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ON A CAPUTO TYPE FRACTIONAL INTEGRO-DIFFERENTIAL INCLUSION*

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Abstract

A Cauchy problem associated to a fractional integro-differential defined by a Caputo type fractional derivative is studied. It is proved the arcwise connectedness of the solution set and that the set of selections corresponding to the solutions of the problem considered is a retract of the space of integrable functions on a given interval.

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1 Introduction

In the last years one may see a strong development of the theory of differential equations and inclusions of fractional order ([4, 11, 13, 14] etc.). The main reason is that fractional differential equations are very useful tools in order to model many physical phenomena. In the fractional calculus there are several fractional derivatives. From them, the fractional derivative introduced by Caputo in [6] allows to use Cauchy conditions which have physical meanings.

A Caputo type fractional derivative of a function with respect to another function ([13]) that extends and unifies several fractional derivatives existing in the literature like Caputo, Caputo-Hadamard, Caputo-Katugampola

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was intensively studied in recent years [1, 2, 3] etc.. Existence results and qualitative properties of the solutions for fractional differential equations defined by this fractional derivative are obtained in [2, 3] and the existence of solutions in the set-valued framework is studied in [9, 10].

The present paper is concerned with the following problem

$$D_C^{\alpha,\psi}x(t) \in F(t, x(t), V(x)(t)) \quad a.e.\ ([0,T]), \quad x(0) = x_0, \tag{1.1}$$

where $\alpha \in (0,1]$, $D_C^{\alpha,\psi}$ is the fractional derivative mentioned above, $x_0 \in \mathbf{R}$ and $F : [0,T] \times \mathbf{R} \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is a set-valued map. $V : C([0,T], \mathbf{R}) \to C([0,T], \mathbf{R})$ is a nonlinear Volterra integral operator defined by $V(x)(t) = \int_0^t k(t, s, x(s))ds$ with $k(.,.,.) : [0,T] \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ a given function.

Our goal is twofold. On one hand, we prove the arcwise connectedness of the solution set of problem (1.1) when the set-valued map is Lipschitz in the second and third variable. On the other hand, under such type of hypotheses on the set-valued map we establish a more general topological property of the solution set of problem (1.1). Namely, we prove that the set of selections of the set-valued map F that correspond to the solutions of problem (1.1) is a retract of $L^1([0,T], \mathbf{R})$. Both results are essentially based on Bressan and Colombo results ([5]) concerning the existence of continuous selections of lower semicontinuous multifunctions with decomposable values.

We recall that in the theory of ordinary differential equations Kneser's theorem states that the solution set of an ordinary differential equation is connected, i.e., it cannot be represented as a union of two closed sets without common points. In the case of differential inclusions, although the solution set multifunction is not, in general, convex valued we are able to prove its arcwise connectedness and therefore, our result may be regarded as an extension of the classical theorem of Kneser.

The results in the present paper extend and unify similar results obtained for classical ordinary differential inclusions ([15, 16]), for fractional differential inclusions defined by Caputo fractional derivative ([7]) and for fractional differential inclusions defined by Caputo-Katugampola fractional derivative ([8]).

The paper is organized as follows: in Section 2 we present the notations, definitions and the preliminary results to be used in the sequel and in Section 3 we prove our main results.

2 Preliminaries

Let $T > 0$, $I := [0, T]$ and denote by $\mathcal{L}(I)$ the σ-algebra of all Lebesgue measurable subsets of I. Let X be a real separable Banach space with the norm $|.|$. Denote by $\mathcal{P}(X)$ the family of all nonempty subsets of X and by $\mathcal{B}(X)$ the family of all Borel subsets of X. If $A \subset I$ then $\chi_A(.) : I \to \{0,1\}$ denotes the characteristic function of A. For any subset $A \subset X$ we denote by $cl(A)$ the closure of A.

The distance between a point $x \in X$ and a subset $A \subset X$ is defined as usual by $d(x, A) = \inf\{|x-a|; a \in A\}$. We recall that Pompeiu-Hausdorff distance between the closed subsets $A, B \subset X$ is defined by $d_H(A,B) = \max\{d^*(A,B), d^*(B,A)\}$, $d^*(A,B) = \sup\{d(a,B);\ a \in A\}$.

As usual, we denote by $C(I,X)$ the Banach space of all continuous functions $x : I \to X$ endowed with the norm $|x|_C = \sup_{t\in I}|x(t)|$ and by $L^1(I,X)$ the Banach space of all (Bochner) integrable functions $x : I \to X$ endowed with the norm $|x|_1 = \int_0^T |x(t)|dt$.

We recall first several preliminary results we shall use in the sequel.

A subset $D \subset L^1(I,X)$ is said to be *decomposable* if for any $u, v \in D$ and any subset $A \in \mathcal{L}(I)$ one has $u\chi_A + v\chi_B \in D$, where $B = I\backslash A$.

We denote by $\mathcal{D}(I,X)$ the family of all decomposable closed subsets of $L^1(I,X)$.

Next (S,d) is a separable metric space; we recall that a multifunction $G : S \to \mathcal{P}(X)$ is said to be lower semicontinuous (l.s.c.) if for any closed subset $C \subset X$, the subset $\{s \in S;\ G(s) \subset C\}$ is closed. The next lemmas may be found in [5].

Lemma 2.1. *If $F : I \to \mathcal{D}(I,X)$ is a lower semicontinuous multifunction with closed nonempty and decomposable values then there exists $f : I \to L^1(I,X)$ a continuous selection from F.*

Lemma 2.2. *Let $G(.,.) : I \times S \to \mathcal{P}(X)$ be a closed-valued $\mathcal{L}(I) \otimes \mathcal{B}(S)$-measurable multifunction such that $G(t,.)$ is l.s.c. for any $t \in I$.*

Then the multifunction $G^*(.) : S \to \mathcal{D}(I,X)$ defined by

$$G^*(s) = \{f \in L^1(I,X); \quad f(t) \in G(t,s) \quad a.e.\ (I)\}$$

is l.s.c. with nonempty closed values if and only if there exists a continuous mapping $q(.) : S \to L^1(I,X)$ such that

$$d(0, G(t,s)) \leq q(s)(t) \quad a.e.\ (I),\ \forall s \in S.$$

Lemma 2.3. *Let $H(.) : S \to \mathcal{D}(I,X)$ be a l.s.c. multifunction with closed decomposable values and let $a(.) : S \to L^1(I,X)$, $b(.) : S \to L^1(I,\mathbf{R})$ be* *continuous such that the multifunction* $F(.) : S \to \mathcal{D}(I, X)$ *defined by*

$$F(s) = cl\{f \in H(s); \quad |f(t) - a(s)(t)| < b(s)(t) \quad a.e. \ (I)\}$$

has nonempty values.

Then $F(.)$ *has a continuous selection.*

Consider $\beta > 0$, $f(.) \in L^1(I, \mathbf{R})$ and $\psi(.) \in C^n(I, \mathbf{R})$ such that $\psi'(t) > 0$ $\forall\, t \in I$.

Definition 2.4. ([13]) a) The ψ - Riemann-Liouville fractional integral of f of order β is defined by

$$I^{\beta,\psi} f(t) = \frac{1}{\Gamma(\beta)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\beta-1} f(s) ds,$$

where Γ is the (Euler's) Gamma function defined by $\Gamma(\beta) = \int_0^\infty t^{\beta-1} e^{-t} dt$.

b) The ψ - Riemann-Liouville fractional derivative of f of order β is defined by

$$D^{\beta,\psi} f(t) = \frac{1}{\Gamma(n - \beta)} (\frac{1}{\psi'(t)} \frac{d}{dt})^n \int_0^t \psi'(s)(\psi(t) - \psi(s))^{n-\beta-1} f(s) ds,$$

where $n = [\beta] + 1$.

c) The ψ - Caputo fractional derivative of f of order β is defined by

$$D_C^{\beta,\psi} f(t) = D^{\beta,\psi}[f(t) - \sum_{k=0}^{n-1} \frac{f_\psi^{[k]}(0)}{k!} (\psi(t) - \psi(0))^k],$$

where $f_\psi^{[k]}(t) = (\frac{1}{\psi'(t)} \frac{d}{dt})^k x(t)$, $n = \beta$ if $\alpha \in \mathbf{N}$ and $n = [\beta] + 1$, otherwise.

We note that if $\beta = m \in \mathbf{N}$ then $D_C^{\beta,\psi} f(t) = f_\psi^{[m]}(t)$ and if $n = [\beta] + 1$ then $D_C^{\beta,\psi} f(t) = \frac{1}{\Gamma(n-\beta)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{n-\alpha-1} f_\psi^{[n]}(s) ds$. Also, if $\psi(t) \equiv t$ one obtains Caputo's fractional derivative, if $\psi(t) \equiv \ln(t)$ one obtains Caputo-Hadamard's fractional derivative and, finally, if $\psi(t) \equiv t^\sigma$ one obtains Caputo-Katugampola's fractional derivative.

In what follows we need the following technical lemma proved in [2] (namely, Theorem 2 in [2]).

Lemma 2.5. *Let* $\alpha \in [0, 1)$ *and* $\psi(.) \in C^1(I, \mathbf{R})$ *with* $\psi'(t) > 0$ $\forall\, t \in I$. *For a given integrable function* $h(.) : I \to \mathbf{R}$, *the unique solution of the initial value problem*

$$D_C^{\alpha,\psi} x(t) = h(t) \quad a.e. \ (I), \quad x(0) = x_0,$$

is given by

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} h(s)ds$$

Definition 2.6. By a solution of the problem (1.1) we mean a function $x \in C(I, \mathbf{R})$ for which there exists a function $h \in L^1(I, \mathbf{R})$ satisfying $h(t) \in F(t, x(t), V(x)(t))$ *a.e.* (I), $D_C^{\alpha,\psi} x(t) = h(t)$ *a.e.* (I) and $x(0) = x_0$.

In this case $(x(.), f(.))$ is called a *trajectory-selection* pair of problem (1.1).

We shall use the following notations for the solution sets and for the selection sets of problem (1.1).

$$\mathcal{S}(x_0) = \{x \in C(I, \mathbf{R}); \quad x(.) \text{ is a solution of } (1.1)\},$$

$$\tilde{f}(t) = x_0 + \tfrac{1}{\Gamma(\alpha)} \textstyle\int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f(s)ds,$$
$$\mathcal{T}(x_0) = \{f \in L^1(I, \mathbf{R}); \quad f(t) \in F(t, \tilde{f}(t), V(\tilde{f})(t)) \quad a.e.\ I\}.$$

3 The main results

In order to prove our topological properties of the solution set of problem (1.1) we need the following hypotheses.

Hypothesis 3.1. i) $F(.,.) : I \times \mathbf{R} \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ *has nonempty closed values and is* $\mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R} \times \mathbf{R})$ *measurable.*

ii) *There exists a mapping* $L(.) \in L^1(I, (0, \infty))$ *such that, for almost all* $t \in I, F(t, ., .)$ *is* $L(t)$*-Lipschitz in the sense that*

$$d_H(F(t, x_1, y_1), F(t, x_2, y_2)) \leq L(t)(|x_1 - x_2| + |y_1 - y_2|) \quad \forall\, x_1, x_2, y_1, y_2 \in \mathbf{R}.$$

iii) *There exists* $p \in L^1(I, \mathbf{R})$ *such that*

$$d_H(\{0\}, F(t, 0, V(0)(t))) \leq p(t) \quad a.e.\ I.$$

iv) $k(., ., .) : I \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ *is a function such that* $\forall x \in \mathbf{R}$, $(t, s) \to k(t, s, x)$ *is measurable.*

v) $|k(t, s, x) - k(t, s, y)| \leq L(t)|x - y| \quad$ *a.e.* $(t, s) \in I \times I, \quad \forall\, x, y \in \mathbf{R}$.

We use next the following notations

$$M(t) := L(t)(1 + \int_0^t L(u)du), \quad t \in I, \quad I^{\alpha,\psi} M := \sup_{t \in I} |I^{\alpha,\psi} M(t)|.$$

Theorem 3.2. *Assume that Hypothesis 3.1 is satisfied and* $I^{\alpha,\psi}M < 1$.

Then for any $\xi_0 \in \mathbf{R}$ *the solution set* $\mathcal{S}(\xi_0)$ *of (1.1) is arcwise connected in the space* $C(I, \mathbf{R})$.

Proof. Let $\xi_0 \in \mathbf{R}$ and $x_0, x_1 \in \mathcal{S}(\xi_0)$. Therefore there exist $f_0, f_1 \in L^1(I, \mathbf{R})$ such that $x_0(t) = \xi_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} f_0(u)du$ and $x_1(t) = \xi_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} f_1(u)du$, $t \in I$.

For $\lambda \in [0,1]$ define

$$x^0(\lambda) = (1-\lambda)x_0 + \lambda x_1 \quad \text{and} \quad g^0(\lambda) = (1-\lambda)f_0 + \lambda f_1$$

Obviously, the mapping $\lambda \mapsto x^0(\lambda)$ is continuous from $[0,1]$ into $C(I, \mathbf{R})$ and since $|g^0(\lambda) - g^0(\lambda_0)|_1 = |\lambda - \lambda_0|.|f_0 - f_1|_1$ it follows that $\lambda \mapsto g^0(\lambda)$ is continuous from $[0,1]$ into $L^1(I, \mathbf{R})$.

Define the set-valued maps

$$\Psi^1(\lambda) = \{v \in L^1(I, \mathbf{R});\ \ v(t) \in F(t, x^0(\lambda)(t), V(x^0(\lambda))(t)) \ \ a.e.\ I\},$$

$$\Phi^1(\lambda) = \begin{cases} \{f_0\} & \text{if } \lambda = 0, \\ \Psi^1(\lambda) & \text{if } 0 < \lambda < 1, \\ \{f_1\} & \text{if } \lambda = 1 \end{cases}$$

and note that $\Phi^1 : [0,1] \to \mathcal{D}(I, \mathbf{R})$ is lower semicontinuous (e.g., [8]).

Next we use the following notation

$$p_0(\lambda)(t) = |g^0(\lambda)(t)| + p(t) + L(t)(|x^0(\lambda)(t)| + \int_0^t L(s)|x^0(\lambda)(s)|ds),$$

$t \in I$, $\lambda \in [0,1]$.

Since

$$\begin{aligned} &|p_0(\lambda)(t) - p_0(\lambda_0)(t)| \leq |\lambda - \lambda_0|[|f_1(t) - f_0(t)| + \\ &L(t)(|x_0(t) - x_1(t)| + \textstyle\int_0^t L(s)|x_0(s) - x_1(s)|ds)] \end{aligned}$$

we deduce that $p_0(.)$ is continuous from $[0,1]$ to $L^1(I, \mathbf{R})$.

At the same time, from Hypothesis 3.1 it follows

$$d(g^0(\lambda)(t), F(t, x^0(\lambda)(t), V(x^0(\lambda))(t)) \leq p_0(\lambda)(t) \quad a.e.\ I. \tag{3.1}$$

Fix $\delta > 0$ and for $m \in \mathbf{N}$ we set $\delta_m = \frac{m+1}{m+2}\delta$.

We shall prove next that there exists a continuous mapping $g^1 : [0,1] \to L^1(I, \mathbf{R})$ with the following properties

a) $g^1(\lambda)(t) \in F(t, x^0(\lambda)(t), V(x^0(\lambda))(t)) \quad a.e.\ I$,

b) $g^1(0) = f_0, \quad g^1(1) = f_1$,
c) $|g^1(\lambda)(t) - g^0(\lambda)(t)| \leq p_0(\lambda)(t) + \delta_0 \frac{\Gamma(\alpha+1)}{(\psi(T))^\alpha} \quad a.e.\ I$.
Define

$$G^1(\lambda) = \mathrm{cl}\{v \in \Phi^1(\lambda); \quad |v(t) - g^0(\lambda)(t)| < p_0(\lambda)(t) + \delta_0 \frac{\Gamma(\alpha+1)}{(\psi(T))^\alpha}, \quad a.e.\ I\}$$

and, by (3.1), we find that $G^1(\lambda)$ is nonempty for any $\lambda \in [0,1]$. Moreover, since the mapping $\lambda \mapsto p_0(\lambda)$ is continuous, we apply Lemma 2.3 and we obtain the existence of a continuous mapping $g^1 : [0,1] \to L^1(I, \mathbf{R})$ such that $g^1(\lambda) \in G^1(\lambda)\ \forall \lambda \in [0,1]$, hence with properties a)-c).

Define now

$$x^1(\lambda)(t) = \xi_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} g^1(\lambda)(u)du, \quad t \in I$$

and note that, since $|x^1(\lambda) - x^1(\lambda_0)|_C \leq \frac{(\psi(T))^\alpha}{\Gamma(\alpha+1)} |g^1(\lambda) - g^1(\lambda_0)|_1$, $x^1(.)$ is continuous from $[0,1]$ into $C(I, \mathbf{R})$.

Set $p_m(\lambda) := (I^{\alpha,\psi} M)^{m-1} (\frac{(\psi(T))^\alpha}{\Gamma(\alpha+1)} |p_0(\lambda)|_1 + \delta_m)$.

We shall prove that for all $m \geq 1$ and $\lambda \in [0,1]$ there exist $x^m(\lambda) \in C(I, \mathbf{R})$ and $g^m(\lambda) \in L^1(I, \mathbf{R})$ with the following properties
i) $g^m(0) = f_0, \quad g^m(1) = f_1$,
ii) $g^m(\lambda)(t) \in F(t, x^{m-1}(\lambda)(t), V(x^{m-1}(\lambda))(t)) \quad a.e.\ I$,
iii) $g^m : [0,1] \to L^1(I, \mathbf{R})$is continuous,
iv) $|g^1(\lambda)(t) - g^0(\lambda)(t)| \leq p_0(\lambda)(t) + \delta_0 \frac{\Gamma(\alpha+1)}{(\psi(T))^\alpha}$,
v) $|g^m(\lambda)(t) - g^{m-1}(\lambda)(t)| \leq M(t) p_m(\lambda), \quad m \geq 2$,
vi) $x^m(\lambda)(t) = \xi_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} g^m(\lambda)(u)du, \quad t \in I$.

Assume that we have already constructed $g^m(.)$ and $x^m(.)$ with i)-vi) and define

$$\Psi^{m+1}(\lambda) = \{v \in L^1(I, \mathbf{R}); \quad v(t) \in F(t, x^m(\lambda)(t), V(x^m(\lambda))(t)) \quad a.e.\ I\},$$

$$\Phi^{m+1}(\lambda) = \begin{cases} \{f_0\} & \text{if } \lambda = 0, \\ \Psi^{m+1}(\lambda) & \text{if } 0 < \lambda < 1, \\ \{f_1\} & \text{if } \lambda = 1. \end{cases}$$

As in the case $m = 1$ we obtain that $\Phi^{m+1} : [0,1] \to \mathcal{D}(I, \mathbf{R})$ is lower semicontinuous.

From ii), v) and Hypothesis 3.1, for almost all $t \in I$, we have

$|x^m(\lambda)(t) - x^{m-1}(\lambda)(t)| \leq \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} |g^m(\lambda)(u) -$
$g^{m-1}(\lambda)(u)| \mathrm{d}u \leq \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} M(u) p_m(\lambda) du =$
$I^{\alpha,\psi} M(t) p_m(\lambda) \leq I^{\alpha,\psi} M p_m(\lambda) < p_{m+1}(\lambda)$.

For $\lambda \in [0,1]$ consider the set

$$G^{m+1}(\lambda) = cl\{v \in \Phi^{m+1}(\lambda); \quad |v(t) - g^m(\lambda)(t)| < M(t)p_{m+1}(\lambda) \quad a.e.\ I\}.$$

In order to prove that $G^{m+1}(\lambda)$ is not empty we note first that $r_m := (I^{\alpha,\psi}M)^m(\delta_{m+1} - \delta_m) > 0$ and by Hypothesis 3.1 and ii) one has

$$\begin{aligned}&d(g^m(t), F(t, x^m(\lambda)(t), V(x^m(\lambda))(t)) \leq L(t)(|x^m(\lambda)(t) - x^{m-1}(\lambda)(t)| + \\ &\textstyle\int_0^t L(s)|x^m(\lambda)(s) - x^{m-1}(\lambda)(s)|ds) \leq L(t)(1 + \int_0^t L(s)ds)|I^{\alpha,\psi}M(t)|p_m(\lambda) \\ &= M(t)(p_{m+1}(\lambda) - r_m) < M(t)p_{m+1}(\lambda).\end{aligned}$$

Moreover, since $\Phi^{m+1} : [0,1] \to \mathcal{D}(I, \mathbf{R})$ is lower semicontinuous and the maps $\lambda \to p_{m+1}(\lambda)$, $\lambda \to h^m(\lambda)$ are continuous we apply Lemma 2.3 and we obtain the existence of a continuous selection g^{m+1} of G^{m+1}.

Therefore,

$$|x^m(\lambda) - x^{m-1}(\lambda)|_C \leq I^{\alpha,\psi}Mp_m(\lambda) \leq (I^{\alpha,\psi}M)^m(\frac{(\psi(T))^\alpha}{\Gamma(\alpha+1)}|p_0(\lambda)|_1 + \delta)$$

and thus $\{x^m(\lambda)\}_{m\in\mathbf{N}}$ is a Cauchy sequence in the Banach space $C(I, \mathbf{R})$, hence it converges to some function $x(\lambda) \in C(I, \mathbf{R})$.

Let $g(\lambda) \in L^1(I, \mathbf{R})$ be such that

$$x(\lambda)(t) = \xi_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}g(\lambda)(u)du, \quad t \in I.$$

The function $\lambda \mapsto \frac{(\psi(T))^\alpha}{\Gamma(\alpha+1)}|p_0(\lambda)|_1 + \delta$ is continuous, so it is locally bounded. Therefore the Cauchy condition is satisfied by $\{x^m(\lambda)\}_{m\in\mathbf{N}}$ locally uniformly with respect to λ and this implies that the mapping $\lambda \to x(\lambda)$ is continuous from $[0,1]$ into $C(I, \mathbf{R})$. Obviously, the convergence of the sequence $\{x^m(\lambda)\}$ to $x(\lambda)$ in $C(I, \mathbf{R})$ implies that $g^m(\lambda)$ converges to $g(\lambda)$ in $L^1(I, \mathbf{R})$.

Finally, from ii), Hypothesis 3.1 and from the fact that the values of F are closed we obtain that $x(\lambda) \in \mathcal{S}(\xi_0)$. From i) and v) we have $x(0) = x_0, x(1) = x_1$ and the proof is complete.

In what follows we use the notations

$$\tilde{u}(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}u(s)ds, \quad u \in L^1(I, \mathbf{R}) \tag{3.2}$$

and

$$p_0(u)(t) = |u(t)| + p(t) + L(t)(|\tilde{u}(t)| + \int_0^t L(s)|\tilde{u}(s)|ds), \quad t \in I \tag{3.3}$$

Let us note that

$$d(u(t), F(t, \tilde{u}(t), V(\tilde{u})(t)) \leq p_0(u)(t) \quad a.e.\ I \tag{3.4}$$

and, since for any $u_1, u_2 \in L^1(I, \mathbf{R})$

$$|p_0(u_1) - p_0(u_2)|_1 \leq (1 + |I^{\alpha,\psi} M(T)|)|u_1 - u_2|_1$$

the mapping $p_0 : L^1(I, \mathbf{R}) \to L^1(I, \mathbf{R})$ is continuous.

Proposition 3.3. *Assume that Hypothesis 3.1 is satisfied and let $\phi : L^1(I, \mathbf{R}) \to L^1(I, \mathbf{R})$ be a continuous map such that $\phi(u) = u$ for all $u \in \mathcal{T}(x_0)$. For $u \in L^1(I, \mathbf{R})$, we define*

$$\Psi(u) = \{u \in L^1(I, \mathbf{R}); \quad u(t) \in F(t, \widetilde{\phi(u)}(t), V(\widetilde{\phi(u)})(t)) \quad a.e.\ I\},$$

$$\Phi(u) = \begin{cases} \{u\} & \text{if } u \in \mathcal{T}(x_0), \\ \Psi(u) & \text{otherwise.} \end{cases}$$

Then the multifunction $\Phi : L^1(I, \mathbf{R}) \to \mathcal{P}(L^1(I, \mathbf{R}))$ is lower semicontinuous with closed decomposable and nonempty values.

The proof of Proposition 3.3 is similar to the proof of Proposition 3.2 in [7].

Theorem 3.4. *Assume that Hypothesis 3.1 is satisfied, consider $x_0 \in \mathbf{R}$ and assume $I^{\alpha,\psi} M < 1$.*

Then there exists a continuous mapping $g : L^1(I, \mathbf{R}) \to L^1(I, \mathbf{R})$ such that

i) $g(u) \in \mathcal{T}(x_0), \quad \forall u \in L^1(I, \mathbf{R})$,

ii) $g(u) = u, \quad \forall u \in \mathcal{T}(x_0)$.

Proof. Fix $\delta > 0$ and for $m \geq 0$ set $\delta_m = \frac{m+1}{m+2}\delta$ and define $p_m(u) := (I^{\alpha,\psi} M)^{m-1}(\frac{(\psi(T))^\alpha}{\Gamma(\alpha+1)}|p_0(u)|_1 + \delta_m)$, where $\tilde{u}$ and $p_0(.)$ are defined in (3.2) and (3.3). By the continuity of the map $p_0(.)$, already proved, we obtain that $p_m : L^1(I, \mathbf{R}) \to L^1(I, \mathbf{R})$ is continuous.

We define $g_0(u) = u$ and we shall prove that for any $m \geq 1$ there exists a continuous map $g_m : L^1(I, \mathbf{R}) \to L^1(I, \mathbf{R})$ that satisfies

a) $$g_m(u) = u, \quad \forall u \in \mathcal{T}(x_0),$$

b) $$g_m(u)(t) \in F(t, \widetilde{g_{m-1}}(u)(t), V(\widetilde{g_{m-1}}(u))(t)) \quad a.e.\ I,$$

c) $$|g_1(u)(t)-g_0(u)(t)|\leq p_0(u)(t)+\delta_0\frac{\Gamma(\alpha+1)}{(\psi(T))^\alpha}\quad a.e.\ I,$$

d) $$|g_m(u)(t)-g_{m-1}(t)|\leq M(t)p_m(u)\quad a.e.\ I,\ m\geq 2.$$

For $u\in L^1(I,\mathbf{R})$, we define

$$\Psi_1(u)=\{v\in L^1(I,\mathbf{R});\quad v(t)\in F(t,\widetilde{u}(t),V(\widetilde{u})(t))\quad a.e.\ I\},$$

$$\Phi_1(u)=\begin{cases}\{u\} & \text{if } u\in\mathcal{T}(x_0),\\ \Psi_1(u) & \text{otherwise}\end{cases}$$

and by Proposition 3.3 (with $\phi(u)=u$) we obtain that $\Phi_1:L^1(I,\mathbf{R})\to\mathcal{D}(I,\mathbf{R})$ is lower semicontinuous. Moreover, due to (3.4), the set

$$G_1(u)=\mathrm{cl}\{v\in\Phi_1(u);\quad |v(t)-u(t)|<p_0(u)(t)+\delta_0\frac{\Gamma(\alpha+1)}{(\psi(T))^\alpha}\quad a.e.\ I\}$$

is not empty for any $u\in L^1(I,\mathbf{R})$. So applying Lemma 2.3, we find a continuous selection $g_1(.)$ of $G_1(.)$ that satisfies a)-c).

Suppose we have already constructed $g_i(.)$, $i=1,\ldots,m$ satisfying a)-d). For $u\in L^1(I,\mathbf{R})$ we define

$$\Psi_{m+1}(u)=\{v\in L^1(I,\mathbf{R});\quad v(t)\in F(t,\widetilde{g_m(u)}(t),V(\widetilde{g_m(u)})(t))\quad a.e.\ I\},$$

$$\Phi_{m+1}(u)=\begin{cases}\{u\} & \text{if } u\in\mathcal{T}(x_0),\\ \Psi_{m+1}(u) & \text{otherwise.}\end{cases}$$

We apply Proposition 3.3 (with $\phi(u)=g_m(u)$) and get that $\Phi_{m+1}(.)$ is lower semicontinuous with closed decomposable and nonempty values. Define the set

$$G_{m+1}(u)=\mathrm{cl}\{v\in\Phi_{m+1}(u);\quad |v(t)-g_{m+1}(u)(t)|<M(t)p_{m+1}(u)\quad a.e.\ I\}.$$

In order to prove that $G_{m+1}(u)$ is not empty we note first that $r_m:=(I^{\alpha,\psi}M)^m(\delta_{m+1}-\delta_m)>0$ and by Hypothesis 3.1 and b) one has

$$d(g_m(t),F(t,\widetilde{g_m(u)}(t),V(\widetilde{g_m(u)})(t))\leq L(t)(|\widetilde{g_m(u)}(t)-\widetilde{g_{m-1}}(u)(t)|+$$
$$\int_0^t L(s)|\widetilde{g_m(u)}(s)-\widetilde{g_{m-1}}(u)(s)|ds\leq M(t)(I^{\alpha,\psi}M)p_m(u)=M(t)(p_{m+1}(u)$$
$$-r_m<M(t)p_{m+1}(u).$$

Thus $G_{m+1}(u)$ is not empty for any $u\in L^1(I,\mathbf{R})$. With Lemma 2.3, we find a continuous selection g_{m+1} of G_{m+1}, satisfying a)-d).

Therefore, we obtain that

$$|g_{m+1}(u) - g_m(u)|_1 \leq (I^{\alpha,\psi} M)^m (\frac{(\psi(T))^\alpha}{\Gamma(\alpha+1)} |p_0(u)|_1 + \delta)$$

and this implies that the sequence $\{g_m(u)\}_{m\in\mathbf{N}}$ is a Cauchy sequence in the Banach space $L^1(I,\mathbf{R})$. Let $g(u) \in L^1(I,\mathbf{R})$ be its limit. The function $u \to |p_0(u)|_1$ is continuous, hence it is locally bounded and the Cauchy condition is satisfied by $\{g_m(u)\}_{m\in\mathbf{N}}$ locally uniformly with respect to u. Hence the mapping $g(.) : L^1(I,\mathbf{R}) \to L^1(I,\mathbf{R})$ is continuous.

From a) it follows that $g(u) = u, \quad \forall u \in \mathcal{T}(x_0)$ and from b) and the fact that F has closed values we obtain that

$$g(u)(t) \in F(t, \widetilde{g(u)}(t), V(\widetilde{g(u)})(t)) \quad a.e.\ I \quad \forall u \in L^1(I,\mathbf{R}).$$

and the proof is complete.

Remark 3.5. We recall that if Y is a Hausdorff topological space, a subspace X of Y is called retract of Y if there is a continuous map $h : Y \to X$ such that $h(x) = x, \ \forall x \in X$.

Therefore, by Theorem 3.4, for any $x_0 \in \mathbf{R}$, the set $\mathcal{T}(x_0)$ of selections of solutions of (1.1) is a retract of the Banach space $L^1(I,\mathbf{R})$.

Remark 3.6. If $\psi(t) \equiv t$ then Theorem 3.2 yields Theorem 3.1 in [7] and Theorem 3.4 covers Theorem 3.3 in [7]; if in our approach $\psi(t) \equiv t^\sigma$ we get Theorem 3.2 and Theorem 3.4 in [8].

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