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THE MINIMIZATION OF THE MEAN SQUARE OF THE DEVIATION OF A RANDOM SIGNAL FROM A GIVEN TARGET*

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Abstract

In this paper we consider the problem of minimization of the mean square value of the deviation of a random signal $z(t_f)$ from a given target ζ. The random signal $z(t_f)$ represents the value at instant time t_f of an output of a controlled dynamical system described by an Itô differential equation. Both the case when the set of admissible controls consist of general nonanticipative stochastic processes and the case when only piecewise constant controls are available are analyzed. We show that in both cases the optimal controls are in affine state feedback forms. Explicit formulae of the gain matrices of the optimal controls are provided.

MSC: 93E20, 93C57, 49N10.

keywords: optimal control model, Itô differential equation, piecewise constant controls, sampled-data system.

1 Introduction

Tracking problems are often encountered in many applications and have received attention from the research community in the past few decades [2, 3, 8, 12, 18]. In the stochastic context this problem was studied in [10]

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as well as in [4] and [5]. Applications of tracking problems may be found in economic policy control [16], process control [1], etc.

Usually, a linear quadratic tracking problem require minimization of the L_2-norm of the deviation of a signal generated by a controlled linear system from a reference signal. For the synthesis of the optimal control, the a priori perfect knowledge of the reference signal is needed. Unfortunately, the perfect knowledge of the reference signal is not always possible, either when the time interval is too long or it requires a large amount of memory. That is why, in many tracking problems are used signals which are easy to be memorate. There are applications in which the signal reference reduces to a target to which the signal generated by the controlled system needs to get as close as possible. Different stochastic algorithms have used in target tracking problems [17, 19] and many others. The Gaussian approximation filters and Monte Carlo filters have applied for solving the target tracking problems [13]. The extended Kalman filter has widely used in nonlinear systems [9]. A target tracking algorithm based on assemble with the Markov chain the Monte Carlo simulation is proposed [11].

In the present work we consider the problem of the minimization of the mean square of the deviation of a random signal from a given target. The random signal is generated by a controlled system having the state space representation described by a system of Ito linear differential equations. We consider two classes of admissible controls:

a) controls described by stochastic processes with bounded energy adapted to the filtration generated by Wiener process appearing in the mathematical model of the controlled system,

b) controls described by piecewise constant stochastic processes.

In both cases we provide explicit formulae of the optimal control.

2 The problem setting

Let $z(t)$, $t_0 \leq t \leq t_f$ be a random signal of the form:

$$z(t) = C(t)x(t) \tag{1}$$

where $x(t)$ is the vector of the states of the controlled system:

$$\begin{aligned} dx(t) &= (A_0(t)x(t) + B_0(t)u(t))dt + (A_1(t)x(t) + B_1(t)u(t))dw(t) \\ x(t_0) &= x_0 \end{aligned} \tag{2}$$

where $u(t)$ is the vector of the control parameters. In (2) $\{w(t)\}_{t \geq 0}$ is a 1-dimensional standard Wiener process defined on a given probability space

$(\Omega, \mathcal{F}, \mathcal{P})$, that is, $w(0) = 0$ and for each $t > 0$, $\mathbb{E}[w(t)] = 0$, $\mathbb{E}[(w(t) - w(s))^2] = t - s$ if $0 \leq s \leq t$.

Regarding the coefficients from (1) and (2) we make the assumption:

$\mathbf{H_1}$) $t \to (A_k(t), B_k(t)) : [t_0, t_f] \to \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$, $k = 0, 1$, $t \to C(t) : [t_0, t_f] \to \mathbb{R}^{p\times n}$ are continuous matrix valued functions.

Throughout $\mathbb{E}[\cdot]$ denotes the mathematical expectation. Roughly speaking, our aim is to show how can be constructed a control $\tilde{u} : [t_0, t_f] \to \mathbb{R}^m$ which minimizes the value of

$$V(x_0, \zeta, u(\cdot)) \triangleq \mathbb{E}[|z(t_f) - \zeta|^2] \tag{3}$$

where $\zeta \in \mathbb{R}^p$ is a given reference (target).

For more rigorous setting of the problem of the minimization of the deviation from the desired target ζ described in (3) let us introduce the class of admissible controls. In this work we shall consider two classes of admissible controls. First, we consider the case when the class $\mathcal{U}_{ad}$ of the admissible controls consists of all measurable stochastic processes $u : [t_0, t_f] \times \Omega \to \mathbb{R}^m$, with the property that for each $t_0 \leq t \leq t_f$, $u(t)$ is $\mathcal{F}_t$-measurable and $\mathbb{E}[\int\limits_{t_0}^{t_f} |u(t)|^2 dt] < \infty$. Here, $\mathcal{F}_t \subset \mathcal{F}$ is the sigma algebra generated by the random variables $w(s)$, $0 \leq s \leq t$. Often, the controls included in $\mathcal{U}_{ad}$ will be called nonanticipative stochastic processes with finite energy.

Also, we shall consider the class of admissible controls that consists of all piecewise constant stochastic processes of the form

$$u(t) = u_k, t_k \leq t < t_{k+1}, 0 \leq k \leq N - 1 \tag{4}$$

where $t_0 < t_1 < ... < t_N = t_f$ is a partition of the interval $[t_0, t_f]$ and $u_k : \Omega \to \mathbb{R}^m$ are random vectors which are $\mathcal{F}_{t_k}$-measurable and $\mathbb{E}[|u_k|^2] < \infty$, $0 \leq k \leq N - 1$.

In the sequel we shall denote $\mathcal{U}_{pc}$ the class of piecewise constant controls of type (4). It is obvious that $\mathcal{U}_{pc} \subset \mathcal{U}_{ad}$.

In the next section we shall analyse the problem of minimization of (3) in the class of admissible controls $\mathcal{U}_{ad}$ and provide explicit formulae of the optimal controls. We shall emphasize a set of sufficient conditions which guarantee the existence of the optimal control.

The problem of minimization of (3) with respect to the class $\mathcal{U}_{pc}$ of piecewise constant controls will be transformed into a linear quadratic optimal control problem for a controlled system with finite jumps. In Section 4 we shall show how can be solved the problem of minimization of a cost of type (3) in the case of a controlled system with finite jumps. Further, we shall show how the obtained results can be used to derive the optimal control in the case of minimization of (3) over the trajectories of the controlled system (2) determined by the admissible controls $\mathcal{U}_{pc}$.

3 The case of nonanticipative admissible controls with finite energy

Let us consider the performance criterion

$$J(x_0,\zeta;u(\cdot)) = \mathbb{E}[|z_u(t_f)-\zeta|^2] + \mathbb{E}[\int_{t_0}^{t_f} u^T(t)R(t)u(t)dt] \tag{5}$$

where $z_u(t_f) = C(t_f)x_u(t_f)$ with $x_u(t)$, $t_0 \le t \le t_f$ being the solution of the initial value problem (2) corresponding to the input $u \in \mathcal{U}_{ad}$. In (5), $t \to R(t) : [t_0,t_f] \to \mathbb{R}^{m\times m}$ is a continuous matrix valued function such that $R(t) = R^T(t)$ for all $t \in [t_0,t_f]$. When $R(t) = 0$ for all $t \in [t_0,t_f]$, the performance criterion (5) reduces to (3). Our aim is to look for conditions that guarantee the existence of the controls $\tilde{u} \in \mathcal{U}_{ad}$ such that

$$J(x_0,\zeta;\tilde{u}(\cdot)) = \min_{u(\cdot)\in\mathcal{U}_{ad}} J(x_0,\zeta;u(\cdot)). \tag{6}$$

Let $t \to P(t) : [t_0,t_f] \to \mathbb{R}^{n\times n}$, $t \to \varphi(t) : [t_0,t_f] \to \mathbb{R}^n$, $t \to \mu(t) : [t_0,t_f] \to \mathbb{R}$ be continuously differentiable functions such that $P(t) = P^T(t)$ for all $t \in [t_0,t_f]$.

Applying the Itô formula in the case of the function

$$\mathbb{V}(t,x) = x^T P(t)x + 2x^T\varphi(t) + \mu(t), \ \ (t,x) \in [t_0,t_f]\times\mathbb{R}^n$$

and to the stochastic process $x_u(t)$ generated by the system (2) we obtain:

$$\mathbb{E}[\int_{t_0}^{t_f} u^T(t)R(t)u(t)dt] + \mathbb{E}[\mathbb{V}(t_f,x_u(t_f))] - \mathbb{V}(t_0,x_0) = \tag{7}$$

$$= \mathbb{E}[\int_{t_0}^{t_f} \begin{pmatrix} x_u(t) \\ 1 \\ u(t) \end{pmatrix}^T \begin{pmatrix} \mathbb{W}_{11}(t) & \mathbb{W}_{12}(t) & \mathbb{W}_{13}(t) \\ \mathbb{W}_{12}^T(t) & \mathbb{W}_{22}(t) & \mathbb{W}_{23}(t) \\ \mathbb{W}_{13}^T(t) & \mathbb{W}_{23}^T(t) & \mathbb{W}_{33}(t) \end{pmatrix} \begin{pmatrix} x_u(t) \\ 1 \\ u(t) \end{pmatrix} dt$$

where we denoted:

$$\mathbb{W}_{11}(t) = \dot{P}(t) + A_0^T(t)P(t) + P(t)A_0(t) + A_1^T(t)P(t)A_1(t) \tag{8a}$$
$$\mathbb{W}_{12}(t) = \dot{\varphi}(t) + A_0^T(t)\varphi(t) \tag{8b}$$
$$\mathbb{W}_{13}(t) = P(t)B_0(t) + A_1^T(t)P(t)B_1(t) \tag{8c}$$
$$\mathbb{W}_{22}(t) = \dot{\mu}(t) \tag{8d}$$
$$\mathbb{W}_{23}(t) = \varphi^T(t)B_0(t) \tag{8e}$$
$$\mathbb{W}_{33}(t) = R(t) + B_1^T(t)P(t)B_1(t). \tag{8f}$$

Let us assume that $P(\cdot)$ and $\varphi(\cdot)$ are such that the following equalities hold for any $t \in [t_0, t_f]$:

$$\begin{aligned}
&(R(t) + B_1^T(t)P(t)B_1(t))(R(t) + B_1^T(t)P(t)B_1(t))^\dagger \times \\
&\times (B_0^T(t)P(t) + B_1^T(t)P(t)A_1(t)) \\
&= \; B_0^T(t)P(t) + B_1^T(t)P(t)A_1(t)
\end{aligned} \tag{9a}$$
$$\begin{aligned}
&(R(t) + B_1^T(t)P(t)B_1(t))(R(t) + B_1^T(t)P(t)B_1(t))^\dagger B_0^T(t)\varphi(t) \\
&= \; B_0^T(t)\varphi(t).
\end{aligned} \tag{9b}$$

Throughout the paper the superscript $\dagger$ denotes the pseudoinverse of a matrix. For precise definitions and useful properties of the pseudoinverse of a matrix refer to [15]. It is worth mentioning that if $\mathbb{W}_{33}(t)$ is invertible for any $t \in [t_0, t_f]$ then the identities (9) are automatically satisfied. They become relevant in the case when there exist $t \in [t_0, t_f]$ such that $\mathbb{W}_{33}(t)$ is not invertible. By direct calculation, one obtains that if the identities (9) are satisfied, then we have:

$$\begin{pmatrix} \mathbb{W}_{11}(t) & \mathbb{W}_{12}(t) & \mathbb{W}_{13}(t) \\ \mathbb{W}_{12}^T(t) & \mathbb{W}_{22}(t) & \mathbb{W}_{23}(t) \\ \mathbb{W}_{13}^T(t) & \mathbb{W}_{23}^T(t) & \mathbb{W}_{33}(t) \end{pmatrix} = \begin{pmatrix} I_n & 0_{n1} & \mathbb{W}_{13}(t)\mathbb{W}_{33}^\dagger(t) \\ 0_{1n} & 1 & \mathbb{W}_{23}(t)\mathbb{W}_{33}^\dagger(t) \\ 0_{mn} & 0_{m1} & I_m \end{pmatrix} \cdot$$
$$\begin{pmatrix} \tilde{\mathbb{W}}_{11}(t) & \tilde{\mathbb{W}}_{12}(t) & 0_{nm} \\ \tilde{\mathbb{W}}_{12}^T(t) & \tilde{\mathbb{W}}_{22}(t) & 0_{1m} \\ 0_{mn} & 0_{m1} & \mathbb{W}_{33}(t) \end{pmatrix} \cdot \begin{pmatrix} I_n & 0_{n1} & 0_{nm} \\ 0_{1n} & 1 & 0_{1m} \\ \mathbb{W}_{33}^\dagger(t)\mathbb{W}_{13}^T(t) & \mathbb{W}_{33}^\dagger(t)\mathbb{W}_{32}^T(t) & I_m \end{pmatrix} \tag{10}$$

where 0_{qr} is the zero matrix of size $q \times r$,

$$\begin{aligned}
\tilde{\mathbb{W}}_{11}(t) &= \mathbb{W}_{11}(t) - \mathbb{W}_{13}(t)\mathbb{W}_{33}^{\dagger}(t)\mathbb{W}_{13}^{T}(t) \\
&= \dot{P}(t) + A_0^T(t)P(t) + P(t)A_0(t) + A_1^T(t)P(t)A_1(t) \\
&- (P(t)B_0(t)+A_1^T(t)P(t)B_1(t))(R(t)+B_1^T(t)P(t)B_1(t))^{\dagger} \qquad \text{(11a)} \\
&\times (B_0^T(t)P(t)+B_1^T(t)P(t)A_1(t)) \\
\tilde{\mathbb{W}}_{12}(t) &= \mathbb{W}_{12}(t) - \mathbb{W}_{13}(t)\mathbb{W}_{33}^{\dagger}(t)\mathbb{W}_{23}^{T}(t) \\
&= \dot{\varphi}(t) + (A_0(t) + B_0(t)\tilde{F}(t))^T\varphi(t) \qquad \text{(11b)} \\
\tilde{\mathbb{W}}_{22}(t) &= \mathbb{W}_{22}(t) - \mathbb{W}_{23}(t)\mathbb{W}_{33}^{\dagger}(t)\mathbb{W}_{23}^{T}(t) \\
&= \dot{\mu}(t) - \varphi^T(t)B_0(t)(R(t)+B_1^T(t)P(t)B_1(t))^{\dagger}B_0^T(t)\varphi(t) \qquad \text{(11c)}
\end{aligned}$$

where we have denoted

$$\tilde{F}(t) = -(R(t) + B_1^T(t)P(t)B_1(t))^{\dagger}(B_0^T(t)P(t) + B_1^T(t)P(t)A_1(t)). \qquad (12)$$

Plugging (10) in (7) we obtain

$$\begin{aligned}
&\mathbb{E}[\int_{t_0}^{t_f} \mu^T(t)R(t)u(t)dt] + \mathbb{E}[\mathbb{V}(t_f, x_u(t_f))] - \mathbb{V}(t_0, x_0) = \\
&\mathbb{E}[\int_{t_0}^{t_f} \begin{pmatrix} x_u(t) \\ 1 \end{pmatrix}^T \begin{pmatrix} \tilde{\mathbb{W}}_{11}(t) & \tilde{\mathbb{W}}_{12}(t) \\ \tilde{\mathbb{W}}_{12}^T(t) & \tilde{\mathbb{W}}_{22}(t) \end{pmatrix} \begin{pmatrix} x_u(t) \\ 1 \end{pmatrix} dt] + \qquad (13) \\
&\mathbb{E}[\int_{t_0}^{t_f} (u(t) - \tilde{u}(t))^T(R(t) + B_1^T(t)P(t)B_1(t))(u(t) - \tilde{u}(t))dt]
\end{aligned}$$

where we denote

$$\tilde{u}(t) = \tilde{F}(t)x_u(t) - (R(t) + B_1^T(t)P(t)B_1(t))^{\dagger}B_0^T(t)\varphi(t). \qquad (14)$$

Further, we assume that $P(\cdot)$, $\varphi(\cdot)$, $\mu(\cdot)$ are the solutions of the terminal value problems (TVPs):

$$\begin{aligned}
&\dot{P}(t) + A_0^T(t)P(t) + P(t)A_0(t) + A_1^T(t)P(t)A_1(t) \\
&\quad -(P(t)B_0(t) + A_1^T(t)P(t)B_1(t))(R(t) + B_1^T(t)P(t)B_1(t))^{\dagger} \times \\
&\quad (B_0^T(t)P(t) + B_1^T(t)P(t)A_1(t)) = 0 \qquad (15) \\
&P(t_f) = C^T(t_f)C(t_f)
\end{aligned}$$

$$\begin{aligned}&\dot{\varphi}(t) + (A_0(t) + B_0(t)\tilde{F}(t))^T \varphi(t) = 0 \\ &\varphi(t_f) = -C^T(t_f)\zeta\end{aligned} \tag{16}$$

$$\begin{aligned}&\dot{\mu}(t) = \varphi^T(t) B_0(t)(R(t) + B_1^T(t)P(t)B_1(t))^\dagger B_0^T(t)\varphi(t) \\ &\mu(t_f) = \zeta^T\zeta.\end{aligned} \tag{17}$$

If the solutions of the TVPs (15) and (16) are well defined on the whole interval $[t_0, t_f]$ and satisfy the conditions (9), then (13) becomes:

$$\begin{aligned}J(x_0, \zeta; u(\cdot)) &= \mathbb{E}[\mathbb{V}(t_f, x_u(t_f))] + \mathbb{E}[\int_{t_0}^{t_f} u^T(t)R(t)u(t)dt] \\ &= x_0^T P(t_0)x_0 + 2x_0^T\varphi(t_0) + \mu(t_0) + \\ &+ \mathbb{E}[\int_{t_0}^{t_f} (u(t) - \tilde{u}(t))^T (R(t) + B_1^T(t)P(t)B_1(t))(u(t) - \tilde{u}(t))dt].\end{aligned} \tag{18}$$

Lemma 3.1 *Assume that the solution $P(\cdot)$ of TVP (15) is defined on the whole interval $[t_0, t_f]$ and satisfies (9a). Assume also that $R(t) \geq 0$ for all $t \in [t_0, t_f]$. Under these conditions $P(t) \geq 0$ for all $t_0 \leq t \leq t_f$.*

Proof. By direct calculations one obtains that (15) may be rewritten in the form of a Lyapunov type differential equation as

$$\begin{aligned}&\dot{P}(t) + (A_0(t) + B_0(t)\tilde{F}(t))^T P(t) + P(t)(A_0(t) + B_0(t)\tilde{F}(t)) + \\ &\quad +(A_1(t) + B_1(t)\tilde{F}(t))^T P(t)(A_1(t) + B_1(t)\tilde{F}(t)) + M(t) = 0\end{aligned} \tag{19}$$

where $\tilde{F}(t)$ is defined in (12) and $M(t) = \tilde{F}^T(t)R(t)\tilde{F}(t) \geq 0$.

The solution of the differential equation (19) has the representation

$$P(t) = \mathbb{T}^*(t_f, t)[C^T(t_f)C(t_f)] + \int_t^{t_f} \mathbb{T}^*(s, t)[M(s)]ds, \quad t_0 \leq t \leq t_f \tag{20}$$

$\mathbb{T}^*(s, t)$ being the adjoint operator of the linear evolution operator defined by the differential equation:

$$\begin{aligned}\dot{S}(t) &= (A_0(t) + B_0(t)\tilde{F}(t))S(t) + S(t)(A_0(t) + B_0(t)\tilde{F}(t))^T + \\ &+(A_1(t) + B_1(t)\tilde{F}(t))S(t)(A_1(t) + B_1(t)\tilde{F}(t))^T.\end{aligned}$$

Based on Theorem 2.6.1 from [5] we deduce that $\mathbb{T}^*(s,t)[X] \geq 0$ for all $t_f \geq s \geq t \geq t_0$ if $X \geq 0$. This allows us to conclude via (20) that $P(t) \geq 0$ for all $t \in [t_0, t_f]$.
This ends the proof. □

Remark 3.1 *Under the assumptions of Lemma 3.1 it follows that:*
(i) $R(t) + B_1^T(t)P(t)B_1(t) \geq 0$, $t \in [t_0, t_f]$.
(ii) $R(t) + B_1^T(t)P(t)B_1(t) > 0$, *if* $R(t) > 0$, $t \in [t_0, t_f]$.

Now we are in a position to prove the main result of this section:

Theorem 3.2 *Assume: a) the solution $P(\cdot)$ of the TVP (15) is well defined on the whole interval $[t_0, t_f]$ and satisfies the conditions (9);*
b) $R(t) \geq 0$, $t \in [t_0, t_f]$.
Under these conditions, the control of type (14) is optimal with respect to the optimal control problem (6). The minimal value of the performance criteria (5) is

$$J(x_0, \zeta; \tilde{u}(\cdot)) = x_0^T P(t_0) x_0 + 2x_0^T \varphi(t_0) + \mu(t_0),$$

$\varphi(\cdot), \mu(\cdot)$ being solutions of TVPs (16) and (17), respectively. Furthermore, if $R(t) > 0$, $t \in t_0, t_f]$ then $\tilde{u}(\cdot)$ defined in (14) is the unique optimal control of the optimization problem (6).

Proof. Substituting (14) in (2) one obtains the closed loop system

$$\begin{aligned} dx(t) &= [(A_0(t) + B_0(t)\tilde{F}(t))x(t) + B_0(t)\Psi(t)]dt + \\ &\quad [(A_1(t) + B_1(t)\tilde{F}(t))x(t) + B_1(t)\Psi(t)]dw(t) \\ x(t_0) &= x_0 \end{aligned} \tag{21}$$

where we have denoted $\Psi(t) \triangleq -(R(t) + B_1^T(t)P(t)B_1(t))^\dagger B_0^T(t)\varphi(t)$.
From the uniqueness of the solution of IVP we obtain that $x_u(t)$ involved in (14) coincides with the solution $\tilde{x}(t)$ of the IVP (21). Hence, the control (14) lies in $\mathcal{U}_{ad}$. Further, Remark 3.1 (i) allows us to obtain from (18) that

$$J(x_0, \zeta; u(\cdot)) \geq x_0^T P(t_0) x_0 + 2x_0^T \varphi(t_0) + \mu(t_0) = J(x_0, \zeta; \tilde{u}(\cdot))$$

for all $u(\cdot) \in \mathcal{U}_{ad}$, which confirms the optimality property of the control of type (14). In the case when $R(t) > 0$, the uniqueness of the optimal control follows combining Remark 3.1 (ii) and the identity (18). Thus the proof ends. □

Remark 3.2 *Since the performance criteria (5) reduces to (3) if $R(t) = 0$, $t \in [t_0, t_f]$, it follows that the results stated in Theorem 4.3 specialized to the case $R(t) \equiv 0$ provides the optimal control $\tilde{u}(\cdot)$ which minimizes the square mean of the deviation of the signal $z(t_f)$ from the desired reference (target) ζ. In this case (14) becomes*

$$\tilde{u}(t) = \tilde{F}(t)\tilde{x}(t) - (B_1^T(t)\tilde{P}(t)B_1(t))^\dagger B_0^T(t)\varphi(t) \tag{22a}$$
$$\tilde{F}(t) = -(B_1^T(t)\tilde{P}(t)B_1(t))^\dagger (B_0^T(t)\tilde{P}(t) + B_1^T(t)\tilde{P}(t)A_1(t)). \tag{22b}$$

In order to have $\tilde{u}(t) \neq 0$ we need to have $B_1(t) \not\equiv 0$, for all t. That is, the considered controlled system contains control multiplicative white noise perturbations.
We shall see in the next section that this require does not appear in the case of piecewise constant admissible controls.

4 The case of piecewise constant admissible controls

Substituting a control of type (4) in (2) and in (5) we obtain

$$\begin{aligned} dx(t) &= (A_0(t)x(t) + B_0(t)u_k)dt + (A_1(t)x(t) + B_1(t)u_k)dw(t), \quad (23)\\ &\quad t_k \leq t < t_{k+1}, 0 \leq k \leq N-1 \\ x(t_0) &= x_0 \end{aligned}$$

and

$$J(x_0, \zeta, u(\cdot)) = \mathbb{E}[|z_u(t_f) - \zeta|^2] + \sum_{k=0}^{N-1} \mathbb{E}[u_k^T \int_{t_k}^{t_{k+1}} R(t)dt\, u_k]. \tag{24}$$

Setting $\mathbf{x}(t) = \begin{pmatrix} x^T(t) & u^T(t) \end{pmatrix}^T$ we may transform the system (23) in a controlled linear system with finite jumps of the form:

$$d\mathbf{x}(t) = \mathcal{A}_0(t)\mathbf{x}(t)dt + \mathcal{A}_1(t)\mathbf{x}(t)dw(t), t_k \leq t \leq t_{k+1} \tag{25a}$$
$$\mathbf{x}(t_k^+) = \mathcal{A}_d\mathbf{x}(t_k) + \mathcal{B}_d u_k, 0 \leq k \leq N-1 \tag{25b}$$

and $\mathbf{x}(t_0) = \begin{pmatrix} x_0^T & 0^T \end{pmatrix}^T$ where:

$$\begin{aligned}
\mathcal{A}_k(t) &= \begin{pmatrix} A_k(t) & B_k(t) \\ 0_{mn} & 0_m \end{pmatrix} \in \mathbb{R}^{(n+m)\times(n+m)}, k = 0, 1 \\
\mathcal{A}_d &= \begin{pmatrix} I_n & 0_{nm} \\ 0_{mn} & 0_m \end{pmatrix} \in \mathbb{R}^{(n+m)\times(n+m)}, \\
\mathcal{B}_d &= \begin{pmatrix} 0_{nm} \\ I_m \end{pmatrix} \in \mathbb{R}^{(n+m)\times m}.
\end{aligned} \tag{26}$$

The functional (25) becomes

$$\mathcal{J}(\mathbf{x}, \zeta, \mathbf{u}) = \mathbb{E}[|z_{\mathbf{u}}(t_f) - \zeta|^2] + \sum_{k=0}^{N-1} \mathbb{E}[u_k^T \mathcal{R}_k u_k] \tag{27}$$

where $\mathcal{R}_k = \int\limits_{t_k}^{t_{k+1}} R(t)dt$ and $z_{\mathbf{u}}(t) = \mathcal{C}(t)\mathbf{x}_{\mathbf{u}}(t)$ with $\mathcal{C}(t) = \begin{pmatrix} C(t) & 0 \end{pmatrix} \in \mathbb{R}^{p\times(n+m)}$ and $\mathbf{x}_{\mathbf{u}}(t)$ is the solution of the initial value problem IVP (26) corresponding to the input $\mathbf{u} = (u_0, u_1, ..., u_{N-1})$.

In order to state the optimization problem associated to the controlled system (26) and the performance criterion (28) we describe the class of admissible controls $\mathcal{U}^d$.

In this section $\mathcal{U}^d$ consists of all finite sequences of random vectors $\mathbf{u} = (u_0, u_1, ..., u_{N-1})$ where $u_k : \Omega \to \mathbb{R}^m$ are $\mathcal{F}_{t_k}$-measurable and $\mathbb{E}[|u_k|^2] < \infty$. Employing (4) one obtains that there exists a one to one correspondence between the class of piecewise admissible controls $\mathcal{U}_{pc}$ and the class $\mathcal{U}^d$ defined before.

Applying Theorem 5.2.1 from [14] on each interval $[t_k, t_{k+1}]$ we obtain:

Proposition 4.1 *For each $\mathbf{x}_0 \in \mathbb{R}^{n+m}$ and any $\mathbf{u} \in \mathcal{U}^d$ the differential equation with finite jumps (25) has a unique solution $\mathbf{x}_{\mathbf{u}}(t) = \mathbf{x}(t, t_0, \mathbf{x}_0, \mathbf{u})$ with the properties:*

(i) $\mathbf{x}_{\mathbf{u}}(\cdot)$ *is left continuous a.s. in each t is $(t_0, t_f]$;*

(ii) *for each $t \in [t_0, t_f], mathbf{x}_{\mathbf{u}}(t)$ is $\mathcal{F}_t$-measurable, and $\mathbb{E}[|\mathbf{x}_{\mathbf{u}}(t)|^2] < \infty$;*

(iii) $\mathbf{x}_{\mathbf{u}}(t_0) = \mathbf{x}_0$.

The optimal control problem we want to solve in this section ask for finding a control $\tilde{\mathbf{u}} \in \mathcal{U}^d$ that satisfies

$$\mathcal{J}(\mathbf{x}_0, \zeta, \tilde{\mathbf{u}}) = \min_{\mathbf{u}\in\mathcal{U}^d} \mathcal{J}(\mathbf{x}_0, \zeta, \mathbf{u}). \tag{28}$$

Let us consider the differential equation with finite jumps on the space $\mathcal{S}_{n+m}$,

$$\dot{X}(t)+\mathcal{A}_0^T(t)X(t)+X(t)\mathcal{A}_0(t)+\mathcal{A}_1^T(t)X(t)\mathcal{A}_1(t)=0, t_k\leq t<t_{k+1} \tag{29a}$$

$$X(t_k^-)=\mathcal{A}_d^T X(t_k)\mathcal{A}_d-\mathcal{A}_d^T X(t_k)\mathcal{B}_d(\mathcal{R}_k+\mathcal{B}_d^T X(t_k)\mathcal{B}_d)^\dagger\mathcal{B}_d^T X(t_k)\mathcal{A}_d, \quad k=0,1,...,N-1 \tag{29b}$$

$$X(t_f^-)=\mathcal{C}^T(t_f)\mathcal{C}(t_f). \tag{29c}$$

Here and after, $\mathcal{S}_q\subset\mathbb{R}^{q\times q}$ denotes the linear space of symmetric matrices of size $q\times q$.

Specializing the result proved in Theorem 6 from [7] to the case of terminal value problem TVP (29), we obtain:

Corollary 4.2 *If $R(t)\geq 0$, for all $t\in[t_0,t_f]$ then the TVP (29) has a unique solution $\tilde{X}:[t_0,t_f]\to\mathcal{S}_{n+m}$ which is right continuous and positive semidefinite in any $t\in[t_0,t_f]$. Furthermore, $\tilde{X}(\cdot)$ satisfies the equalities:*

$$(\mathcal{R}_k+\mathcal{B}_d^T\tilde{X}(t_k)\mathcal{B}_d)(\mathcal{R}_k+\mathcal{B}_d^T\tilde{X}(t_k)\mathcal{B}_d)^\dagger\mathcal{B}_d^T\tilde{X}(t_k)\mathcal{A}_d=\mathcal{B}_d^T\tilde{X}(t_k)\mathcal{A}_d, \tag{30}$$

$k=0,1,...,N-1.$

We set

$$\tilde{F}_d(k)\triangleq-(\mathcal{R}_k+\mathcal{B}_d^T\tilde{X}(t_k)\mathcal{B}_d)^\dagger\mathcal{B}_d^T\tilde{X}(t_k)\mathcal{A}_d. \tag{31}$$

Let $\tilde{\xi}(\cdot)$ be the solution of the following problem with given terminal values

$$\dot{\xi}(t)+\mathcal{A}_0^T(t)\xi(t)=0, \quad t_k\leq t<t_{k+1} \tag{32a}$$

$$\xi(t_k^-)=(\mathcal{A}_d+\mathcal{B}_d\tilde{F}_d(k))^T\xi(t_k), \quad k=0,1,...,N-1 \tag{32b}$$

$$\tilde{\xi}(t_N^-)=-\mathcal{C}^T(t_f)\zeta \tag{32c}$$

Since (32) is a linear differential equation with finite jumps one obtains that $\tilde{\xi}(t)$ is well defined and right continuous for any $t\in[t_0,t_f]$.

The next result provides a solution of the optimal control problem (28).

Theorem 4.3 *Assume: a) $R(t)\geq 0$ for all $t\in[t_0,t_f]$;*

b) the solutions $\tilde{X}(\cdot)$ and $\tilde{\xi}(\cdot)$ of the TVPs (29) and (32) respectively, satisfy the condition:

$$(\mathcal{R}_k+\mathcal{B}_d^T\tilde{X}(t_k)\mathcal{B}_d)(\mathcal{R}_k+\mathcal{B}_d^T\tilde{X}(t_k)\mathcal{B}_d)^\dagger\mathcal{B}_d^T\tilde{\xi}(t_k)=\mathcal{B}_d^T\tilde{\xi}(t_k), \quad k=0,1,...,N-1. \tag{33}$$

We consider the control $\tilde{\mathbf{u}} = (\tilde{u}_0, \tilde{u}_1, ..., \tilde{u}_{N-1})$ *with*

$$\tilde{u}_k = \tilde{F}_d(k)\tilde{\mathbf{x}}(t_k) - (\mathcal{R}_k + \mathcal{B}_d^T\tilde{X}(t_k)\mathcal{B}_d)^\dagger \mathcal{B}_d^T\tilde{\xi}(t_k) \tag{34}$$

where $\tilde{\mathbf{x}}(t_k)$ *are the values at the time instance* t_k *of the solution of the closed-loop system obtained when (34) is substituted in (25).*

Under the considered assumptions the control $\tilde{\mathbf{u}}$ *lies in* $\mathcal{U}^d$ *and achieves the minimum value of the cost function (27). The minimal value of the cost (27) is:*

$$\begin{aligned}\mathcal{J}(\mathbf{x}_0, \zeta, \tilde{\mathbf{u}}) = \mathbf{x}_0^T\tilde{X}(t_0^-)\mathbf{x}_0 + 2\mathbf{x}_0^T\tilde{\xi}(t_0^-) + \zeta^T\zeta - \\ \sum_{k=0}^{N-1}\tilde{\xi}^T(t_k)\mathcal{B}_d(\mathcal{R}_k + \mathcal{B}_d^T\tilde{X}(t_k)\mathcal{B}_d)^\dagger \mathcal{B}_d^T\tilde{\xi}(t_k).\end{aligned} \tag{35}$$

Moreover if $R(t) > 0$ *for all* $t \in [t_0, t_f]$*, then* $\tilde{\mathbf{u}}$ *described in (34) is the unique optimal control of the problem (28).*

Proof. Applying the Itô formula in the case of the function $\mathbb{V}_d(t, \mathbf{x}) \triangleq \mathbf{x}^T\tilde{X}(t)\mathbf{x} + 2\mathbf{x}\tilde{\xi}(t)$ and to the stochastic process $\mathbf{x}(\cdot)$ generated by (25) on intervals of the form $[t_k', t_k''] \subset [t_k, t_{k+1}]$ and letting $t_k' \to t_k$ and $t_k'' \to t_{k+1}$ we obtain via (29a) and (32a) that

$$\begin{aligned}\mathbb{E}[\mathbf{x}^T(t_{k+1})\tilde{X}(t_{k+1}^-)\mathbf{x}(t_{k+1}) + 2\mathbf{x}^T(t_{k+1})\tilde{\xi}(t_{k+1}^-)] = \\ = \mathbb{E}[\mathbf{x}^T(t_k^+)\tilde{X}(t_k)\mathbf{x}(t_k^+) + 2\mathbf{x}^T(t_k^+)\tilde{\xi}(t_k)].\end{aligned} \tag{36}$$

To obtain this equality we also used the fact that $t \to \tilde{X}(t)$, $t \to \tilde{\xi}(t)$ are right continuous and $t \to \mathbf{x}(t)$ are left continuous almost surely. Employing (25b) we rewrite (37) in the form:

$$\begin{aligned}\mathbb{E}[\mathbb{V}_d(t_{k+1}^-, \mathbf{x}(t_{k+1}))] = \mathbb{E}[\mathbf{x}^T(t_k)\mathcal{A}_d^T\tilde{X}(t_k)\mathcal{A}_d\mathbf{x}(t_k)] + 2\mathbb{E}[\mathbf{x}^T(t_k)\tilde{\xi}(t_k)] \\ +2\mathbb{E}[u_k^T\mathcal{B}_d^T\tilde{X}(t_k)\mathcal{A}_d\mathbf{x}(t_k) + u_k^T\mathcal{B}_d^T\tilde{\xi}(t_k)] + \mathbb{E}[u_k^T\mathcal{B}_d^T\tilde{X}(t_k)\mathcal{B}_d u_k], \\ k = 0, 1, ..., N-1.\end{aligned}$$

Further (30) and (33) allow us to use the square completion technique to obtain via (29b), (31) and (32b)

$$\begin{aligned}\mathbb{E}[u_k^T\mathcal{R}_k u_k] + \mathbb{E}[\mathbb{V}_d(t_{k+1}^-, \mathbf{x}(t_{k+1}))] - \mathbb{E}[\mathbb{V}_d(t_k^-, \mathbf{x}(t_k))] = \\ = \mathbb{E}[(u_k - \tilde{F}_d(k)\mathbf{x}(t_k) + (\mathcal{R}_k + \mathcal{B}_d^T\tilde{X}(t_k)\mathcal{B}_d)^\dagger\mathcal{B}_d^T\tilde{\xi}(t_k))^T(\mathcal{R}_k + \\ +\mathcal{B}_d^T\tilde{X}(t_k)\mathcal{B}_d)(u_k - \tilde{F}_d(k)\mathbf{x}(t_k) + (\mathcal{R}_k + \mathcal{B}_d^T\tilde{X}(t_k)\mathcal{B}_d)^\dagger\mathcal{B}_d^T\tilde{\xi}(t_k))] - \\ -\tilde{\xi}^T(t_k)\mathcal{B}_d(\mathcal{R}_k + \mathcal{B}_d^T\tilde{X}(t_k)\mathcal{B}_d)^\dagger\mathcal{B}_d^T\tilde{\xi}(t_k).\end{aligned} \tag{37}$$

Summing from $k = 0$ up to $k = N - 1$ in (37) and taking into account (29c) and (32c) we get:

$$\mathcal{J}(\mathbf{x}_0, \zeta, \mathbf{u}_0) = \mathbf{x}_0^T \tilde{X}(t_0^-)\mathbf{x}_0 + 2\mathbf{x}_0^T \tilde{\xi}(t_0^-) + \mu_d + \\ + \sum_{k=0}^{N-1} \mathbb{E}[(u_k - \hat{u}_k)^T (\mathcal{R}_k + \mathcal{B}_d^T \tilde{X}(t_k)\mathcal{B}_d)(u_k - \hat{u}_k)] \tag{38}$$

for all $\mathbf{u} = (u_0, u_1, ..., u_{N-1}) \in \mathcal{U}^d$, where we denoted

$$\hat{u}_k = \tilde{F}_d(k)\mathbf{x}_\mathbf{u}(t_k) - (\mathcal{R}_k + \mathcal{B}_d^T \tilde{X}(t_k)\mathcal{B}_d)^\dagger \mathcal{B}_d^T \tilde{\xi}(t_k) \tag{39}$$

and

$$\mu_d = \zeta^T \zeta - \sum_{k=0}^{N-1} \tilde{\xi}^T(t_k)\mathcal{B}_d(\mathcal{R}_k + \mathcal{B}_d^T \tilde{X}(t_k)\mathcal{B}_d)^\dagger + \mathcal{B}_d^T \tilde{\xi}(t_k). \tag{40}$$

From the assumption (a) and Corollary 4.2 we may infer that

$$\mathcal{R}_k + \mathcal{B}_d^T \tilde{X}(t_k)\mathcal{B}_d \geq 0, \ k = 0, 1, ..., N-1. \tag{41}$$

On the other hand, Proposition 4.1 allows us to conclude that the control $\hat{\mathbf{u}} = (\hat{u}_0, \hat{u}_1, ..., \hat{u}_{N-1})$ with $\hat{u}_k$ defined in (39) lies in $\mathcal{U}_d$. Combining (38) and (41) we deduce that

$$\mathcal{J}(\mathbf{x}_0, \zeta, \mathbf{u}) \geq \mathbf{x}_0^T \tilde{X}(t_0^-)\mathbf{x}_0 + 2\mathbf{x}_0^T \tilde{\xi}(t_0^-) + \mu_d = \mathcal{J}(\mathbf{x}_0, \zeta, \hat{\mathbf{u}}) \tag{42}$$

for all $\mathbf{u} \in \mathcal{U}^d$. The uniqueness of the solution of the initial value problem (25) allows us to infer that $\mathbf{x}_\mathbf{u}(t) = \tilde{\mathbf{x}}(t)$, $t \in [t_0, t_f]$. This means that $\tilde{u}_k$ defined in (34) coincides with $\hat{u}_k$ introduced in (39). This together with (42) confirms the fact that the control (34) solves the optimal control problem (28). Also, from (40) and (42) we obtain that the minimal value of the cost function (27) is given by (35). Additionally, if $R(t) > 0$, $t \in [t_0, t_f]$, then $\mathcal{R}_k > 0$ and (41) becomes

$$\mathcal{R}_k + \mathcal{B}_d^T \tilde{X}(t_k)\mathcal{B}_d > 0, \ 0 \leq k \leq N-1 \tag{43}$$

which leads to the uniqueness of the optimal control. Thus the proof ends. □

Remark 4.1 *Let us remark that if (43) is fulfilled then (33) is automatically satisfied.*

Let $\begin{pmatrix} \tilde{X}_{11}(t) & \tilde{X}_{12}(t) \\ \tilde{X}_{12}^T(t) & \tilde{X}_{22}(t) \end{pmatrix}$ be the partition of $\tilde{X}(t)$ with $\tilde{X}_{11}(t) \in \mathcal{S}_n$ and $\tilde{X}_{22}(t) \in \mathcal{S}_m$. Using the partition (26) of the coefficients of (29) we obtain the following partition of TVP (29):

$$\begin{aligned}
&\dot{X}_{11}(t) + A_0^T(t)X_{11}(t) + X_{11}(t)A_0(t) + A_1^T(t)X_{11}(t)A_1(t) = 0 \\
&\dot{X}_{12}(t) + A_0^T(t)X_{12}(t) + X_{11}(t)B_0(t) + A_1^T(t)X_{11}(t)B_1(t) = 0 \qquad (44a) \\
&\dot{X}_{22}(t) + B_0^T(t)X_{12}(t) + X_{12}^T(t)B_0(t) + B_1^T(t)X_{11}(t)B_1(t) = 0,\ t_k \leq t < t_{k+1}
\end{aligned}$$

$$\begin{aligned}
&X_{11}(t_k^-) = X_{11}(t_k) - X_{12}(t_k)(\mathcal{R}_k + X_{22}(t_k))^\dagger X_{12}^T(t_k) \\
&X_{12}(t_k^-) = 0 \qquad (44b) \\
&X_{22}(t_k^-) = 0, 0 \leq k \leq N-1
\end{aligned}$$

$$\begin{aligned}
&X_{11}(t_N^-) = C^T(t_f)C(t_f) \\
&X_{12}(t_N^-) = 0 \qquad (44c) \\
&X_{22}(t_N^-) = 0.
\end{aligned}$$

The partition of the solution $\tilde{X}(t)$ leads to the following partition of (31) $\tilde{F}_d(k) = \begin{pmatrix} \tilde{\mathbb{F}}_d(k) & 0 \end{pmatrix}$ where

$$\tilde{\mathbb{F}}_d(k) = -(\mathcal{R}_k + \tilde{X}_{22}(t_k))^\dagger \tilde{X}_{12}^T(t_k). \qquad (45)$$

Let $\begin{pmatrix} \tilde{\xi}_1^T(t) & \tilde{\xi}_2^T(t) \end{pmatrix}^T$ be the partition of the solution $\tilde{\xi}(t)$ of the TVP (32) such that $\tilde{\xi}_1(t) \in \mathbb{R}^n$ and $\tilde{\xi}_2(t) \in \mathbb{R}^m$. We have the following partition of (32):

$$\begin{aligned}
\dot{\xi}_1(t) + A_0^T(t)\xi_1(t) &= 0 \\
\dot{\xi}_2(t) + B_0^T(t)\xi_1(t) &= 0 \qquad (46a)
\end{aligned}$$

$$\begin{aligned}
\xi_1(t_k^-) &= \xi_1(t_k) + \tilde{\mathbb{F}}_d^T(k)\xi_2(t_k) \\
\xi_2(t_k^-) &= 0, k = 0, 1, ..., N-1 \qquad (46b)
\end{aligned}$$

$$\begin{aligned}
\xi_1(t_N^-) &= -C^T(t_f)\zeta \\
\xi_2(t_N^-) &= 0. \qquad (46c)
\end{aligned}$$

Now we are in position to provide the solution of the optimal control problem that ask for the minimization of the cost (5) along of the trajectories of the system (2) corresponding to piecewise constant controls of type (4).

Theorem 4.4 *Assume: a) $R(t) \geq 0$, for all $t \in [t_0, t_f]$.*

b) The solutions of TVPs (44) and (46) satisfy the condition

$$(\mathcal{R}_k + \tilde{X}_{22}(t_k))(\mathcal{R}_k + \tilde{X}_{22}(t_k))^\dagger \tilde{\xi}_2(t_k) = \tilde{\xi}_2(t_k),\ 0 \leq k \leq N-1. \qquad (47)$$

We consider the piecewise constant control

$$\tilde{u}(t) = \tilde{\mathbb{F}}_d(k)\tilde{x}(t_k) + \tilde{\nu}(k),\ t_k \leq t < t_{k+1}, 0 \leq k \leq N-1 \tag{48}$$

where $\tilde{\mathbb{F}}_d(k)$ *are defined in (46) and*

$$\tilde{\nu}(k) = -(\mathcal{R}_k + \tilde{X}_{22}(t_k))^{\dagger}\tilde{\xi}_2(t_k) \tag{49}$$

while $\tilde{x}(t_k)$ *are the values at instance time* t_k *of the solution of the initial value problem (IVP)*

$$\begin{aligned} dx(t) \quad = \quad & [A_0(t)x(t) + B_0(t)(\tilde{\mathbb{F}}_d(k)x(t_k) + \tilde{\nu}(k))]dt + \\ & +[A_1(t)x(t) + B_1(t)(\tilde{\mathbb{F}}_d(k)x(t_k)\tilde{\nu}(k))]dw(t), \end{aligned} \tag{50}$$

$t_k \leq t < t_{k+1}, k = 0,1,...,N-1$. *Under the considered assumptions the control* $\tilde{u}(\cdot)$ *defined by (48)-(50) achieves the minimum of the cost functional (5) in the class of piecewise constant controls* $\mathcal{U}_{pc}$. *The minimal value of (5) is*

$$\begin{aligned} J(x_0, \zeta, \tilde{u}(\cdot)) = & \, x_0^T \tilde{X}_{11}(t_0^-)x_0 + 2x_0^T\tilde{\xi}_1(t_0^-) + \zeta^T\zeta \\ & - \sum_{k=0}^{N-1} \tilde{\xi}_2^T(t_k)(\mathcal{R}_k + \tilde{X}_{22}(t_k))^{\dagger}\tilde{\xi}_2(t_k). \end{aligned} \tag{51}$$

Moreover if $R(t) > 0$, $t \in [t_0, t_f]$ *then the control defined by (48)-(50) is the unique optimal control of the optimization problems described by the cost (5) in the class of piecewise constant admissible controls.*

Proof follows immediately from Theorem 4.3, taking into account that for any control $u(\cdot)$ of type (4) we have $J(x_0, \zeta, u(\cdot)) = \mathcal{J}(\mathbf{x}_0, \zeta, \mathbf{u})$ where $\mathbf{u} = (u_0, u_1, ..., u_{N-1})$, u_k being the random vectors arising in the formula of type (4) of the control $u(\cdot)$. □

Since the performance criterion (3) is obtained from (5) for $R(t) \equiv 0$ we obtain immediately.

Corollary 4.5 *Let* $(\tilde{X}_{11}(\cdot), \tilde{X}_{12}(\cdot), \tilde{X}_{22}(\cdot))$ *be the solution of TVP (44) written for* $\mathcal{R}_k = 0$, $0 \leq k \leq N-1$. *Let* $(\tilde{\xi}_1(\cdot), \tilde{\xi}_2(\cdot))$ *be the solution of TVP:*

$$\begin{aligned} \dot{\xi}_1(t) + A_0^T(t)\xi_1(t) &= 0 \\ \dot{\xi}_2(t) + B_0^T(t)\xi_1(t) &= 0 \\ t_k \leq t &< t_{k+1} \end{aligned} \tag{52a}$$

$$\begin{aligned} \xi_1(t_k^-) &= \xi_1(t_k) - \tilde{X}_{12}(t_k)\tilde{X}_{22}^{\dagger}(t_k)\xi_2(t_k) \\ \xi_2(t_k^-) &= 0, 0 \leq k \leq N-1 \end{aligned} \tag{52b}$$

$$\begin{aligned} \xi_1(t_N^-) &= -C^T(t_f)\zeta \\ \xi_2(t_N^-) &= 0. \end{aligned} \tag{52c}$$

Assume that the following equalities are fulfilled:

$$\tilde{X}_{22}(t_k)\tilde{X}_{22}^{\dagger}(t_k)\tilde{\xi}_2(t_k) = \tilde{\xi}_2(t_k), \quad 0 \leq k \leq N-1. \tag{53}$$

We consider the piecewise constant control:

$$\tilde{\tilde{u}}(t) = -\tilde{X}_{22}^{\dagger}(t_k)(\tilde{X}_{12}^T(t_k)\tilde{\tilde{x}}(t_k) + \tilde{\xi}_2(t_k)), \tag{54}$$

$t_k \leq t < t_{k+1}$*, where* $\tilde{\tilde{x}}(t_k)$ *are the values at instance time* t_k *of the solution of IVP*

$$\begin{aligned} d\tilde{\tilde{x}}(t) &= [A_0(t)\tilde{\tilde{x}}(t) - B_0(t)\tilde{X}_{22}^{\dagger}(t_k)(\tilde{X}_{12}^T(t_k)\tilde{\tilde{x}}(t_k) + \tilde{\xi}_2(t_k))]dt + \\ &\quad [A_1(t)\tilde{\tilde{x}}(t) - B_1(t)\tilde{X}_{22}^{\dagger}(t_k)(\tilde{X}_{12}^T(t_k)\tilde{\tilde{x}}(t_k) + \tilde{\xi}_2(t_k))]dw(t), \quad (55) \\ &\quad t \leq t < t_{k+1}, 0 \leq k \leq N-1 \\ \tilde{\tilde{x}}(t_0) &= x_0. \end{aligned}$$

Under the considered assumptions the control (54)-(55) minimizes the value of the deviation (3) of the signal $z(t_f)$ *from the reference target* ζ *with respect to the class of piecewise constant controls* $\mathcal{U}_{pc}$*. The minimal value of the deviation (3) is given by*

$$\begin{aligned} V(x_0, \zeta, \tilde{\tilde{u}}(\cdot)) &= x_0^T\tilde{X}_{11}(t_0^-)x_0 + 2x_0^T\tilde{\xi}_1(t_0^-) + \\ &+\zeta^T\zeta - \sum_{k=0}^{N-1}\tilde{\xi}_2^T(t_k)\tilde{X}_{22}^T(t_k)\tilde{\xi}_2(t_k) \end{aligned} \tag{56}$$

Remark 4.2 *a) The condition (53) is automatically satisfied if* $B_0(t) \equiv 0$ *because in this case* $\tilde{\xi}_2(t) = 0$*, for all* $t \in [t_0, t_f]$*.*

b) If the condition (53) is not satisfied for some $k \in \{0, 1, ..., N-1\}$ *then one replaces the performance (3) by a perturbed one of type (5) with* $R(t) > 0$ *and one applies Theorem 4.4 to obtain an optimal control.*

The next example shows that sometimes the approach based on the piecewise constant controls is preferable to the one based on stochastic processes with bounded energy.

Example 4.3 Consider the system (1)-(2) in the special case $n = m = 1$, $[t_0, t_f] = [0, 1]$, $A_0(t) = 0, B_1(t) = 0, B_0(t) = b_0 \in \mathbb{R} - \{0\}$, $A_1(t) = a_1 \in \mathbb{R} - \{0\}$, $C(t) = 1, t \in [0, 1]$. In this case (1)-(2) takes the form:

$$\begin{aligned} z(t) &= x(t) \\ dx(t) &= b_0 u(t)dt + a_1 x(t)dw(t). \end{aligned} \tag{57}$$

The performance criterion (3) becomes:

$$V(x_0, \zeta; u(\cdot)) = \mathbb{E}[|x(1) - \zeta|^2]. \tag{58}$$

A. Minimization by nonanticipative controls with bounded energy

The Riccati type equation (15) reduces to

$$\dot{P}(t) + a_1^2 P(t) = 0, \quad P(1) = 1.$$

Its solution is

$$P(t) = e^{a_1^2(1-t)}. \tag{59}$$

The condition (9a) reduces to

$$b_0 P(t) = 0, \ t \in [0, 1]. \tag{60}$$

From (59) it sees that (60) is not satisfied. Hence, Theorem 3.2 is not applicable to solve the problem of minimization of the deviation (58) in the class $\mathcal{U}_{ad}$ of nonanticipative stochastic processes with bounded energy.

B. The approach by piecewise constant controls.

The system (32) takes the special form:

$$\dot{X}_{11}(t) + a_1^2 X_{11}(t) = 0$$

$$\dot{X}_{12}(t) + b_0 X_{11}(t) = 0$$

$$\dot{X}_{22}(t) + 2b_0 X_{12}(t) = 0$$

for $t_k \le t < t_{k+1}$.

$$X_{11}(t_k^-) = X_{11}(t_k) - \frac{X_{12}^2(t_k)}{X_{22}(t_k)} \text{ if } X_{22}(t_k) \ne 0$$

or

$$X_{11}(t_k^-) = X_{11}(t_k) \quad \text{if} \quad X_{22}(t_k) = 0$$

$$X_{12}(t_k^-) = X_{22}(t_k^-) = 0$$

for $0 \le k \le N-1$.

$$X_{11}(1) = 1, \;\; X_{12}(1) = X_{22}(1) = 0.$$

Assume that $t_{k+1} - t_k = h > 0, 0 \le k \le N-1$. We have

$$X_{11}(t) = e^{a_1^2(t_{k+1}-t)} X_{11}(t_{k+1}^-)$$

$$X_{12}(t) = \frac{b_0}{a_1^2} X_{11}(t_{k+1}^-)(e^{a_1^2(t_{k+1}-t)} - 1)$$

$$X_{22}(t) = 2\frac{b_0^2}{a_1^4} X_{11}(t_{k+1}^-)(e^{a_1^2(t_{k+1}-t)} - 1) - 2\frac{b_0^2}{a_1^2} X_{11}(t_{k+1}^-)(t_{k+1} - t).$$

One obtains $X_{22}(t_k) = 2\frac{b_0^2}{a_1^4} X_{11}(t_{k+1}^-)(e^{a_1^2 h} - ha_1^2 - 1) > 0$ if $X_{11}(t_{k+1}^-) > 0$. By direct calculation one obtains

$$X_{11}(t_k^-) = \gamma X_{11}(t_{k+1}^-)(e^{2ha_1^2} - 2a_1^2 h e^{a_1^2 h} - 1) > 0.$$

Thus, inductively, one obtains that $X_{22}(t_k) > 0$ and therefore the condition (41) from the above Corollary is fulfilled.

5 A Numerical Experiment

Let us consider the academic example of equation (1)-(2). In order to form the block matrix coefficients of (26) we apply the following matrices for $t \in [0,1]\,, n = 4,\; m = 2$ (using Matlab notations) :

A_0=[1 0 3 0; -4 2 0 -10; -14 8.5 -2.5 0; 0 -2 0 -10];, $A_0 \in \mathbb{R}^{4\times 4}$

A_1=[0 2 0. -1; 0 0 -3 1.5; -1.45 0.6 -2 0; 0 -3 0 5]; $A_1 \in \mathbb{R}^{4\times 4}$

$B_0 = [1\;0; 2\;5; -1\;4; 2\;6]; \quad B_1 = [0\;1; -1.5\;-3; 2\;-4; 2\;0];, B_0, B_1 \in \mathbb{R}^{4\times 2}$,

$C = [1.0\;\; -0.25\;\; -0.75\;\; -0.5]; \in \mathbb{R}^{1\times n}$,

$R = [0.45\;0; 0\;0.75]; \in \mathbb{R}^{2\times 2}$

and $\zeta = 1.5$;

The computation of the solution $\tilde{X}(\cdot)$ of the TVP (29) and the gain matrices $\tilde{F}_d(k)$ from (31) , can be done using algorithms derived in [6, 7].

The optimal controls (at the point $t_k = k * h$), $\mathbb{F}_d(k), k = 0, 1, ..., N-1 = 9$ are

$$\mathbb{F}_d(0) = \begin{pmatrix} 0.2958 & 0.5098 & -0.1588 & -0.6806 \\ 0.1392 & -0.1988 & -0.4888 & 0.0924 \end{pmatrix},$$

$$\mathbb{F}_d(1) = \begin{pmatrix} 0.2966 & 0.5071 & -0.1585 & -0.6785 \\ 0.1396 & -0.2002 & -0.4887 & 0.0937 \end{pmatrix},$$

$$\dots\dots\dots$$

$$\mathbb{F}_d(8) = \begin{pmatrix} 0.2416 & 0.4167 & -0.1944 & -0.5877 \\ 0.2010 & -0.2935 & -0.4663 & 0.1718 \end{pmatrix},$$

$$\mathbb{F}_d(9) = \begin{pmatrix} -0.1311 & 0.9683 & -0.2949 & -0.9427 \\ 0.3496 & -0.3983 & -0.4695 & 0.1956 \end{pmatrix}.$$

In order to compute the piecewise constant control $\tilde{u}(t)$ (48) we need the vector $\tilde{\nu}(k)$, which can be computed via (49) The values of $\tilde{\nu}(k)$, are

$$\tilde{\nu}(0) = \begin{pmatrix} 0.0009 \\ -0.0019 \end{pmatrix},$$

$$\tilde{\nu}(1) = \begin{pmatrix} 0.0012 \\ -0.0027 \end{pmatrix},$$

$$\dots\dots\dots$$

$$\tilde{\nu}(8) = \begin{pmatrix} -0.0132 \\ 0.0565 \end{pmatrix},$$

$$\tilde{\nu}(9) = \begin{pmatrix} 0.0423 \\ -0.0141 \end{pmatrix}.$$

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