...,m$. For a sequence $\left\{w_m^i\right\}_{i=0}^m$, we denote $\delta w_m^{i+1} = \dfrac{w_m^{i+1} - w_m^i}{h_m}$ and for a continuous function $z \in C([0,T];X)$ with values in a normed space X, we use the notation $z_i^m = z(t_i^m)$, $i = 0,...,m$. Using the Riesz representation theorem, we can introduce the operator $\mathcal{F} : V \to V$ defined by

$$(\mathcal{F}v, w)_V = (\mathcal{A}\varepsilon(v), \varepsilon(w))_{\mathcal{Q}}, \ \forall v, w \in V. \tag{39}$$

From (17) and (39), it follows that the operator $\mathcal{F}$ satisfies

$$m_{\mathcal{A}} \|w_1 - w_2\|_V^2 \leq (\mathcal{F}w_1 - \mathcal{F}w_2, w_1 - w_2)_V, \ \forall w_1, w_2 \in V, \tag{40}$$

$$\|\mathcal{F}w_1 - \mathcal{F}w_2\|_V \leq L_{\mathcal{A}} \|w_1 - w_2\|_V, \ \forall w_1, w_2 \in V. \tag{41}$$

We consider the following incremental problems $\mathcal{P}_m^{i+1}$, $i \in \{0, ..., m-1\}$.

Problem 3 ($\mathcal{P}_m^{i+1}$)**.** *Find a function $u_m^{i+1} \in V$ such that*

$$\left\{\begin{array}{l} \left(\mathcal{F}u_m^{i+1}, w - \delta u_m^{i+1}\right)_V + \left(h_m \displaystyle\sum_{j=0}^{i} \mathcal{B}_{i+1,j}^m, \varepsilon\left(w - \delta u_m^{i+1}\right)\right)_{\mathcal{Q}} \\ +\psi(u_m^{i+1}, w) - \psi(u_m^{i+1}, \delta u_m^{i+1}) \\ \geq \left(f_{i+1}^m, w - \delta u_m^{i+1}\right)_V, \text{ for all } w \in V, \end{array}\right. \tag{42}$$

where

$$u_m^0 = u_0, \tag{43}$$

u_m^j is the unique solution of the problem $\mathcal{P}_m^j$, $j = 1, ..., i$,

$$\mathcal{B}_{i+1,j}^m = \mathcal{B}\left(t_{i+1}^m - t_j^m\right) \varepsilon\left(u_m^j\right), \ i = 0, ..., m-1, \ j = 0, ..., i, \tag{44}$$

$$f_i^m = f(t_i^m), \ i = 0, ..., m. \tag{45}$$

Now, by setting $w = \dfrac{v - u_m^i}{h_m}$ in (42), it follows that the problem $\mathcal{P}_m^{i+1}$ is formally equivalent to the following problem.

Problem 4 ($\mathcal{Q}_m^{i+1}$)**.** *Find a function $u_m^{i+1} \in V$ such that*

$$\begin{cases} \left(\mathcal{F}u_m^{i+1}, v - u_m^{i+1}\right)_V + \left(h_m \sum\limits_{j=0}^{i} \mathcal{B}_{i+1,j}^m, \varepsilon\left(v - u_m^{i+1}\right)\right)_{\mathcal{Q}} \\ +\psi(u_m^{i+1}, v - u_m^i) - \psi(u_m^{i+1}, u_m^{i+1} - u_m^i) \\ \geq \left(f_{i+1}^m, v - u_m^{i+1}\right)_V, \textit{ for all } v \in V, \end{cases} \tag{46}$$

where u_m^0, $\left\{\mathcal{B}_{i+1,j}^m\right\}$ and $\left\{f_{i+1}^m\right\}$ are given by (43)-(45) and u_m^j is the unique solution of problem $\mathcal{P}_m^j$, $j = 1, ..., i$.

Lemma 3. *Problem $\mathcal{P}_m^{i+1}$, $i \in \{0\ , ...,\ m-1\}$, has a unique solution.*

Proof. From (40)-(41), it follows that the operator $\mathcal{F}$ is strongly monotone and Lipschitz continuous. On the other hand let $\eta \in V$. Using (23) and (28), it follows that the functional $\varphi : V \to \mathbb{R}$ defined by

$$\varphi(v) = \left(h_m \sum_{j=0}^{i} \mathcal{B}_{i+1,j}^m, \varepsilon(v)\right)_{\mathcal{Q}} + \psi(\eta, v - u_m^i), \text{ for all } v \in V,$$

is a proper convex and continuous function. Therefore, using a standard result on elliptic variational inequalities of the second kind, (see [7, p.60]), we deduce that the following problem. Find $u_{m\eta}^{i+1} \in V$ such that

$$\begin{cases} \left(\mathcal{F}u_{m\eta}^{i+1}, v - u_{m\eta}^{i+1}\right)_V + \left(h_m \sum\limits_{j=0}^{i} \mathcal{B}_{i+1,j}^m, \varepsilon\left(v - u_{m\eta}^{i+1}\right)\right)_{\mathcal{Q}} \\ +\psi(\eta, v - u_m^i) - \psi(\eta, u_{m\eta}^{i+1} - u_m^i) \\ \geq \left(f_{i+1}^m, v - u_{m\eta}^{i+1}\right)_V, \text{ for all } v \in V, \end{cases} \tag{47}$$

has a unique solution $u_{m\eta}^{i+1} \in V$. To continue, we define the operator $\Psi : V \to V$ by

$$\Psi(\eta) = u_{m\eta}^{i+1}, \text{ for all } \eta \in V. \tag{48}$$

Let η_1, $\eta_2 \in V$. Using the notation $u_1 = u^{i+1}_{m\eta_1}$ and $u_2 = u^{i+1}_{m\eta_2}$ in (47), we get

$$\begin{aligned}(\mathcal{F}u_1 - \mathcal{F}u_2, u_1 - u_2)_V \quad &\le \quad \psi(\eta_1, u_2 - u^i_m) - \psi(\eta_1, u_1 - u^i_m) \\ &\quad + \psi(\eta_2, u_1 - u^i_m) - \psi(\eta_2, u_2 - u^i_m),\end{aligned}$$

which together with (24) and (40), implies that

$$m_{\mathcal{A}} \left\| u_1 - u_2 \right\|^2_V \le c_0^2 \left(L_\tau + L_\nu \right) \left\| \eta_1 - \eta_2 \right\|_V \left\| u_1 - u_2 \right\|_V .$$

Thus, using (48), we obtain

$$\left\| \Psi\eta_1 - \Psi\eta_2 \right\|_V \le \frac{c_0^2 \left(L_\tau + L_\nu \right)}{m_{\mathcal{A}}} \left\| \eta_1 - \eta_2 \right\|_V .$$

This last inequality implies that if $c_0^2 \left(L_\tau + L_\nu \right) < m_{\mathcal{A}}$, then Ψ is a contraction function in the Banach space V. Therefore, Ψ has a unique fixed point $\eta^* \in V$. We have now all the ingredients to prove Lemma 3. Let η^* be the unique fixed point of Ψ and let $u^{i+1}_m = \eta^* = u^{i+1}_{m\eta^*}$ be the unique solution of the problem (47) for $\eta = \eta^*$, then we deduce that u^{i+1}_m is a solution to the problem $\mathcal{Q}^{i+1}_m$ which is equivalent to the problem $\mathcal{P}^{i+1}_m$. The uniqueness of the solution is a consequence of the uniqueness of the fixed point of the operator Ψ and of the uniqueness of the solution of the problem (47). $\square$

In the rest of this paper, the same letter c will be used to denote different positive constants which do not depend on $m \in \mathbb{N}^*$ nor on $t \in (0, T)$.

Second step. In this step we have the following result.

Lemma 4. *There exists $c > 0$, such that for all $m \in \mathbb{N}^*$,*

$$\left\| u^{i+1}_m \right\|_V \le c,\ i = \ 0, ...,\ m - 1. \tag{49}$$

$$\left\| \delta u^{i+1}_m \right\|_V \le c,\ i = \ 0, ...,\ m - 1. \tag{50}$$

Proof. It follows from (43) and (22), that there exists $c > 0$, such that

$$\left\| u^0_m \right\|_V \le c,\ \forall\ m \in \mathbb{N}^*.$$

Taking $v = 0_V$ in (46) yields

$$\begin{aligned}\left(\mathcal{F}u^{i+1}_m, u^{i+1}_m \right)_V \quad &\le \quad \psi(u^{i+1}_m, -u^i_m) - \psi(u^{i+1}_m, u^{i+1}_m - u^i_m) \\ &\quad - \left(h_m \sum_{j=0}^{i} \mathcal{B}^m_{i+1,j}, \varepsilon\left(u^{i+1}_m \right) \right)_{\mathcal{Q}} + \left(f^m_{i+1}, u^{i+1}_m \right)_V ,\end{aligned}$$

and using (25), (33), (40), (44), we get

$$m_{\mathcal{A}} \left\| u_m^{i+1} \right\|_V^2 \leq c_0^2 \left(L_\tau + L_\nu \right) \left\| u_m^{i+1} \right\|_V^2 + c_0 \left(L_\tau + L_\nu \right) \| g \|_{L^2(\Gamma_3)} \left\| u_m^{i+1} \right\|_V +$$
$$+ \left(c h_m \sum_{j=0}^{i} \left\| u_m^j \right\|_V \right) \left\| u_m^{i+1} \right\|_V + \left\| f_{i+1}^m \right\|_V \left\| u_m^{i+1} \right\|_V + \| \mathcal{F} \left(0_V \right) \|_V \left\| u_m^{i+1} \right\|_V ,$$

which with (37) and (31) gives

$$\left\| u_m^{i+1} \right\|_V \leq c h_m \sum_{j=0}^{i} \left\| u_m^j \right\|_V + c.$$

Now, applying Lemma 2 in the last inequality, we obtain (49). Setting $v = u_m^0$ in (46) for i=0, and $w = u_m^1 - u_m^0$ in (36), adding the two inequalities, we obtain

$$\left\{ \begin{array}{l} \left(\mathcal{F} u_m^1 - \mathcal{F} u_m^0, u_m^1 - u_m^0 \right)_V \leq \psi(u_m^0, u_m^1 - u_m^0) - \psi(u_m^1, u_m^1 - u_m^0) \\ - \left(h_m \mathcal{B}_{1,0}^m, \varepsilon \left(u_m^1 - u_m^0 \right) \right)_{\mathcal{Q}} + \left(f_1^m - f_0^m, u_m^1 - u_m^0 \right)_V . \end{array} \right. \tag{51}$$

We use now (51), (40), (33), (44) and (27), to see that

$$\left\{ \begin{array}{l} m_{\mathcal{A}} \left\| u_m^1 - u_m^0 \right\|_V^2 \leq c_0^2 \left(L_\tau + L_\nu \right) \left\| u_m^1 - u_m^0 \right\|_V^2 \\ + c h_m \left\| u_m^0 \right\|_V \left\| u_m^1 - u_m^0 \right\|_V + \| f_1^m - f_0^m \|_V \left\| u_m^1 - u_m^0 \right\|_V , \end{array} \right.$$

and thanks to (31) and (37), we get

$$\begin{aligned} \left\| \frac{u_m^1 - u_m^0}{h_m} \right\|_V & \leq c + c \left\| \frac{f_1^m - f_0^m}{h_m} \right\|_V \\ & \leq c + c \left\| \dot{f} \right\|_{L^\infty(0,T;V)} . \end{aligned}$$

Thus, we obtain

$$\left\| \delta u_m^1 \right\|_V \leq c. \tag{52}$$

Now, for $i \in \{1, ..., m-1\}$, taking $w = 0_V$ in problem $\mathcal{P}_m^{i+1}$, and

$w = \dfrac{u_m^{i+1} - u_m^{i-1}}{h_m}$ in problem $\mathcal{P}_m^i$, adding the two inequalities, we obtain

$$\begin{cases} \left(\mathcal{F}u_m^{i+1} - \mathcal{F}u_m^i, \delta u_m^{i+1}\right)_V \leq h_m \left(\sum_{j=0}^{i-1} \left(\mathcal{B}_{i,j}^m - \mathcal{B}_{i+1,j}^m\right), \varepsilon\left(\delta u_m^{i+1}\right) \right)_{\mathcal{Q}} \\ -h_m \left(\mathcal{B}_{i+1,i}^m, \varepsilon\left(\delta u_m^{i+1}\right)\right)_{\mathcal{Q}} + \\ + \left(\psi(u_m^i, \dfrac{u_m^{i+1} - u_m^{i-1}}{h_m}) - \psi(u_m^i, \dfrac{u_m^i - u_m^{i-1}}{h_m}) \right) - \psi(u_m^{i+1}, \dfrac{u_m^{i+1} - u_m^i}{h_m}) \\ + \left(f_{i+1}^m - f_i^m, \delta u_m^{i+1}\right)_V . \end{cases} \tag{53}$$

It follows from (53), (40), (32), (33), (44), (26) and (27) that

$$\begin{cases} m_{\mathcal{A}} \left\|u_m^{i+1} - u_m^i\right\|_V^2 \leq c h_m^2 \left(\sum_{j=0}^{i-1} \left\|u_m^j\right\|_V \right) \left\|u_m^{i+1} - u_m^i\right\|_V + \\ + c h_m \left\|u_m^i\right\|_V \ \left\|u_m^{i+1} - u_m^i\right\|_V + c_0^2 \left(L_\tau + L_\nu\right) \left\|u_m^{i+1} - u_m^i\right\|_V^2 \\ + \left\|f_{i+1}^m - f_i^m\right\|_V \left\|u_m^{i+1} - u_m^i\right\|_V , \end{cases}$$

which, with (31) and (37), gives

$$\left\|\delta u_m^{i+1}\right\|_V \leq c h_m \left(\sum_{j=0}^{i-1} \left\|u_m^j\right\|_V \right) + c \left\|u_m^i\right\|_V + c \left\|\dot{f}\right\|_{L^\infty(0,T;V)} . \tag{54}$$

Now, (50) is a consequence of (49), (52) and (54). □

Third step. In this step we construct an approximate solution to the problem (34)-(35) and we provide some estimate results. To this end, for each $m \in \mathbb{N}^*$, let u_m^j be the unique solution of the Problem $\mathcal{P}_m^j$, $j = 1, ..., m$. We introduce the following functions $u_m : [0,T] \to V$, $\tilde{u}_m : [0,T] \to V$, $\mathcal{B}_m : [0,T] \to \mathcal{Q}$ and $f_m : [0,T] \to V$ defined, respectively, by

$$u_m(0) = u_0,\ u_m(t) = u_m^i + \left(t - t_i^m\right) \delta u_m^{i+1},\ \forall t \in \left(t_i^m, t_{i+1}^m\right],\ i = 0, ..., m-1, \tag{55}$$

$$\tilde{u}_m(0) = u_0,\ \tilde{u}_m(t) = u_m^{i+1},\ \forall t \in \left(t_i^m, t_{i+1}^m\right],\ i = 0, ..., m-1, \tag{56}$$

$$\mathcal{B}_m(0) = 0_{\mathcal{Q}},\ \mathcal{B}_m(t) = h_m \sum_{j=0}^{i} \mathcal{B}_{i+1,j}^m,\ \forall t \in \left(t_i^m, t_{i+1}^m\right],\ i = 0, ..., m-1, \quad (57)$$

$$f_m(0) = f(0),\ f_m(t) = f_{i+1}^m,\ \forall t \in \left(t_i^m, t_{i+1}^m\right],\ i = 0, ..., m-1, \quad (58)$$

where u_m^0, $\left\{\mathcal{B}_{i+1,j}^m\right\}$ and $\left\{f_{i+1}^m\right\}$ are given by (43)-(45). Using (55), we deduce that the function u_m has a derivative function given by

$$\dot{u}_m(t) = \delta u_m^{i+1},\ \forall t \in \left(t_i^m, t_{i+1}^m\right),\ i = 0, ..., m-1. \quad (59)$$

We have the following estimate results.

Lemma 5. *There exists $c > 0$, such that for all $m \in \mathbb{N}^*$,*

$$\begin{aligned}
\left\|\tilde{u}_m(t)\right\|_V &\leq c,\ \forall t \in [0, T], && (60)\\
\left\|u_m(t)\right\|_V &\leq c,\ \forall t \in [0, T], && (61)\\
\left\|\dot{u}_m(t)\right\|_V &\leq c,\ a.e. t \in [0, T], && (62)\\
\left\|\tilde{u}_m(t) - u_m(t)\right\|_V &\leq ch_m,\ \forall t \in [0, T], && (63)\\
\left\|f_m(t) - f(t)\right\|_V &\leq ch_m,\ \forall t \in [0, T], && (64)\\
\left\|u_m(t) - u_m(s)\right\|_V &\leq c\left|t - s\right|,\ \forall t, s \in [0, T], && (65)\\
\left\|u_m(t) - u_m(s)\right\|_{L^2\left(\Gamma_3;\mathbb{R}^d\right)} &\leq c\left|t - s\right|,\ \forall t, s \in [0, T]. && (66)
\end{aligned}$$

Proof. Use arguments similar to those in [10, proof of Lemma 5]. □

Lemma 6. *There exists $c > 0$, such that for all m, $n \in \mathbb{N}^*$ with $m > n$,*

$$\left\|\mathcal{B}_m(t) - \mathcal{B}_n(t)\right\|_{\mathcal{Q}} \leq c \int_0^t \left\|u_m(s) - u_n(s)\right\|_V ds + ch_n,\ \forall t \in [0, T]. \quad (67)$$

Proof. Let m, $n \in \mathbb{N}^*$ with $m > n$. It is obvious that (67) holds for $t = 0$. Now, let $t \in (0, T]$, then, there are three cases, (i) $t \in (t_0^m, t_1^m] \cap (t_0^n, t_1^n]$, (ii) $t \in \left(t_q^m, t_{q+1}^m\right] \cap (t_0^n, t_1^n]$ with $q \in \{1, ..., m-1\}$, (iii) $t \in \left(t_q^m, t_{q+1}^m\right] \cap \left(t_p^n, t_{p+1}^n\right]$ with $q \in \{1, ..., m-1\}$ and $p \in \{1, ..., n-1\}$. Using (43), (44), (56), (57) and (33), we get

$$\begin{aligned}
\left\|\mathcal{B}_m(t) - \mathcal{B}_n(t)\right\|_{\mathcal{Q}} &= \left\|h_m \mathcal{B}\left(t_1^m\right) \varepsilon\left(u_m^0\right) - h_n \mathcal{B}\left(t_1^n\right) \varepsilon\left(u_n^0\right)\right\|_{\mathcal{Q}}\\
&\leq ch_m + ch_n,\ \forall t \in (t_0^m, t_1^m] \cap (t_0^n, t_1^n]. \qquad (68)
\end{aligned}$$

On the other hand, let $t \in \left(t_q^m, t_{q+1}^m\right]$ with $q \in \{1, ..., m-1\}$, it follows from (44), (57) and (56), that

$$\mathcal{B}_m(t) = \sum_{j=1}^{q} \int_{t_{j-1}^m}^{t_j^m} \mathcal{B}\left(t_{q+1}^m - t_j^m\right) \varepsilon\left(\tilde{u}_m(s)\right) ds + h_m \mathcal{B}\left(t_{q+1}^m\right) \varepsilon\left(u_m^0\right),$$

which gives

$$\begin{aligned}
\mathcal{B}_m(t) &= \sum_{j=1}^{q} \int_{t_{j-1}^m}^{t_j^m} \left(\mathcal{B}\left(t_{q+1}^m - t_j^m\right) - \mathcal{B}\left(t_{q+1}^m - s\right)\right) \varepsilon\left(\tilde{u}_m(s)\right) ds \\
&\quad + \sum_{j=1}^{q} \int_{t_{j-1}^m}^{t_j^m} \left(\mathcal{B}\left(t_{q+1}^m - s\right) - \mathcal{B}(t-s)\right) \varepsilon\left(\tilde{u}_m(s)\right) ds + \int_0^t \mathcal{B}(t-s) \varepsilon\left(\tilde{u}_m(s)\right) ds \\
&\quad + \int_t^{t_q^m} \mathcal{B}(t-s) \varepsilon\left(\tilde{u}_m(s)\right) ds + h_m \mathcal{B}\left(t_{q+1}^m\right) \varepsilon\left(u_m^0\right). \qquad (69)
\end{aligned}$$

Thus, for $t \in \left(t_q^m, t_{q+1}^m\right] \cap (t_0^n, t_1^n]$ with $q \in \{1, ..., m-1\}$, we have

$$\begin{aligned}
&\|\mathcal{B}_m(t) - \mathcal{B}_n(t)\|_{\mathcal{Q}} \leq c \sum_{j=1}^{q} \int_{t_{j-1}^m}^{t_j^m} \left|s - t_j^m\right| \|\tilde{u}_m(s)\|_V ds + c \sum_{j=1}^{q} \int_{t_{j-1}^m}^{t_j^m} \left|t_{q+1}^m - t\right| \|\tilde{u}_m(s)\|_V ds \\
&+ \int_0^{t_q^m} \|\mathcal{B}(t-s) \varepsilon\left(\tilde{u}_m(s)\right)\|_{\mathcal{Q}} ds + \left\|h_m \mathcal{B}\left(t_{q+1}^m\right) \varepsilon\left(u_m^0\right)\right\|_{\mathcal{Q}} + \left\|h_n \mathcal{B}\left(t_1^n\right) \varepsilon\left(u_n^0\right)\right\|_{\mathcal{Q}} \\
&\leq c \sum_{j=1}^{q} h_m^2 + c \sum_{j=1}^{q} h_m^2 + c \int_0^{t_q^m} \|\tilde{u}_m(s)\|_V ds + c h_m + c h_n \\
&\leq c \int_0^{t_1^n} \|\tilde{u}_m(s)\|_V ds + c h_m + c h_n \\
&\leq c h_n. \qquad (70)
\end{aligned}$$

Now, using (69) and (32)-(33), we get

$$\begin{aligned}
&\|\mathcal{B}_m(t) - \mathcal{B}_n(t)\|_{\mathcal{Q}} \leq c \sum_{j=1}^{q} \int_{t_{j-1}^m}^{t_j^m} \left|s - t_j^m\right| \|\tilde{u}_m(s)\|_V ds + c \sum_{j=1}^{p} \int_{t_{j-1}^n}^{t_j^n} \left|s - t_j^n\right| \|\tilde{u}_n(s)\|_V ds \\
&+ c \sum_{j=1}^{q} \int_{t_{j-1}^m}^{t_j^m} \left|t_{q+1}^m - t\right| \|\tilde{u}_m(s)\|_V ds + c \sum_{j=1}^{p} \int_{t_{j-1}^n}^{t_j^n} \left|t_{p+1}^n - t\right| \|\tilde{u}_n(s)\|_V ds \\
&+ c \int_0^t \|\tilde{u}_m(s) - \tilde{u}_n(s)\|_V ds + c h_m + c h_n,
\end{aligned}$$

from this and using (60) and (63), we get, for $t \in \left(t_q^m, t_{q+1}^m\right] \cap \left(t_p^n, t_{p+1}^n\right]$ with $q \in \{1, ..., m-1\}$ and $p \in \{1, ..., n-1\}$,

$$\begin{aligned}\|\mathcal{B}_m(t) - \mathcal{B}_n(t)\|_{\mathcal{Q}} &\leq c\sum_{j=1}^{q} h_m^2 + c\sum_{j=1}^{p} h_n^2 + c\sum_{j=1}^{q} h_m^2 + +c\sum_{j=1}^{p} h_n^2 \\ &\quad +c\int_0^t \|\tilde{u}_m(s) - \tilde{u}_n(s)\|_V \, ds + ch_m + ch_n \\ &\leq c\int_0^t \|u_m(s) - u_n(s)\|_V \, ds + ch_m + ch_n,\end{aligned}$$

which, with (70) and (68), gives (67). □

Fourth step. In this step we prove some convergence results.

Lemma 7. *There exists a function $u \in W^{1,2}(0, T; V)$ and two subsequences of $\{u_m\}$ and $\{\tilde{u}_m\}$ again denoted by $\{u_m\}$ and $\{\tilde{u}_m\}$, respectively, such that*

$$u_m \rightharpoonup u \text{ weakly in } L^2(0, T; V). \tag{71}$$

$$\dot{u}_m \rightharpoonup \dot{u} \text{ weakly in } L^2(0, T; V). \tag{72}$$

$$\varepsilon(\dot{u}_m) \rightharpoonup \varepsilon(\dot{u}) \text{ weakly in } L^2(0, T; \mathcal{Q}). \tag{73}$$

$$u_m \to u \text{ strongly in } C\left([0, T]; L^2\left(\Gamma_3; \mathbb{R}^d\right)\right). \tag{74}$$

$$u_m \to u \text{ strongly in } C([0, T]; V). \tag{75}$$

$$\tilde{u}_m \to u \text{ strongly in } L^2(0, T; V). \tag{76}$$

Proof. To proof (71)-(74), we use Lemma 5 and compactness arguments similar to those in [10, Lemma 7]. It follows from (46), (26), (56), (57), (44) and (58) that $\{\mathcal{B}_m\}$, $\{\tilde{u}_m\}$ and $\{f_m\}$ satisfy the following inequality

$$\left\{\begin{aligned} &(\mathcal{F}\tilde{u}_m(t), v - \tilde{u}_m(t))_V + (\mathcal{B}_m(t), \varepsilon(v - \tilde{u}_m(t)))_{\mathcal{Q}} \\ &+\psi(\tilde{u}_m(t), v - \tilde{u}_m(t)) \geq (f_m(t), v - \tilde{u}_m(t))_V, \end{aligned}\right. \tag{77}$$

for all $v \in V$ and for all $t \in [0, T]$. Now, let m, $n \in \mathbb{N}^*$, such that $m > n > T$, by taking $(\mathcal{B}_m, \tilde{u}_m, f_m, v) = (\mathcal{B}_m, \tilde{u}_m, f_m, \tilde{u}_n)$, $(\mathcal{B}_m, \tilde{u}_m, f_m, v) = (\mathcal{B}_n, \tilde{u}_n, f_n, \tilde{u}_m)$ in (77) and adding the two inequalities, we get

$$\left\{\begin{aligned} &(\mathcal{F}\tilde{u}_m(t) - \mathcal{F}\tilde{u}_n(t), \tilde{u}_m(t) - \tilde{u}_n(t))_V \leq (\mathcal{B}_n(t) - \mathcal{B}_m(t), \varepsilon(\tilde{u}_m(t) - \tilde{u}_n(t)))_{\mathcal{Q}} \\ &+\psi(\tilde{u}_m(t), \tilde{u}_n(t) - \tilde{u}_m(t)) + \psi(\tilde{u}_n(t), \tilde{u}_m(t) - \tilde{u}_n(t)) \\ &+(f_m(t) - f_n(t), \tilde{u}_m(t) - \tilde{u}_n(t))_V, \ \forall t \in [0, T], \end{aligned}\right.$$

which combined with (29), (40), (60) and using the inequality

$$ab \le \frac{a^2}{m_{\mathcal{A}}} + \frac{m_{\mathcal{A}}}{4} b^2, \ \forall a, \ b \in \mathbb{R},$$

leads us to

$$\begin{aligned} \|\tilde{u}_m(t) - \tilde{u}_n(t)\|_V^2 \ &\le \ c\,\|\mathcal{B}_n(t) - \mathcal{B}_m(t)\|_{\mathcal{Q}}^2 + c\,\|\tilde{u}_m(t) - \tilde{u}_n(t)\|_{L^2(\Gamma_3;\mathbb{R}^d)} \\ &\quad + c\,\|f_m(t) - f(t)\|_V^2 + c\,\|f(t) - f_n(t)\|_V^2 . \qquad (78) \end{aligned}$$

Using (63) and (16), we deduce that

$$\begin{aligned} \|\tilde{u}_m(t) - \tilde{u}_n(t)\|_{L^2(\Gamma_3;\mathbb{R}^d)} \ &\le \ \|\tilde{u}_m(t) - u_m(t)\|_{L^2(\Gamma_3;\mathbb{R}^d)} + \|u_m(t) - u_n(t)\|_{L^2(\Gamma_3;\mathbb{R}^d)} \\ &\quad + \|u_n(t) - \tilde{u}_n(t)\|_{L^2(\Gamma_3;\mathbb{R}^d)} \\ &\le \ \|u_m(t) - u_n(t)\|_{L^2(\Gamma_3;\mathbb{R}^d)} + ch_m + ch_n. \qquad (79) \end{aligned}$$

Now, it follows from (64), (67), (78) and (79), that

$$\begin{aligned} \|\tilde{u}_m(t) - \tilde{u}_n(t)\|_V^2 \ &\le \ c\,\|u_m(t) - u_n(t)\|_{L^2(\Gamma_3;\mathbb{R}^d)} + c\int_0^t \|u_n(s) - u_m(s)\|_V^2 \, ds \\ &\quad + ch_m + ch_n, \end{aligned}$$

and using the inequality

$$\|u_m(t) - u_n(t)\|_V^2 \le c\,\|u_m(t) - \tilde{u}_m(t)\|_V^2 + c\,\|\tilde{u}_m(t) - \tilde{u}_n(t)\|_V^2 + c\,\|\tilde{u}_n(t) - u_n(t)\|_V^2 ,$$

we get

$$\begin{aligned} \|u_m(t) - u_n(t)\|_V^2 \ &\le \ c\,\|u_m(t) - u_n(t)\|_{L^2(\Gamma_3;\mathbb{R}^d)} + c\int_0^t \|u_n(s) - u_m(s)\|_V^2 \, ds \\ &\quad + ch_m + ch_n, \ \forall t \in [0, T] . \end{aligned}$$

Using Lemma 1 in the last inequality leads to

$$\begin{aligned} \|u_m(t) - u_n(t)\|_V^2 &\le c\,\|u_m(t) - u_n(t)\|_{L^2(\Gamma_3;\mathbb{R}^d)} + c\int_0^t \|u_m(s) - u_n(s)\|_{L^2(\Gamma_3;\mathbb{R}^d)} \, ds \\ &\quad + ch_m + ch_n, \end{aligned}$$

from this, we obtain

$$\|u_m - u_n\|_{C([0,T];V)}^2 \le c\,\|u_m - u_n\|_{C\left([0,T];L^2(\Gamma_3;\mathbb{R}^d)\right)} + ch_n.$$

This last inequality and (74) imply that $\{u_m\}$ is a Cauchy sequence in $C\left([0, T] ; V\right)$ and, threfore, by using the convergence (71), we obtain (75). Now, the convergence (76) is a consequence of (63) and (75). □

In the rest of this paper $\{u_m\}, \{\mathcal{B}_m\}, \{\tilde{u}_m\}$ and $\{f_m\}$ represent appropriate subsequences of $\{u_m\}$, $\{\mathcal{B}_m\}$, $\{\tilde{u}_m\}$ and $\{f_m\}$ such that the convergences (71)-(76) hold.

Lemma 8. *The following convergences hold.*

$$\mathcal{F}\tilde{u}_m \to \mathcal{F}u \text{ strongly in } L^2(0,T;V). \tag{80}$$

$$f_m \to f \text{ strongly in } L^2([0,T];V). \tag{81}$$

$$\mathcal{B}_m \to \widetilde{\mathcal{B}} \text{ strongly in } L^2(0,T;\mathcal{Q}), \tag{82}$$

where the function $\widetilde{\mathcal{B}} : [0,T] \to \mathcal{Q}$ is defined by

$$\widetilde{\mathcal{B}}(t) = \int_0^t \mathcal{B}(t-s)\,\varepsilon(u(s))\,ds, \ \forall\ t \in [0,T]. \tag{83}$$

Proof. Obviously, (41) and (76) imply (80). Using (58) and (64) we get (81). Now, it follows from (43), (44), (56), (57), (83) and (33), that

$$\begin{aligned} \left\|\mathcal{B}_m(t) - \widetilde{\mathcal{B}}(t)\right\|_{\mathcal{Q}} &\le \left\|h_m \mathcal{B}(t_1^m)\,u_m^0\right\|_{\mathcal{Q}} + c\int_0^t \|u(s)\|_V \\ &\le c h_m, \forall t \in (t_0^m, t_1^m]. \end{aligned} \tag{84}$$

On the other hand, for each $t \in \left(t_i^m, t_{i+1}^m\right]$, $i = 1, ..., m-1$, using (44), (56), (57) and (83), we obtain

$$\begin{aligned} \left\|\mathcal{B}_m(t) - \widetilde{\mathcal{B}}(t)\right\|_{\mathcal{Q}} \le \sum_{j=1}^{i} \int_{t_{j-1}^m}^{t_j^m} &\left\|\mathcal{B}\left(t_{i+1}^m - t_j^m\right)\varepsilon\left(\tilde{u}_m(s)\right)\right. \\ &\left.-\mathcal{B}(t-s)\,\varepsilon(u(s))\right\|_{\mathcal{Q}} ds + h_m\left\|\mathcal{B}\left(t_{i+1}^m\right)\varepsilon\left(u_m^0\right)\right\|_{\mathcal{Q}} \\ &+ \int_{t_i^m}^t \left\|\mathcal{B}(t-s)\,\varepsilon(u(s))\right\|_{\mathcal{Q}} ds. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} \left\|\mathcal{B}_m(t) - \widetilde{\mathcal{B}}(t)\right\|_{\mathcal{Q}} \le &\sum_{j=1}^{i} \int_{t_{j-1}^m}^{t_j^m} \left\|\left[\mathcal{B}\left(t_{i+1}^m - t_j^m\right) - \mathcal{B}(t-s)\right]\varepsilon\left(\tilde{u}_m(s)\right)\right\|_{\mathcal{Q}} ds \\ &+ \sum_{j=1}^{i} \int_{t_{j-1}^m}^{t_j^m} \left\|\mathcal{B}(t-s)\,\varepsilon\left(\tilde{u}_m(s) - u(s)\right)\right\|_{\mathcal{Q}} ds \\ &+ h_m\left\|\mathcal{B}\left(t_{i+1}^m\right)\varepsilon\left(u_m^0\right)\right\|_{\mathcal{Q}} + \int_{t_i^m}^t \left\|\mathcal{B}(t-s)\,\varepsilon(u(s))\right\|_{\mathcal{Q}} ds, \end{aligned}$$

and keeping in mind (32)-(33) and (60), we can show that

$$\left\|\mathcal{B}_m(t) - \widetilde{\mathcal{B}}(t)\right\|_{\mathcal{Q}} \le c \sum_{j=1}^{i} \int_{t_{j-1}^m}^{t_j^m} \left(\left|t_{i+1}^m - t\right| + \left|s - t_j^m\right|\right) ds$$

$$+ c \sum_{j=1}^{i} \int_{t_{j-1}^m}^{t_j^m} \|\tilde{u}_m(s) - u(s)\|_V \, ds + ch_m \left\|u_m^0\right\|_V + ch_m \|u\|_{C([0,T];V)}$$

$$\le c \int_0^t \|\tilde{u}_m(s) - u(s)\|_V \, ds + ch_m.$$

This last inequality with (84), gives

$$\left\|\mathcal{B}_m - \widetilde{\mathcal{B}}\right\|_{L^2(0,T;\mathcal{Q})} \le c \|\tilde{u}_m - u\|_{L^2(0,T;V)} + ch_m. \tag{85}$$

Passing to the limit as $m \to +\infty$ in (85) by using (76), we get (82). □

The following properties hold.

Lemma 9. *For all $v \in L^2(0,T;V)$, we have*

$$\lim_{m\to+\infty} \int_0^T \psi(\tilde{u}_m(s), v(s)) ds = \int_0^T \psi(u(s), v(s)) ds. \tag{86}$$

$$\lim_{m\to+\infty} \int_0^T \left[\psi(\tilde{u}_m(s), \dot{u}_m(s)) - \psi(u(s), \dot{u}_m(s))\right] ds = 0. \tag{87}$$

$$\liminf_{m\to+\infty} \int_0^T \psi(\tilde{u}_m(s), \dot{u}_m(s)) ds \ge \int_0^T \psi(u(s), \dot{u}(s)) ds. \tag{88}$$

Proof. Using (27), we deduce the following

$$\left|\int_0^T \left[\psi(\tilde{u}_m(s), v(s)) - \psi(u(s), v(s))\right] ds\right| \le c \|\tilde{u}_m - u\|_{L^2(0,T;V)} \|v\|_{L^2(0,T;V)}, \tag{89}$$

for all $v \in L^2(0,T;V)$. From (76), (89) and (62), it follows that the convergences (86)-(87) hold. To continue, let $\Phi : L^2(0,T;V) \to \mathbb{R}$ be the functional defined by

$$\Phi(v) = \int_0^T \psi(u(s), v(s)) ds, \ \forall v \in L^2(0,T;V). \tag{90}$$

Using (23), (28) and (90), we find that Φ is convex and continuous, which implies that Φ is weakly lower semicontinuous, see [2]. Thus, from (72), we get

$$\liminf_{m\to+\infty} \Phi\left(\dot{u}_m\right) \geq \Phi\left(\dot{u}\right). \tag{91}$$

On the other hand, one has

$$\begin{aligned}\int_0^T \psi(\tilde{u}_m(s),\dot{u}_m(s))ds &= \int_0^T \left[\psi(\tilde{u}_m(s),\dot{u}_m(s))ds - \psi(u(s),\dot{u}_m(s))\right]ds \\ &\quad +\Phi\left(\dot{u}_m\right).\end{aligned} \tag{92}$$

Therefore, taking into account (87) and (91) when passing to the lim inf as $m \to +\infty$ in (92), we obtain (88). □

Fifth step. We have now all the ingredients to prove Theorem 1.

Proof of Theorem 1. Let $t \in (0,T)$, let $r > 0$, such that $t + r \in (0,T)$. For each $w \in V$, we define a function $v \in L^2(0,T;V)$ by

$$v(s) = \begin{cases} w & \text{for } s \in (t,t+r) \\ \\ \dot{u}(s) & \text{elsewhere,}\end{cases}$$

Now, we use (42), (44), (56), (57), (58), (59), to obtain the following inequality

$$\begin{cases} \displaystyle\int_0^T \left(\mathcal{F}\tilde{u}_m(s), v(s)-\dot{u}_m(s)\right)_V ds + \int_0^T \left(\mathcal{B}_m(s), \varepsilon\left(v(s)-\dot{u}_m(s)\right)\right)_{\mathcal{Q}} ds+ \\ \displaystyle +\int_0^T \psi(\tilde{u}_m(s),v(s))ds - \int_0^T \psi(\tilde{u}_m(s),\dot{u}_m(s))ds \\ \displaystyle \geq \int_0^T \left(f_m(s), v(s)-\dot{u}_m(s)\right)_V ds. \end{cases} \tag{93}$$

Passing to the lim sup as $m \to +\infty$ in (93), by using Lemma 8, Lemma 9,

(72) and (73), we obtain

$$\left\{\begin{array}{l} \dfrac{1}{r}\displaystyle\int_t^{t+r}(\mathcal{F}u(s), w-\dot{u}(s))_V\,ds\ +\dfrac{1}{r}\displaystyle\int_t^{t+r}\left(\widetilde{\mathcal{B}}(s), \varepsilon\,(w-\dot{u}(s))\right)_{\mathcal{Q}}ds \\ +\dfrac{1}{r}\displaystyle\int_t^{t+r}\left[\psi(u(s),w)-\psi(u(s),\dot{u}(s))\right]ds \\ \geq \dfrac{1}{r}\displaystyle\int_t^{t+r}(f(s), w-\dot{u}(s))_V\,ds,\ \text{for all } w\in V. \end{array}\right. \tag{94}$$

Since $u_m(t) \to u(t)$ strongly in V, $\forall t \in [0,T]$, it follows from (55) that $u(0) = u_0$. Now, letting $r \to 0$ in (94) and keeping in mind (39) and (83), we conclude that u is a solution of the problem (34)-(35). On the other hand, using (65), we obtain

$$\begin{aligned} \|u(t)-u(s)\|_V &\leq \|u(t)-u_m(t)\|_V + \|u_m(t)-u_m(s)\|_V + \|u_m(s)-u(s)\|_V \\ &\leq \|u(t)-u_m(t)\|_V + c\,|t-s| + \|u_m(s)-u(s)\|_V\,, \\ &\forall t,s\in[0,T]\,. \end{aligned}$$

Passing to the limit as $m \to +\infty$, we get

$$\|u(t)-u(s)\|_V \leq c\,|t-s|\,,\ \forall t,s\in[0,T]\,.$$

Thus, u satisfies the regularity (38). Finally, it is easy to see that the function σ defined by (8) has the regularity $\sigma \in W^{1,\infty}(0,T;\mathcal{Q}_1)$. □

5 Conclusion

In this paper, we have studied a quasistatic contact problem with normal compliance associated to a version of Coulomb's law of dry friction, for viscoelastic materials with long memory. We have shown the existence of a weak solution under a smallness assumption depending only on the normal compliance functions, the elasticity operator and on the geometry of the problem. The uniqueness of the solution remains, as far as we know, an open question.

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