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STATIC OUTPUT FEEDBACK REVISITED*

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Abstract

The synthesis problem of static output feedback controllers within the anisotropic-norm setup is revisited. A tractable synthesis approach involving iterations over a convex optimisation problem is suggested, similarly to existing results for the H_∞-norm minimisation case. The results are formulated by a couple of Linear Matrix Inequalities coupled via a bilinear equality, revealing, as in the H_∞ case the duality of between the control-type and filtering type LMIs and allowing a tractable iterative method to cope with practical static output feedback synthesis problems. The resulting optimisation scheme is then applied to a flight control problem, where the merit of the anisotropic norm setup is shown to provide a useful trade-off between closed loop response and feedback gains.

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1 Introduction

The problems of optimal control and filtering received much attention over the years. Solutions for these problems were presented by Kwakernaak and Sivan [10]. Modelling errors were considered in [16]. When the external input signals are of white noise type, H_2-norm minimisation is applied, leading to the Kalman filter [6] and Linear Quadratic Gaussian (LQG) control.

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An alternative modelling of the exogenous inputs is based on deterministic bounded energy signals associated with the H_∞-norm based framework ([20]) applicable both to filtering ([5], [15]) and control ([21]). Many practical applications require an intermediate solution between H_2 and H_∞. Since H_2 is not entirely suitable when signals are strongly coloured and H_∞ may result in poor performance when these signals are weakly coloured (e.g. white noise), mixed H_2/H_∞ norm minimisation becomes useful (see, e.g. [2], [12]). Another option to accomplish a compromise between the H_2 and the H_∞ norms is to use the so-called *a-anisotropic norm* ([7], [19], [9]) defined as follows: consider the discrete-time stable system denoted by F with the state-space equations

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t), t = 0, 1, ... \end{aligned} \tag{1.1}$$

where $A \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times m}$, $C_f \in \mathcal{R}^{p\times n}$, $D_f \in \mathcal{R}^{p\times m}$. By definition, the a-anisotropic norm of F is

$$|||F|||_a = \sup_{G\in\mathcal{G}_a} \frac{\|FG\|_2}{\|G\|_2}, \tag{1.2}$$

where G denotes a discrete-time stable filter of form

$$\begin{aligned} x_f(t+1) &= A_f x_f(t) + B_f v(t) \\ w(t) &= C_f x_f(t) + D_f v(t), t = 0, 1, ... \end{aligned} \tag{1.3}$$

with m inputs and m outputs, where the inputs $v \in \mathcal{R}^m$ are independent Gaussian white noises. In equation (1.2), $\mathcal{G}_a$ denotes the set of all systems (1.3) with the *mean anisotropy* $\bar{A}(G) \le a$. The mean anisotropy of stationary Gaussian sequences has been introduced in [7] and it represents an entropy theoretic measure of the deviation of a probability distribution from Gaussian distributions (for more details, see for instance [7] and ([13])). Based on the Szegö-Kolmogorov theorem ([13]), in [8] it is proved that the mean anisotropy of a signal generated by an m-dimensional Gaussian white noise $v(t)$ with zero mean and identity covariance applied to a stable linear system G with m outputs has the form

$$\bar{A}(G) = -\frac{1}{2} \ln \det \left(\frac{mE\left[\tilde{w}(0)\tilde{w}(0)^T\right]}{Tr\left(E\left[w(0)w(0)^T\right]\right)} \right), \tag{1.4}$$

where $E[\tilde{w}(0)\tilde{w}(0)^T]$ is the covariance of the prediction error $\tilde{w}(0) := w(0) - E[w(0)|(w(k), k < 0]$. In the case when the output w of the filter G is a zero mean Gaussian white noise (i.e. its optimal estimate is just zero), $w(0)$ cannot be estimated from its past values and $\tilde{w}(0) = w(0)$ leading to $\bar{A}(G) = 0$. The relationship between the H_2, H_∞ and the a-anisotropic norms are given by the following inequalities (see, for instance [19]):

$$\frac{1}{\sqrt{m}}||F||_2 = |||F|||_0 \leq |||F|||_a \leq ||F||_\infty = \lim_{a\to\infty} |||F|||_a \tag{1.5}$$

showing that the a-anisotropic norm may be regarded as a relaxation of the H_∞ norm. In a study case presented in [17] it is concluded that using the a-anisotropic norm instead of H_∞ norm one may reduce the controller gains and the controls effort. In [22] a static output feedback design have been considered aiming to minimise the a-anisotropic norm of the resulting closed-loop system. The main result proved in [22] states that the optimal static output feedback gain may be obtained solving a non-convex optimisation problem.

The aim of the present paper is to derive solvability conditions for the static output feedback problem with respect to the anisotropic norm expressed in terms of convex optimisation conditions suitable for tractable numerical implementation.

The paper is organised as follows: in Section 2 some preliminaries and known results are briefly presented. In Section 3 the static output feedback minimisation problem with respect to the anisotropic norm is formulated and an iterative solution is derived. Section 4 presents a case study together with comments on the numerical results. Finally, Section 5 includes some concluding remarks.

Notation. Throughout the paper the superscript 'T' stands for matrix transposition, $\mathcal{R}$ denotes the set of scalar real numbers whereas $\mathcal{Z}_+$ stands for the non-negative integers. Moreover, $\mathcal{R}^n$ denotes the n dimensional Euclidean space, $\mathcal{R}^{n\times m}$ is the set of all $n \times m$ real matrices, and the notation $P > 0$ ($P \geq 0$), for $P \in \mathcal{R}^{n\times n}$ means that P is symmetric and positive definite (positive semi-definite). The trace of a matrix Z is denoted by $Tr(Z)$, and $|v|$ denotes the Euclidian norm of an n-dimensional vector v. Finally note that the terms Lyapunov and Riccati equations in this paper, refer to generalised versions of the standard equations appearing in the H_2 and H_∞ control literature.

2 Preliminaries and Motivation

Consider the following discrete-time system F, described by

$$\begin{array}{rcl} x(t+1) & = & Ax(t)+Bw(t) \\ y(t) & = & Cx(t)+Dw(t),\ t=0,1,... \end{array} \tag{2.1}$$

In the following, some known useful results are briefly reminded.

Definition 1. *The H_2-type norm of the system (2.1) is defined as*

$$\|F\|_2 = \left[\lim_{\ell\to\infty} \frac{1}{\ell} \sum_{t=0}^{\ell} E\left[y^T(t)y(t)\right]\right]^{\frac{1}{2}},$$

where $\{y(t)\}_{t\in\mathcal{Z}_+}$ is the output of the system (1.3) with zero initial conditions generated by the sequence $\{w(t)\}_{t\in\mathcal{Z}_+}$ of independent random vectors with the property that $E\left[w(t)\right]=0$ and $E\left[w(t)w^T(t)\right]=I_m$, $\{w(t)\}_{t\in\mathcal{Z}_+}$.

The next result provides a method to compute the H_2 norm of the system of (2.1) (see e.g. ([4]).

Lemma 1. *The H_2 type norm of the system (2.3) is given by*

$$\|F\|_2 = \left(Tr\left(B^T XB + D^T D\right)\right)^{\frac{1}{2}}$$

where $X \geq 0$ is the solution of the generalised Lyapunov equation $X = A^T XA + C^T C$.

Definition 2. *The H_∞ norm of the stable discrete-time system of form (2.1) is defined as*

$$\|F\|_\infty = \sup_{\theta\in[0,2\pi)} \lambda_{\max}^{\frac{1}{2}}\left(F^T\left(e^{-j\theta}\right)F\left(e^{j\theta}\right)\right),$$

where $\lambda_{\max}$ denotes the maximal eigenvalue and $F(\cdot)$ is the transfer function of the system.

The H_∞ norm is characterised by the following result, well-known as the Bounded Real Lemma (BRL).

Lemma 2. *The stable system (2.1) has the norm $\|F\|_\infty < \gamma$ for a certain $\gamma > 0$ if and only if the Riccati equation*

$$\begin{aligned} P = & A^T PA + \left(A^T PB + C^T D\right)\left(\gamma^2 I - B^T PB - D^T D\right)^{-1} \\ & \cdot \left(A^T PB + C^T D\right)^T + C^T C \end{aligned}$$

has a stabilizing solution $P \geq 0$ such that $\Psi_{1/\gamma^2} := \gamma^2 I - B^T PB - D^T D > 0$.

It is reminded ([4]) that a symmetric solution P of the above algebraic Riccati equation is called a *stabilising solution* if the system $x(t+1) = (A+BK)\,x(t)$ is stable, where by definition $K := \Psi_{1/\gamma^2}^{-1}\left(A^T PB + C^T D\right)^T$.

The following BRL-like result characterising the anisotropic norm has been proved in ([8]).

Theorem 1. *The system of (1.3) satisfies* $|||F|||_a \leq \gamma$ *for a given* $\gamma > 0$ *if and only if there exists* $q \in \left(0, \min\left(\gamma^{-2}, \quad \|F\|_\infty^{-2}\right)\right)$ *such that the Riccati equation*

$$\begin{aligned} X \;&=\; A^T XA + \left(A^T XB + C^T D\right)\left(\tfrac{1}{q}I - B^T XB - D^T D\right)^{-1} \\ &\quad \times \left(A^T XB + C^T D\right)^T + C^T C \end{aligned} \tag{2.2}$$

has a stabilising solution $X \geq 0$ *satisfying the following conditions*

$$\Psi_q := \frac{1}{q}I - B^T XB - D^T D > 0 \tag{2.3}$$

and

$$\det\left(\frac{1}{q} - \gamma^2\right)\Psi_q^{-1} \leq e^{-2a}. \tag{2.4}$$

We conclude this section by providing motivation to use the anisotropic norm. To this end, we denote $\eta = \sqrt{1/q}$ and restate the result of Theorem 3 above, as $||F||_\infty < \eta$ so that $det(\eta^2 - \gamma^2)\Psi_q^{-1} \leq e^{-2a}$. Namely, $\eta^2 - \gamma^2 \leq (det\Psi_q)^{1/m} e^{-2a/m}$. Using the general inequality (see [3]) $(det\Psi_q)^{1/m} \leq \frac{Tr\Psi_q}{m}$ valid for any $\Psi_q \geq 0$, and noting that $\Psi_q = \eta^2 I - B^T XB - D^T D$, the following motivating result of [22] was obtained.

Lemma 3. *Consider the system* F *of (1.1). Let* η *and* σ *respectively satisfy*

$$||F||_\infty < \eta \text{ and } ||F||_2 < \sigma$$

The a-anisotropic norm of the system of (1.3) is then upper bounded by the following linear interpolation between its H_∞ *and* H_2 *norms. Namely,*

$$\gamma^2 \geq \eta^2(1 - e^{-2a/m}) + \frac{\sigma^2}{m}e^{-2a/m}$$

We, therefore, see that in view of (1.5) one may interpret the a-anisotropic norm, the following approximate relation

$$|||F|||_a^2 \approx |||F|||_\infty^2(1 - e^{-2a/m}) + |||F|||_0^2 e^{-2a/m}$$

providing a useful insight to the a-anisotropic norm, which approximation can be interpreted also as just mixed H_∞/H_2 *optimisation, however, in the exact proportions dictated by the Lemma, in terms of* $e^{-2a/m}$.

3 Static Output Feedback

Consider the following plant

$$x(t+1) \quad = \quad Ax(t) + B_1 w(t) + B_2 u(t) \tag{3.1}$$

where we seek for a stabilising static control matrix K, such that $u(t) = Ky(t)$, where $y(t) = C_2 x(t)$ will minimize

$$z(t) \quad = \quad C_1 x(t) + D_{12} u(t) + D_{11} w(t) \tag{3.2}$$

in the sense of bounded anisotropic norm.

Define the cost function associated to the above problem

$$J(K) = |||\mathcal{F}_{c\ell}(K)|||_a \tag{3.3}$$

where $\mathcal{F}_{c\ell}(K)$ denotes the closed loop system obtained from (3.1) and (3.2) with the static output feedback $u_2(t) = Ky(t)$, having the realisation

$$\begin{aligned} x(t+1) &= (A + B_2 K C_2) x(t) + B_1 w(t) \\ z(t) &= (C_1 + D_{12} K C_2) x(t) + D_{11} w(t) \,. \end{aligned}$$

Using Theorem 1, if follows that the above closed loop system $\mathcal{F}_{c\ell}(K)$ is stable and it has the a-anisotropic norm less than a given $\gamma > 0$ if and only if there exist a $q \in \left(0, \min(\gamma^{-2}, \|F_{c\ell}\|_\infty^{-2})\right)$ and a symmetric matrix $X > 0$ such that

$$\begin{bmatrix} \mathcal{E}_1(X,K) & \mathcal{E}_2(X,K) \\ (1,2)^T & -\frac{1}{q}I + B_1^T X B_1 + D_{11}^T D_{11} \end{bmatrix} < 0 \tag{3.4}$$

where one denoted

$$\begin{aligned} \mathcal{E}_1(X,K) :=& - X + (A + B_2 K C_2)^T X (A + B_2 K C_2) + (C_1 + D_{12} K C_2)^T \\ & \cdot (C_1 + D_{12} K C_2) \end{aligned}$$

$$\mathcal{E}_2(X,K) := (A + B_2 K C_2)^T X B_1 + (C_1 + D_{12} K C_2)^T D_{11}$$

and

$$\frac{1}{q} - \gamma^2 < e^{-\frac{2a}{m}} \left(\det \left(\frac{1}{q} I - B_1^T B_1 - D_{11}^T D_{11} \right) \right)^{\frac{1}{m}} . \tag{3.5}$$

Based on Schur complements arguments, in [22] it follows that the inequality (3.4) is equivalent with the condition

$$\mathcal{Z} + \mathcal{P}^T K \mathcal{Q} + \mathcal{Q}^T K^T \mathcal{P} < 0, \tag{3.6}$$

where by definition

$$\mathcal{Z} := \begin{bmatrix} -X & 0 & A^T X & C_1^T \\ 0 & -\frac{1}{q}I & B_1^T X & D_{11}^T \\ XA & XB_1 & -X & 0 \\ C_1 & D_{11} & 0 & -I \end{bmatrix}, \mathcal{P}^T := \begin{bmatrix} 0 \\ 0 \\ XB_2 \\ D_{12} \end{bmatrix}, \ \mathcal{Q}^T := \begin{bmatrix} C_2^T \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{3.7}$$

Further, using the so-called Projection lemma (see e.g. [14]), one obtains that the inequality (3.6) is feasible with respect to K if and only if the following conditions are accomplished

$$W_{\mathcal{P}}^T Z W_{\mathcal{P}} < 0 \tag{3.8}$$

and

$$W_{\mathcal{Q}}^T Z W_{\mathcal{Q}} < 0, \tag{3.9}$$

where $W_{\mathcal{P}}$ and $W_{\mathcal{Q}}$ are any bases of the null spaces of $\mathcal{P}$ and $\mathcal{Q}$, respectively. Since a base of the null space of $\mathcal{P}$ is

$$W_{\mathcal{P}} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & X^{-1}W_1 \\ 0 & 0 & W_2 \end{bmatrix} \tag{3.10}$$

where $W := \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}$ is the orthogonal complement of $\begin{bmatrix} B_2^T & D_{12}^T \end{bmatrix}$. Similarly, a base of the null space of $\mathcal{Q}$ is

$$W_{\mathcal{Q}} = \begin{bmatrix} W_3 & 0 & 0 \\ W_4 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \tag{3.11}$$

where $V := \begin{bmatrix} W_3 \\ W_4 \end{bmatrix}$ is the orthogonal complement of $\begin{bmatrix} C_2 & 0 \end{bmatrix}$. In order to simplify the inequality of (3.8) we next express

$$W_P^T = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & W_1^T & W_2^T \end{bmatrix} \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & X^{-1} & 0 \\ 0 & 0 & 0 & I \end{bmatrix},$$

namely

$$W_P^T = \begin{bmatrix} I & 0 \\ 0 & W^T \end{bmatrix} \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & X^{-1} & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

with the above definition of W. Therefore, (3.8) is simply expressed as

$$\begin{bmatrix} I & 0 \\ 0 & W^T \end{bmatrix} \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{12}^T & \mathcal{M}_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & W \end{bmatrix} < 0 \tag{3.12}$$

where $\mathcal{M}$ is given by

$$\mathcal{M} := \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{12}^T & \mathcal{M}_{22} \end{bmatrix}$$

with

$$\begin{aligned} \mathcal{M}_{11} &= \begin{bmatrix} -X & 0 \\ 0 & -\frac{1}{q} I \end{bmatrix}, \mathcal{M}_{12} = \begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix}^T, \\ \mathcal{M}_{22} &= \begin{bmatrix} -X^{-1} & 0 \\ 0 & -I \end{bmatrix}. \end{aligned} \tag{3.13}$$

Then, from (3.12), using the Schur complement of $\mathcal{M}_{11}$ it follows that

$$W^T(\mathcal{M}_{22} - \mathcal{M}_{12}^T \mathcal{M}_{11}^{-1} \mathcal{M}_{12})W < 0\,. \tag{3.14}$$

Substituting the definition for $\mathcal{M}_{ij}, i, j = 1, 2$ and recalling the definition $\eta^2 = \frac{1}{q}$ we obtain the following convenient form of (3.14)

$$W^T \begin{bmatrix} -Y + AYA^T + B_1 B_1^T & AYC_1^T + B_1 D_{11}^T \\ C_1 Y A^T + D_{11} B_1^T & -\Phi_Y \end{bmatrix} W < 0 \tag{3.15}$$

where

$$\Phi_Y := \eta^2 I - C_1 Y C_1^T - D_{11} D_{11}^T$$

and where we have defined

$$\eta^{-2} Y = X^{-1}\,. \tag{3.16}$$

We next repeat the same lines to simplify (3.9) as well. To this end, we partition

$$W_Q = \begin{bmatrix} V & 0 \\ 0 & I \end{bmatrix}$$

and readily obtain using Schur complements, that (3.9) is equivalent to

$$V^T(\mathcal{N}_{11} - \mathcal{N}_{12}\mathcal{N}_{22}^{-1}\mathcal{N}_{12}^T)V < 0$$

where

$$\mathcal{N}_{11} = \begin{bmatrix} -X & 0 \\ 0 & -\frac{1}{q}I \end{bmatrix}, \mathcal{N}_{12} = \begin{bmatrix} XA & XB_1 \\ C_1 & D_{11} \end{bmatrix}^T, \mathcal{N}_{22} = \begin{bmatrix} -X & 0 \\ 0 & -I \end{bmatrix} \tag{3.17}$$

We, therefore, obtain the following form of (3.9)

$$V^T \begin{bmatrix} -X + A^TXA + C_1^TC_1 & A^TXB_1 + C_1^TD_{11} \\ B_1^TXA^{+}D_{11}^TC_1 & -\Phi_X \end{bmatrix} V < 0 \tag{3.18}$$

where $\Phi_X := \eta^2 I - B_1^TXB_1 - D_{11}^TD_{11}$. We summarise the above results in the following result.

Theorem 2. *The closed loop system $\mathcal{F}_{c\ell}(K)$ is stable and it has the a-anisotropic norm less than a given $\gamma > 0$ if there exist symmetric matrices $X > 0$ and $Y > 0$ and a scalar η satisfying the dual LMIs (3.15) and (3.18) together with the convex condition*

$$\eta^2 - det(\Phi_X)^{1/m}e^{-2a/m} < \gamma^2 \tag{3.19}$$

and the additional bilinear condition

$$XY = \eta^2 I\,. \tag{3.20}$$

If the conditions of the above theorem are satisfied then the static output gain may be obtained solving the linear matrix inequality (3.6) with respect to K.

However, the above requires a solution of a set of Bilinear Matrix Inequalities (BMI) due to the $XY = \eta^2 I$ equality. One way to tackle the BMI is to adopt a by first relaxing $XY = \eta^2 I$ by

$$\begin{bmatrix} X & \eta I \\ \eta I & Y \end{bmatrix} > 0 \tag{3.21}$$

Then if one minimises $Tr\{XY\}$, the bilinear constraint of (3.20) is satisfied. To this end, a sequential linearisation algorithm (see e.g. [24]) can be used. In the initialisation step, the convex problem comprised of the inequalities (2.4), (3.21) and (2.2) is solved for a given $\gamma > 0$, and $k = 0$, $X_k = 0$ and

$Y_k = 0$ are set. Next step where k is set to $k+1$ and X, Y are found so as to minimise

$$f_k := Tr\{X_k Y + X Y_k\}$$

subject to (3.15) and (3.18). Then $X_k = X$ and $Y_k = Y$ are set. This step is repeated until f_k is small enough. A related algorithm requiring also line search but with improved convergence properties has been suggested in [11] and will be applied in the calculations below of the numerical example of the next section. To this end we note that one could choose also $X = \tilde{Y}^{-1}$ rather than $\eta^{-2} Y = X^{-1}$. In such a case, the inequality (3.15) is replaced by

$$W^T \begin{bmatrix} -\tilde{Y} + A\tilde{Y}A^T + qB_1B_1^T & A\tilde{Y}C_1^T + qB_1D_{11}^T \\ C_1YA^T + qD_{11}B_1^T & -\Phi_{\tilde{Y}} \end{bmatrix} W < 0$$

where $\Phi_{\tilde{Y}}$ is defined to be

$$\Phi_{\tilde{Y}} := I - C_1\tilde{Y}C_1^T - qD_{11}D_{11}^T$$

and (3.21) becomes

$$\begin{bmatrix} X & I \\ I & \tilde{Y} \end{bmatrix} > 0 \tag{3.22}$$

Although this different choice of Y reveals the duality between the control and filtering type inequalities in a less obvious manner, it is more convenient to deal with. Note that to apply [11] one needs to define a new variable $q = \eta^{-2}$ where in addition to $X\tilde{Y} = I$ also the scalar valued bilinear equality constraint $\eta^2 q = 1$ has to be satisfied. To this end, one needs also to consider the relaxed version

$$\begin{bmatrix} \eta^2 & 1 \\ 1 & q \end{bmatrix} > 0 \tag{3.23}$$

so that the minimization steps involve now searching for X, Y, η^2, q are found so as to minimise

$$f_k := Tr\{X_k Y + X Y_k\} + \eta_k^2 q + \eta^2 q_k$$

subject to (3.15), (3.18), (3.22) and (3.23).

4 Application to Flight Control

We next consider the numerical example of [25] with the synthesis of pitch control loop for the F4E aircraft. Consider

$$\frac{\mathrm{d}}{\mathrm{dt}}\begin{bmatrix} \mathrm{N_z} \\ \mathrm{q} \\ \delta_\mathrm{e} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \mathrm{a}_{13} \\ a_{21} & a_{22} & \mathrm{a}_{23} \\ 0 & 0 & -30 \end{bmatrix}\begin{bmatrix} \mathrm{N_z} \\ \mathrm{q} \\ \delta_\mathrm{e} \end{bmatrix} + \begin{bmatrix} \mathrm{b}_1 \\ 0 \\ 30 \end{bmatrix}\mathrm{u} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\omega$$

$$\mathrm{z} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\mathrm{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\mathrm{u}$$

$$\mathrm{y} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\mathrm{x}$$

The state-vector consists, of the load-factor N_z, the pitch-rate q and elevon angle δ_e. The latter relates to the elevon command u via a first-order servo model of a bandwidth of $30rad/\sec$. The parameters $a_{i,j}, i = 1, 2; j = 1, 2, 3, b_1$ are given in [25] at the four operating points listed in the following table:

Operating point	1	2	3	4
Mach number	.5	.9	.85	1.5
Altitude (ft)	5000	35000	5000	35000
a_{11}	-.9896	-.6607	-1.702	-.5162
a_{12}	17.41	18.11	50.72	29.96
a_{13}	96.15	84.34	263.5	178.9
a_{21}	.2648	.08201	.2201	-.6896
a_{22}	-.8512	-.6587	-1.418	-1.225
a_{23}	-11.39	-10.81	-31.99	-30.38
b_1	-97.78	-272.2	-85.09	-175.6

Table 7: The parameters of the four operating points.

Discrete-time representation of the above systems have been obtained with the sampling period $T_s = 0.001\,\mathrm{sec}$. The static output controller $u = Ky$ with $K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$ will be designed for each of the four operating points, applying Theorem 2, applying the iterative procedure by [11] to deal with the bilinear equality. We will first take mean anisotropy level of a that tends to ∞, to obtain the H_∞ controller. Next we take $a = 0.2$. The upper bound on γ for H_∞ control was taken as envelope point dependent $\gamma_{points\infty} = \begin{bmatrix} 0.3 & 0.6 & 1 & 0.25 \end{bmatrix}$. The results of those designs are depicted in Fig. 1 and Fig. 2 where closed-loop singular values as well as the design bound and the actual norm (H_∞ and a-anisotropic) are shown. As could be expected, the maximum singular values are somewhat lower for the H_∞ design. The singular values of the closed-loop system in the H_∞ design, and for the a-anistropic design, are presented comparatively in Fig. 3. The gains of those two designs are given in Tables 8 and 9 respectively. A close scrutiny of the gains and the singular values reveals that the sub-optimal a-anisotropic design allow a fine trade-off between the maximum singular value and the gain values. Namely, considerably lower gain vector norms are obtained at the cost of a rather small sacrifice in the maximum singular value.

Operating point	1	2	3	4
Mach number	.5	.9	.85	1.5
Altitude (ft)	5000	35000	5000	35000
K_1	0.9740	1.8608	19.0757	1.7339
K_2	2.4035	0.4528	155.5975	3.5705

Table 8: The H_∞ gains of the four operating points.

Operating point	1	2	3	4
Mach number	.5	.9	.85	1.5
Altitude (ft)	5000	35000	5000	35000
K_1	1.0394	2.2678	0.7369	1.3616
K_2	0.9204	0.2091	5.9818	0.9821

Table 9: The sub-optimal a-anisotropic gains of the four operating points.

5 Conclusions

A synthesis scheme for static output feedback controllers has been derived, under the setup of a-anisotropic norm which is based on an intermediate topology between H_2 and H_∞. Given a required norm-bound, set of Linear Matrix Inequalities, along with a geometric-mean convex inequality, and an additional bilinear equality, characterise sub-optimal controllers as in[18], however revealing the duality in the style of [26] between the control and filtering type Linear Matrix Inequalities. The latter dual form is convenient to use with the iterative algorithm of [11] to alleviate the bilinear equality. The resulting control design procedure is useful e.g. in the aerospace industry for flight control loops, where controllers with classical "cook-book" structures in the style of Proportional-Integral-Derivative controllers. An example from the field of flight control is given comparing H_∞ design with

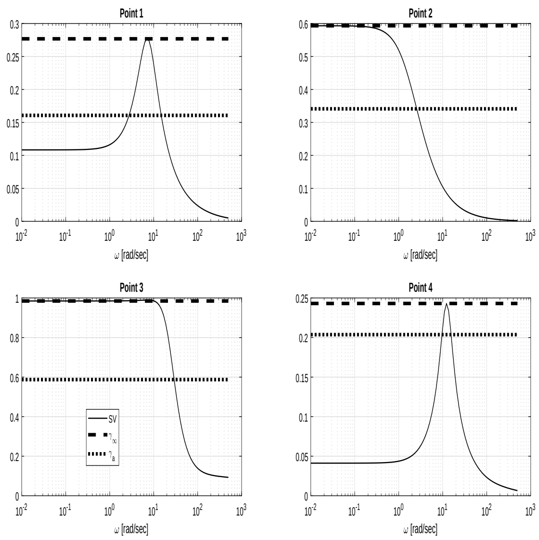

Figure 1: $T_{zw} - H_{\infty} design$

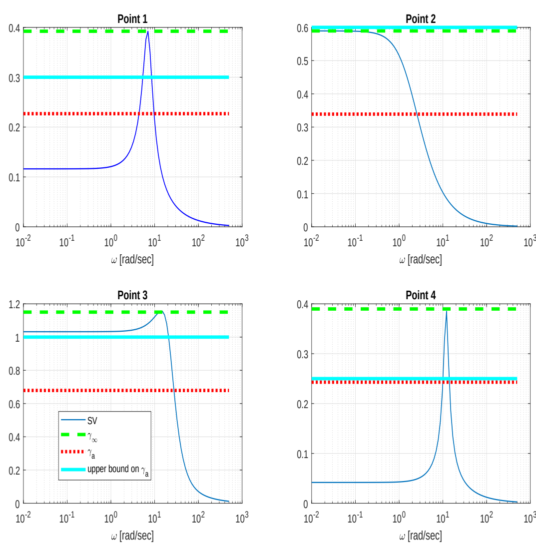

Figure 2: T_{zw} sub-optimal a-anisotropic design

a couple of a-anisotropic designs applying the above mentioned iterative algorithm of [11]. The results suggest that the latter offers a considerable reduction in the gains at the cost of small increase in the closed-loop max-

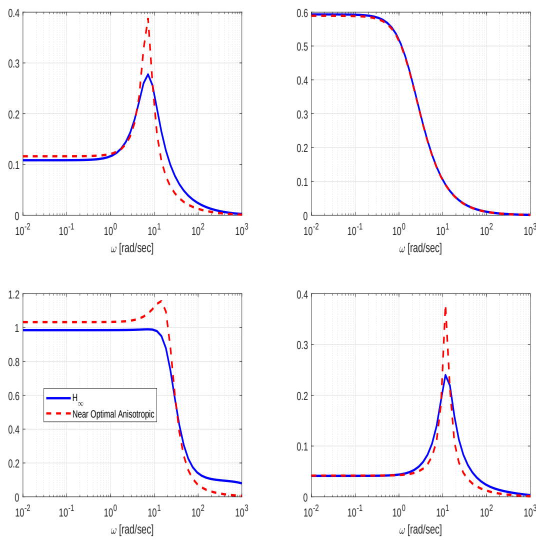

Figure 3: T_{zw} singular values comparison for H_∞ and sub-optimal a-anisotropic design

imum singular values. Those results encourage further experimenting with the a-anisotropic setup for static-output feedback control and motivates further research also along the lines of [17] and [18] and the references therein.

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