Ann. Acad. Rom. Sci.
Ser. Math. Appl.
ISSN 2066-6594 Vol. 13, No. 1-2/2021

EXISTENCE OF FIXED POINTS AND BEST PROXIMITY POINTS OF p-CYCLIC BOYD-WONG CONTRACTIONS*

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DOI https://doi.org/10.56082/annalsarscimath.2021.1-2.111

Abstract

We introduce a new contraction map called p-cyclic Boyd-Wong contraction, defined on the union of p ($p \geq 2$) non empty subsets of a metric space. We give sufficient conditions for the existence of a unique fixed point, best proximity point or periodic point for the map and an iterative method is used to approximate the fixed point and the best proximate point.

MSC: 46H10

keywords: p-cyclic maps, p-cyclic contractions, p-cyclic non expansive maps, best proximity points.

1 Introduction and preliminaries

There are many interesting and useful generalizations of the celebrated Banach contraction theorem. Some of them are given in the literature ([1] to [10]). One of them is given by Boyd and Wong in [1]. The contraction given by Banach is essentially uniformly continuous whereas the contraction given by Boyd and Wong is upper semi-continuous from the right. The continuity condition of the contraction map is thus relaxed. In [6], the following type of maps are introduced, where the maps are not continuous.

*Accepted for publication in revised form on December 1-st, 2020

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Let (X, d) be a metric space. Let $A_1, A_2, \ldots, A_p$ be nonempty subsets of X. Consider $F : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ satisfying the following condition:

$$F(A_1) \subseteq A_2,\ F(A_2) \subseteq A_3,\ \ldots, F(A_{p-1}) \subseteq A_p \ \text{ and } \ F(A_p) \subseteq A_1.$$

Thus the map F takes points of a set to the next set and the last to the first, forming a cycle. In [3], these maps are called as ***p*-cyclic maps**. In [6], a contraction condition similar to the Banach contraction was imposed on F. That is, for some $k,\ 0 < k < 1$,

$$d(Tx, Ty) \leq k\ d(x, y),\ \text{for all } x \in A_i,\ y \in A_{i+1},\ 1 \leq i \leq p,$$

where $A_{p+1} = A_1$. This condition on the map entails $\bigcap_{i=1}^{p} A_i$ to be nonempty, and clearly F restricted to this intersection is a Banach contraction. Thus in [6], Kirk et al. extended the Banach contraction theorem for p-cyclic maps. Also, in [6], the Boyd-Wong's theorem is extended in this direction and obtained the following fixed point theorem:

Theorem 1. *([6]) Let $\{A_i\}_{i=1}^{p}$ be nonempty and closed subsets of a complete metric space (X, d) and $f : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ be a p-cyclic map satisfying the following condition(where $A_{p+1} = A_1$):*

$$d(f(x), f(y)) \leq \psi(d(x, y)),\ \forall x \in A_i,\ y \in A_{i+1},\ \textit{for } 1 \leq i \leq p,$$

where $\psi : [0, \infty) \to [0, \infty)$ is upper semi-continuous from the right and satisfies $\psi(t) < t,\ for\ \ t > 0,\ \psi(0) = 0$. Then f has a unique fixed point.

Recall that $\psi : [0, \infty) \to [0, \infty)$ is upper semicontinuous from the right if, $r_j \downarrow r \Rightarrow \limsup_j \psi(r_j) \leq \psi(r),\ \ where\ \ r_j \in [0, \infty), j = 1, 2.....$

In [2], Eldred and Veeramani introduced a notion of ***cyclic contractions***, which are defined on the union of two nonempty subsets A and B of a metric space such that $T(A) \subseteq B$ and $T(B) \subseteq A$ and for some $k,\ 0 < k < 1$,

$$d(Tx, Ty) \leq k\ d(x, y) + (1 - k)\ \mathrm{dist}(A, B),\ x \in A,\ y \in B,$$

where $dist(A, B) = inf\{d(x, y) : x \in A, \ \ y \in B\}$.

Sufficient conditions were given in [2] to obtain a point called ***best proximity point*** of T, that is a point $x \in A \cup B$ such that

$$d(x, Tx) = \mathrm{dist}(A, B).$$

To obtain a best proximity point, the underlying space needed to be a uniformly convex Banach space and the sets needed to be convex. A question that naturally arises is that, whether the fixed point theorem 1 can be extended to best proximity point theorem? In this paper, we give a positive answer to this question. We introduce a contraction map called p-cyclic Boyd-Wong contraction and give sufficient conditions for the existence and convergence of a unique fixed point, best proximity point and periodic point of the map.

The following results proved in [2] in a uniformly convex Banach space setting are useful to prove best proximity point results in this paper.

Lemma 1. *([2]) Let A be a nonempty closed and convex subset and B be a nonempty closed subset of a uniformly convex Banach space. Let $\{x_n\}$ and $\{z_n\}$ be sequences in A and $\{y_n\}$ be a sequence in B satisfying:*

(1) $\| x_n - y_n \| \longrightarrow \; dist(A, B)$

(2) $\| z_n - y_n \| \longrightarrow \; dist(A, B)$.

Then $\| x_n - z_n \| \longrightarrow 0$.

A p-cyclic mapping T is said to be a p**-cyclic non expansive map** ([3]) if for $x \in A_i, \quad y \in A_{i+1}$, we have

$$d(Tx, Ty) \leq d(x, y)$$

.

Remark 1. *If $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ is a p-cyclic non expansive map, then in [3](Lemma 3.3) it is proved that*

$$\mathrm{dist}(A_i, A_{i+1}) = \mathrm{dist}(A_{i+1}, A_{i+2}) = \mathrm{dist}(A_1, A_2). \tag{1.1}$$

If, moreover, $\xi \in A_i$ is a best proximity point in A_i, then in [3](Remark 3.4) it is proved that $T^j \xi$ is a best proximity point in A_{i+j}, for $j = 1, 2, ..., (p-1)$.

2 Main Results

First we define the map p-cyclic Boyd-Wong contraction as follows:

Definition 1. *Let $A_1, A_2, \ldots, A_p$ be nonempty subsets of a metric space X, and $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ be a p-cyclic map. Suppose,*

$$d(Tx, Ty) \le \psi(d(x, y) - dist(A_i, A_{i+1})) + dist(A_i, A_{i+1}),$$

$x \in A_i,\ y \in A_{i+1},\ 1 \le i \le$ p, where $\psi : [0, \infty) \to [0, \infty)$ is upper semicontinuous from the right and satisfies $\psi(t) < t,\ for\ t > 0$ and $\psi(0) = 0$. Then we call T, a p-cyclic Boyd-Wong contraction.

We give the following lemma which follows from the definition of p-cyclic Boyd-Wong contraction.

Lemma 2. *Let $A_1, A_2, \ldots, A_p$ be nonempty subsets of a metric space (X, d), and $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ be a p-cyclic Boyd-Wong contraction map. Then the following hold:*

(a) T is p-cyclic non expansive map, and hence $dist(A_i, A_{i+1}) = dist(A_1, A_2)$, for all i

(b) For $x,\ y \in A_i,\ d(T^{pn}x, T^{pn+1}y) \to dist(A_i, A_{i+1}),$ as $n \longrightarrow \infty$

(c) For $x,\ y \in A_i,\ d(T^{pn-1}x, T^{pn}y) \longrightarrow dist(A_i, A_{i+1})$

(d) For $x,\ y \in A_i,\ d(T^{pn+p}x, T^{pn+1}y) \longrightarrow dist(A_i, A_{i+1})$

(e) For $x,\ y \in A_i,\ d(T^{pn-p}x, T^{pn+1}y) \longrightarrow dist(A_i, A_{i+1})$

(f) For $x,\ y \in A_i,\ d(T^{pn}x, T^{pn+p+1}y) \longrightarrow dist(A_i, A_{i+1})$.

Proof. (a) Let $x \in A_i$ and $y \in A_{i+1}$. Now,

$$d(Tx, Ty) \le \psi(d(x, y) - \mathrm{dist}(A_i, A_{i+1})) + \mathrm{dist}(A_i, A_{i+1}).$$

Case(i): $d(x, y) > \mathrm{dist}(A_i, A_{i+1})$. Then $d(x, y) - \mathrm{dist}(A_i, A_{i+1}) > 0$. Since $\psi(t) < t$ for $t > 0$, we have

$$\begin{aligned} d(Tx, Ty) &< d(x, y) - \mathrm{dist}(A_i, A_{i+1}) + \mathrm{dist}(A_i, A_{i+1}) \\ &= d(x, y). \end{aligned}$$

Therefore,

$$d(Tx, Ty) < d(x, y). \tag{2.1}$$

Case(ii): $d(x,y) = \text{dist}(A_i, A_{i+1})$.
Then $\psi(d(x,y) - dist(A_i, A_{i+1})) = \psi(0) = 0$. Therefore

$$d(Tx,Ty) = \text{dist}(A_i, A_{i+1})$$

That is,

$$d(Tx,Ty) = d(x,y) \tag{2.2}$$

From (2.1) and (2.2), we have

$$d(Tx,Ty) \leq d(x,y) \quad for\ all \quad x \in A_i \quad and \quad y \in A_{i+1},\ 1 \leq i \leq p.$$

Hence T is a p-cyclic non expansive map.

(b) Let $x,\ y \in A_i$. By applying (a) p times,

$$d(T^{p(n+1)}x, T^{p(n+1)+1}y) \leq d(T^{pn}x, T^{pn+1}y).$$

Therefore, $\{d(T^{pn}x, T^{pn+1}y)\}$ is a non-increasing sequence bounded below by $\text{dist}(A_i, A_{i+1})$. Hence $d(T^{pn}x, T^{pn+1}y) \longrightarrow r \geq \text{dist}(A_i, A_{i+1})$,
where $r = inf_n\{d(T^{pn}x, T^{pn+1}y)\}$.
Claim: $r = \text{dist}(A_i, A_{i+1})$.

Case(1): $d(T^{pn}x, T^{pn+1}y) = \text{dist}(A_i, A_{i+1})$ for some n.
Then by p-cyclic non expansiveness of the map,

$$d(T^{pn+k}x, T^{pn+k+1}y) = \text{dist}(A_i, A_{i+1}), \quad for\ all\ k = 1, 2, ...$$

Therefore, $d(T^{pn}x, T^{pn+1}y) \longrightarrow \text{dist}(A_i, A_{i+1})$.

Case(2): $d(T^{pn}x, T^{pn+1}y) > \text{dist}(A_i, A_{i+1})$, for all n.
Then $d(T^{pn}x, T^{pn+1}y) - \text{dist}(A_i, A_{i+1}) > 0$, for all n. Now,

$$\begin{aligned} d(T^{p(n+1)}x, T^{p(n+1)+1}y) &\leq d(T^{pn+1}x, T^{pn+2}y) \\ &\leq \psi(d(T^{pn}x, T^{pn+1}y) - \text{dist}(A_i, A_{i+1})) \\ &\quad + \text{dist}(A_i, A_{i+1}) \end{aligned}$$

$$d(T^{p(n+1)}x, T^{p(n+1)+1}y) - dist(A_i, A_{i+1}) \leq \psi(d(T^{pn}x, T^{pn+1}y) - \text{dist}(A_i, A_{i+1})).$$

Then taking the limit as $n \longrightarrow \infty$,

$$\lim_{n\to\infty} d(T^{p(n+1)}x, T^{p(n+1)+1}y) - dist(A_i, A_{i+1})$$
$$\leq \limsup_{n\to\infty} \psi(d(T^{pn}x, T^{pn+1}y) - \text{dist}(A_i, A_{i+1})).$$

Since $d(T^{p(n+1)}x, T^{p(n+1)+1}y) - dist(A_i, A_{i+1}) \downarrow r - dist(A_i, A_{i+1})$, we have

$$r - \text{dist}(A_i, A_{i+1}) \leq \psi(r - \text{dist}(A_i, A_{i+1})).$$

If $r > \text{dist}(A_i, A_{i+1})$, then taking $t = r - \text{dist}(A_i, A_{i+1}) > 0$, we get $t \leq \psi(t)$, which is a contradiction to the definition of ψ, where $\psi(t) < t,\ for\ t > 0$. Hence $r = \text{dist}(A_i, A_{i+1})$.

(c) Let $x,\ y \in A_i$. Let $s_n = d(T^{pn-1}x, T^{pn}y)$. Applying p-cyclic non expansiveness of T p-times, we get

$$s_{n+1} = d(T^{pn+p-1}x, T^{pn+p}y) \leq d(T^{pn-1}x, T^{pn}y) = s_n.$$

Therefore, $\{s_n\}$ is non increasing and bounded below by $dist(A_{i-1}, A_i)$. Hence $s_n \longrightarrow r \geq dist(A_{i-1}, A_i)$. Proceeding in a similar way as in (b), we can prove that

$$r = dist(A_{i-1}, A_i) = dist(A_i, A_{i+1}),\ by\ (a).$$

(d) Taking $T^p x$ in place of x in (b), we have
$d(T^{pn+p}x, T^{pn+1}y) \longrightarrow \text{dist}(A_i, A_{i+1})$.

(e) To prove that $d(T^{pn-p}x, T^{pn+1}y) \longrightarrow \text{dist}(A_i, A_{i+1})$, let

$$s_n = d(T^{pn-p}x, T^{pn+1}y).$$

Then $s_{n+1} = d(T^{pn}x, T^{pn+p+1}y)$. Using p-cyclic non-expansiveness of T, p times, $s_{n+1} \leq s_n, \forall n$. Proceeding as in (b),
$d(T^{pn-p}x, T^{pn+1}y) \longrightarrow \text{dist}(A_i, A_{i+1})$.

(f) Taking $T^p y$ in place of y in (b), we have
$d(T^{pn}x, T^{pn+p+1}y) \longrightarrow \text{dist}(A_i, A_{i+1})$. □

Note that if T satisfies

$$d(Tx, Ty) \leq \psi(d(x, y) - \text{dist}(A_i, A_{i+1})) + \text{dist}(A_i, A_{i+1}),$$

then

$$d(Tx, Ty) \leq \psi(d(x, y))d(x, y) + (1 - \psi(d(x, y)))\text{dist}(A_i, A_{i+1}),$$

which can be compared with the p-cyclic contraction map.

Combining Lemma 1 with Lemma 2, the following proposition is obtained on a uniformly convex Banach space setting.

Proposition 1. *Let $A_1, A_2, \ldots, A_{\mathrm{p}}$ be non empty, closed and convex subsets of a uniformly convex Banach space X. Let $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ be a* p-*cyclic Boyd-Wong contraction. Then for any $x \in A_i$, the following hold:*

(a) $\| T^{pn}x - T^{pn+p}x \| \longrightarrow 0$

(b) $\| T^{pn}x - T^{pn-p}x \| \longrightarrow 0$

(c) $\| T^{pn+1}x - T^{pn+p+1}x \| \longrightarrow 0.$

Theorem 2. *Let X be a uniformly convex Banach space. Let $A_1, A_2, \ldots, A_{\mathrm{p}}$ be non empty, closed and convex subsets of X. Let $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ be a* p-*cyclic Boyd-Wong contraction. Then for each i, $1 \leq i \leq p$, there exists a unique $z_i \in A_i$, such that for any $x \in A_i$, $\{T^{pn}x\}$ converges to z_i, and such that z_i is a best proximity point and unique periodic point of T in A_i. Also, $T^j z_i = z_{i+j}$ is a best proximity point and unique periodic point of T in A_{i+j} for $j = 1, 2, \ldots, p-1$.*

Proof. If $\operatorname{dist}(A_i, A_{i+1}) = 0$, for some i, then

$$\| Tx - Ty \| \leq \psi(\| x - y \|), x \in A_i, y \in A_{i+1}, 1 \leq i \leq p.$$

Hence by Theorem 1, T has a unique fixed point.
Assume that $\operatorname{dist}(A_i, A_{i+1}) > 0$. Let $x \in A_i$. We show that $\{T^{pn}x\}$ is a Cauchy sequence. By Lemma 2 (b),

$$\| T^{pn}x - T^{pn+1}x \| \to \operatorname{dist}(A_i, A_{i+1}).$$

If for given $\epsilon > 0$ there exists an $n_0 \in \mathbb{N}$, such that

$$\| T^{pm}x - T^{pn+1}x \| < \text{ dist } (A_i, A_{i+1}) + \epsilon, m > n > n_0; \tag{2.3}$$

then by Lemma 1, for given $\epsilon > 0$, there exists an $n_1 \in \mathbb{N}$ such that

$$\| T^{pm}x - T^{pn}x \| < \epsilon, \quad \textit{for all } m > n > n_1$$

and therefore, the sequence $\{T^{pn}x\}$ is a Cauchy sequence. Hence, it is enough to prove the claim for given $\epsilon > 0$, there exists an $n_0 \in \mathbb{N}$ satisfying (2.3). On the contrary, suppose that there exists an $\epsilon_0 > 0$, and for $k = 1, 2, \ldots,$, there exists $m_k > n_k > k$, such that

$$\|T^{pm_k}x - T^{pn_k+1}x\| \geq \operatorname{dist}(A_i, A_{i+1}) + \epsilon_0. \tag{2.4}$$

Upon choosing m_k to be the least integer greater than n_k to satisfy (2.4), we have,

$$\|T^{pm_k-p}x - T^{pn_k+1}x\| < \mathrm{dist}(A_i, A_{i+1}) + \epsilon_0.$$

Now for each k,

$$\begin{aligned}
\mathrm{dist}(A_i, A_{i+1}) + \epsilon_0 &\leq \|T^{pm_k}x - T^{pn_k+1}x\| \\
&\leq \|T^{pm_k}x - T^{pm_k-p}x\| + \|T^{pm_k-p}x - T^{pn_k+1}x\| \\
&< \|T^{pm_k}x - T^{pm_k-p}x\| + dist(A_i, A_{i+1}) + \epsilon_0.
\end{aligned}$$

Since by Proposition 1 (b),

$$\lim_{k\to\infty} \|T^{pm_k}x - T^{pm_k-p}x\| = 0,$$

we have
$\mathrm{dist}(A_i, A_{i+1}) + \epsilon_0 \leq \lim_{k\to\infty} \|T^{pm_k}x - T^{pn_k+1}x\| \leq dist(A_i, A_{i+1}) + \epsilon_0.$
Hence

$$\lim_{k\to\infty} \|T^{pm_k}x - T^{pn_k+1}x\| = \mathrm{dist}(A_i, A_{i+1}) + \epsilon_0. \tag{2.5}$$

Consequently for each k,

$$\begin{aligned}
\|T^{pm_k}x - T^{pn_k+1}x\| &\leq \|T^{pm_k}x - T^{pm_k+p}x\| + \|T^{pm_k+p}x - T^{pn_k+p+1}x\| \\
&\quad + \|T^{pn_k+p+1}x - T^{pn_k+1}x\| ...(*)
\end{aligned}$$

By p-cyclic non-expansiveness of T,

$$\begin{aligned}
&\|T^{pm_k+p}x - T^{pn_k+p+1}x\| \leq \|T^{pm_k+1}x - T^{pn_k+2}x\| \\
&\leq \psi(\|T^{pm_k}x - T^{pn_k+1}x\| - \mathrm{dist}(A_i, A_{i+1})) + \mathrm{dist}(A_i, A_{i+1}).
\end{aligned}$$

Since $\|T^{pm_k}x - T^{pn_k+1}x\| - \mathrm{dist}(A_i, A_{i+1}) \downarrow \epsilon_0$,

$$\limsup_{k\to\infty} \psi(\|T^{pm_k}x - T^{pn_k+1}x\| - \mathrm{dist}(A_i, A_{i+1})) \leq \psi(\epsilon_0)$$

and by Proposition 1 (a) and (c),

$$\lim_{k\to\infty} \|T^{pm_k}x - T^{pm_k+p}x\| = 0 \quad and \quad \lim_{k\to\infty} \|T^{pn_k+p+1}x - T^{pn_k+1}x\| = 0.$$

Applying all the above in (*), as $k \longrightarrow \infty$, we get, by equation (2.5),

$$\begin{aligned}
\lim_{k\to\infty} \|T^{pm_k}x - T^{pn_k+1}x\| &\leq \psi(\epsilon_0) + \mathrm{dist}(A_i, A_{i+1}) \\
\mathrm{dist}(A_i, A_{i+1}) + \epsilon_0 &\leq \psi(\epsilon_0) + \mathrm{dist}(A_i, A_{i+1})
\end{aligned}$$

Therefore, $\epsilon_0 \leq \psi(\epsilon_0)$. Since $\epsilon_0 > 0$, $\psi(\epsilon_0) < \epsilon_0$, thereby we arrive at a contradiction. Hence the claim is proved. Therefore $\{T^{pn}x\}$ is a Cauchy sequence in A_i and so converges to a $z_i \in A_i$. Now, for each n,

$$\operatorname{dist}(A_i, A_{i-1}) \leq \|z_i - T^{pn-1}x\| \leq \|z_i - T^{pn}x\| + \|T^{pn}x - T^{pn-1}x\|$$

which tends to $\operatorname{dist}(A_i, A_{i-1})$ as $n \longrightarrow \infty$. Hence

$$\lim_{n\to\infty} \|z_i - T^{pn-1}x\| = \operatorname{dist}(A_i, A_{i-1}) = \operatorname{dist}(A_i, A_{i+1}).$$

Now, for each n, $\operatorname{dist}(A_i, A_{i+1}) \leq \|T^{pn}x - Tz_i\| \leq \|T^{pn-1}x - z_i\|$, which tends to $\operatorname{dist}(A_i, A_{i+1})$ as $n \to \infty$. Therefore

$$\lim_{n\to\infty} \|T^{pn}x - Tz_i\| = \operatorname{dist}(A_i, A_{i+1}).$$

That is, $\|z_i - Tz_i\| = \operatorname{dist}(A_i, A_{i+1})$. Hence z_i is a best proximity point of T in A_i.

Next, we prove that, $T^p z_i = z_i$. By p-cyclic non expansiveness,

$$\begin{aligned}
\| z_i - T^{p+1}z_i\| &= \lim_n \|T^{pn+p}x - T^{p+1}z_i\| \\
&\leq \lim_n \|T^{pn}x - Tz_i\| \\
&\leq \|z_i - Tz_i\| = \operatorname{dist}(A_i, A_{i+1}).
\end{aligned}$$

Therefore, $\|z_i - T^{p+1}z_i\| = \operatorname{dist}(A_i, A_{i+1})$. Since $\|z_i - Tz_i\| = \operatorname{dist}(A_i, A_{i+1})$ and A_{i+1} is a convex subset of X, X being strictly convex, $T^{p+1}z_i = Tz_i$. Now,

$$\|T^p z_i - Tz_i\| = \|T^p z_i - T^{p+1}z_i\| \leq \|z_i - Tz_i\| = \operatorname{dist}(A_i, A_{i+1}).$$

Since A_i is convex, $T^p z_i = z_i$.

To prove that z_i is the unique periodic point, let $y \in A_i$ be such that $y \neq x$. Then by what we have proved above, $T^{pn}y \longrightarrow \eta \in A_i$, such that $\|\eta - T\eta\| = dist(A_i, A_{i+1})$. In a similar way as above $T^p\eta = \eta$ and $T^{p+1}\eta = T\eta$.

Claim: $\|z_i - T\eta\| = dist(A_i, A_{i+1})$.

Suppose $\|z_i - T\eta\| > dist(A_i, A_{i+1})$. Then $\|z_i - T\eta\| - dist(A_i, A_{i+1}) > 0$. Now,

$$\begin{aligned}
\|Tz_i - T^2\eta\| &\leq \psi(\|z_i - T\eta\| - dist(A_i, A_{i+1})) + dist(A_i, A_{i+1}) \\
&< \|z_i - T\eta\| - dist(A_i, A_{i+1}) + dist(A_i, A_{i+1}) \\
&= \|T^p z_i - T^{p+1}\eta\| \\
&\leq \|Tz_i - T^2\eta\|.
\end{aligned}$$

Therefore $\|Tz_i - T^2\eta\| < \|Tz_i - T^2\eta\|$, which is a contradiction. Hence the claim is proved. Now since

$$\|\eta - T\eta\| = dist(A_i, A_{i+1}) = \|z_i - T\eta\|,$$

and since A_i is a convex subset of a strictly convex space, η, $z_i \in A_i$ are best approximations to $T\eta$ imply $\eta = z_i$. Therefore for each $x \in A_i$, the sequence $\{T^{pn}x\}$ converges to a unique z_i, which is a best proximity point and unique periodic point of T in A_i. By Remark 1, $T^j z_i = z_{i+j}$ is the best proximity point and unique periodic point of T in A_{i+j}, for $j = 1, 2, ..., p-1$. □

3 Example

Consider the uniformly convex Banach space $\mathbb{R}^2$ endowed with the norm $\|(x_1, y_1) - (x_2, y_2)\| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.
Let $A_1 = \{(0, 1 + x) : 0 \leq x \leq 1\}$, $A_2 = \{(1 + x, 0) : 0 \leq x \leq 1\}$, $A_3 = \{(0, -(1 + x)) : 0 \leq x \leq 1\}$ and $A_4 = \{(-(1 + x), 0) : 0 \leq x \leq 1\}$.
Then A_i are closed and convex subsets of $\mathbb{R}^2$, $\forall\ \ i = 1\ \ to\ \ 4$. Note that $dist(A_i, A_{i+1}) = \sqrt{2}\ \forall\ \ i = 1\ \ to\ \ 4$.
Define $T : \cup_{i=1}^{4} A_i \to \cup_{i=1}^{4} A_i$ as follows:
$T(0, 1 + x) = (1 + \frac{x}{10}, 0)$, $T(1 + x, 0) = (0, -(1 + \frac{x}{10}))$, $T(0, -(1 + x)) = (-(1 + \frac{x}{10}), 0)$ and $T(-(1 + x), 0) = (0, (1 + \frac{x}{10}))$.
Clearly $T(A_i) \subseteq A_{i+1}$, $\forall\ \ i = 1\ \ to\ \ 4$. Define $\psi : [0, \infty] \to [0, \infty)$ as
$\psi(t) = \begin{cases} \frac{t}{3}, & t \in [0, 1); \\ \frac{n^2}{n^2+1}, & t \in [n, n + 1) \end{cases}$ Then ψ is upper semi continuous from the right and $\psi(t) < t, \quad t > 0$.
Let $z_1 = (0, 1 + y) \in A_1$, $z_2 = (1 + x, 0) \in A_2$, $z_3 = (0, -(1 + y))$ and $z_4 = (-(1 + x), 0)$ where $x, y \in [0, 1]$.
Now $\forall\ \ i = 1\ \ to\ \ 4$, $\|z_i - z_{i+1}\| = \sqrt{(1 + x)^2 + (1 + y)^2}$ and $\|Tz_i - Tz_{i+1}\| = \sqrt{(1 + \frac{x}{10})^2 + (1 + \frac{y}{10})^2}$.
Then it is an easy exercise to check that T is a p-cyclic Boyd-Wong contraction and thus all the conditions of theorem 2 are satisfied. For any $x \in A_i$, for any i, $\ i = 1\ \ to\ \ 4$, the sequence $\{T^{4n}x\}$ converges to a best proximity point. Thus if $x \in A_1$ then $\{T^{4n}x\}$ converges to $\xi_1 = (0, 1) \in A_1$ which is a best proximity point of T in A_1 and $T(\xi) = \xi_2 = (1, 0)$, $T^2(\xi) = \xi_3 = (0, -1)$ and $T^3(\xi) = \xi_4 = (-1, 0)$ are the unique best proximity points as well as periodic points of period 4 in A_1, A_2, A_3, A_4 respectively. Thus this illustrates the theorem 2.

Acknowledgement: The author wishes to thank the anonimous referee for his/her suggestions for the improvement of the paper.

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