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APPROXIMATION BY POLYNOMIAL POSSIBILISTIC INTEGRAL OPERATORS*

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Dedicated to Dr. Vasile Drăgan on the occasion of his 70th anniversary

Abstract

In this paper we obtain quantitative estimates in approximation by the so-called polynomial possibilistic operators of Durrmeyer type and of Kantorovich type, whose expressions are obtained from their classical correspondents by replacing the usual integral with the possibilistic integral.
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1 Introduction

Recently, in a series of papers we have started the study of the approximation properties of some nonlinear integral operators obtained from the linear ones by replacing the classical Lebesgue integral by its nonlinear extension called

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Choquet integral with respect to a monotone and submodular set function, see, e.g., [6], [10], [11], [12], [7], [8], [9]. For large classes of functions, all these nonlinear operators give better estimates of approximation than their classical correspondents.

In particular, we have studied the approximation properties of the nonlinear Durrmeyer-Choquet and Kantorovich-Choquet polynomial operators, with the expressions obtained from their classical counterparts by replacing the linear Lebesgue integral by the nonlinear Choquet integral and formally given by

$$\overline{D}_{n,\Gamma_n}(f)(x) = \sum_{k=0}^{n} p_{n,k}(x) \cdot \frac{(C)\int_0^1 f(t)t^k(1-t)^{n-k}d\mu_{\lambda_{n,k}}(t)}{(C)\int_0^1 t^k(1-t)^{n-k}d\mu_{\lambda_{n,k}}(t)},$$

and

$$\overline{K}_{n,\Gamma_n}(f)(x) = \sum_{k=0}^{n} p_{n,k}(x) \cdot (C)\int_0^1 f\left(\frac{k+t}{n+1}\right) d\mu_{\lambda_{n,k}(t)},$$

where $p_{n,k}(x) = \binom{n}{k}x^k(1-x)^{n-k}$, $(C)\int_0^1$ denotes the Choquet integral and $\Gamma_n = \{\mu_{\lambda_{n,k}}\}_{k=0}^n$ is a family of monotone set functions (capacities), see, e.g., [10], [11], [8].

The study in the present paper is based on the main idea of replacing the usual integrals by the possibilistic integral in the expressions of some classical approximation operators and shortly described below.

Section 2 contains some preliminaries in the possibility theory.

We deal in Section 3 with the study of the approximation properties for the polynomial possibilistic operators of Durrmeyer type and of Kantorovich type, obtained by replacing in the above expressions the Choquet integral $(C)\int_0^1$ with the possibilistic integral $(Pos)\int_0^1$, that is with the expressions formally given by

$$D_{n,\Gamma_n}(f)(x) = \sum_{k=0}^{n} p_{n,k}(x) \cdot \frac{(Pos)\int_0^1 f(t)t^k(1-t)^{n-k}d\mu_{\lambda_{n,k}}(t)}{(Pos)\int_0^1 t^k(1-t)^{n-k}d\mu_{\lambda_{n,k}}(t)}, \tag{1}$$

and

$$K_{n,\Gamma_n}(f)(x) = \sum_{k=0}^{n} p_{n,k}(x) \cdot (Pos)\int_0^1 f\left(\frac{k+t}{n+1}\right) d\mu_{\lambda_{n,k}}(t), \tag{2}$$

where $\Gamma_n = \{\mu_{\lambda_{n,k}}\}_{k=0}^n$ is a family of possibility measures.

It is also worth mentioning that in a long series of papers finally collected into the book [1], we have proved that the nonlinear integral in possibility theory can generate the so-called max-product approximation operators, like

those of Bernstein type, Szász type, Baskakov type, of Bleimann-Butzer-Hahn type, of Meyer-König-Zeller type, of Picard type, Weierstrass type, Cauchy-Poisson type, etc, with even better approximation properties than their classical counterparts. Due to their interpretations as possibilistic expectations of some discrete possibilistic variables endowed with various discrete possibilistic distributions, all these operators are of discrete type, see, e.g., [5].

2 Preliminaries in Possibility Theory

In this section we summarize some known concepts in possibility theory which will be useful for the next considerations. As it is easily seen, in fact they are the corresponding concepts for those in probability theory, like random variable, probability distribution, mean value, probability, so on. For details, see e.g. [4] or [3].

Definition 2.1. Let Ω be a non-empty, discrete (i.e. at most countable) or non-discrete set.

(i) A possibilistic (fuzzy) variable X is an application $X : \Omega \to \mathbb{R}$. If Ω is a discrete set, then X is called discrete possibilistic variable. If Ω is not discrete, then X is called non-discrete possibilistic variable.

(ii) A possibility distribution (on Ω), is a function $\lambda : \Omega \to [0,1]$, such that $\sup\{\lambda(s); s \in \Omega\} = 1$.

(iii) The possibility expectation of a possibilistic variable X (on Ω), with the possibility distribution λ is defined by $M_{sup}(X) = \sup_{s\in\Omega} X(s)\lambda(s)$. The possibility variance of X is defined by

$$V_{sup}(X) = \sup\{(X(s) - M_{sup}(X))^2\lambda(s); s \in \Omega\}.$$

(iv) If Ω is a non-empty set, then a possibility measure is a mapping $P : \mathcal{P}(\Omega) \to [0,1]$, satisfying the axioms $P(\emptyset) = 0$, $P(\Omega) = 1$ and $P(\bigcup_{i\in I} A_i) = \sup\{P(A_i); i \in I\}$ for all $A_i \in \Omega$, and any I, an at most countable family of indices (if Ω is finite then obviously I must be finite too). Note that if $A, B \subset \Omega$, satisfy $A \subset B$, then by the last property it easily follows that $P(A) \leq P(B)$ and that $P(A \bigcup B) \leq P(A) + P(B)$.

It is well-known (see e.g. [4]) that any possibility distribution λ on Ω, induces a possibility measure $P_\lambda : \mathcal{P}(\Omega) \to [0,1]$, given by the formula $P_\lambda(A) = \sup\{\lambda(s); s \in A\}$, for all $A \subset \Omega$.

(v) (see e.g. [3]) The possibilistic integral of $f : \Omega \to \mathbb{R}_+$ on $A \subset \Omega$, with respect to the possibilistic measure P_λ induced by the possibilistic

distribution λ, is defined by

$$(Pos)\int_A f(t)dP_\lambda(t) = \sup\{f(t)\cdot\lambda(t); t\in A\}.$$

It is worth noting that for the possibility expectation and possibility variance of X (which is supposed with positive values) defined at the above point (iii), we can write the integral formulas

$$M_{sup}(X) = (Pos)\int_\Omega X(t)dP_\lambda(t),\ V_{sup}(X) = \int_\Omega (X(t)-M_{sup}(X))^2 dP_\lambda(t),$$

where P_λ is the possibility measure induced by the possibility distribution λ.

3 Approximation by Possibilistic Durrmeyer and Kantorovich Polynomial Operators

In this section we study the quantitative approximation results in terms of the modulus of continuity, for some particular possibilistic Durrmeyer and Kantorovich polynomial operators.

Let us denote $p_{n,k}(x) = \binom{n}{k}x^k(1-x)^{n-k}$. By considering the root $\frac{k}{n}$ of $p'_{n,k}(x)$, it is easy to see that $\max\{p_{n,k}(t); t\in[0,1]\} = k^k n^{-n}(n-k)^{n-k}\binom{n}{k}$.

Now, choosing in formula (1), $\lambda_{n,k}=1$ for all $n\in\mathbb{N}$, $k\in\{0,...,n\}$, they are possibility distributions on $[0,1]$ and denoting by $\mu_{\lambda_{n,k}}$ the possibility measure induced by $\lambda_{n,k}$, for any bounded $f:[0,1]\to\mathbb{R}_+$, the possibilistic Durrmeyer polynomial non-linear operators attached to f, will be defined for all $n\in\mathbb{N}$ by the formula

$$\begin{aligned} D_n(f)(x) &= (1-x)^n\cdot\sup\{f(t)(1-t)^n; t\in[0,1]\} \\ &+\sum_{k=1}^{n-1} p_{n,k}(x)\cdot\frac{\sup\{f(t)t^k(1-t)^{n-k}; t\in[0,1]\}}{k^k n^{-n}(n-k)^{n-k}} \\ &+x^n\cdot\sup\{f(t)t^n; t\in[0,1]\}. \end{aligned}$$

Notice that the above formula was mentioned for the first time in Remark 5.3 in [11], but without to make any study on it.

Firstly, we need the following two lemmas.

Lemma 3.1. *Denote by $B_+[0,1]$ the space of all bounded function defined on $[0,1]$ with values in $\mathbb{R}_+$. For all $f,g\in B_+[0,1]$, $a\geq 0$, $n\in\mathbb{N}$ and $x\in[0,1]$, we have*

$$D_n(af)(x) = aD_n(f)(x),\ D_n(f+g)(x)\leq D_n(f)(x)+D_n(g)(x)$$

and if $f(x) \leq g(x)$ *for all* $x \in [0,1]$ *then* $D_n(f)(x) \leq D_n(g)(x)$, *for all* $x \in [0,1]$.

Proof. It is immediate from the formula of $D_n(f)$ and from the properties of the supremum. □

Lemma 3.2. (Lemma 4.1 in [11]) *Let* $n \in \mathbb{N}$ *and* $x \in [0,1]$. *Denoting*

$$A_{n,k}(x) := \sup\{|t-x|t^k(1-t)^{n-k}; t \in [0,1]\} =$$

$$\max\{\sup\{(t-x)t^k(1-t)^{n-k}; t \in [x,1]\}, \sup\{(x-t)t^k(1-t)^{n-k}; t \in [0,x]\}\},$$

for all $k = 0, ..., n$ *we have*

$$A_{n,k}(x) = \max\{(t_2-x)t_2^k(1-t_2)^{n-k}, (x-t_1)t_1^k(1-t_1)^{n-k}\},$$

with $\Delta = (nx+k+1)^2 - 4kx(n+1) = (nx-k)^2 + 2x(n-k) + 2k(1-x) + 1 \geq 1$ *and* t_1, t_2 *given by*

$$t_1 = \frac{nx+k+1-\sqrt{\Delta}}{2(n+1)}, \ t_2 = \frac{nx+k+1+\sqrt{\Delta}}{2(n+1)}.$$

In particular, the above formula gives $A_{n,0}(x) = x$, $A_{n,n}(x) = 1-x$.

Corollary 3.3. *We have*

$$0 \leq t_2 - x \leq \frac{1}{2} \cdot (k/n - x + 2(1-x)/n + |x-k/n| + \sqrt{2x}/\sqrt{n} + \sqrt{2k}/n)$$

and

$$0 \leq x - t_1 \leq \frac{1}{2} \cdot (x - k/n + 2x/n + |x-k/n| + \sqrt{2x}/\sqrt{n} + \sqrt{2k}/n)$$

Proof. By Lemma 3.2 we obtain

$$0 \leq t_2 - x = \frac{(k+1)/n - x - 2x/n + \sqrt{[x+(k+1)/n]^2 - 4x(k/n)[(n+1)/n]}}{2(1+1/n)}$$

$$\leq \frac{1}{2} \cdot \left(k/n - x + (1-2x)/n + \sqrt{[x+(k+1)/n]^2 - 4x(k/n)}\right)$$

$$= \frac{1}{2} \cdot (k/n - x + (1-2x)/n + \sqrt{(x-k/n)^2 + 2x/n + 2k/n^2 + 1/n^2})$$

$$\leq \frac{1}{2} \cdot (k/n - x + (1-2x)/n + |x-k/n| + \sqrt{2x}/\sqrt{n} + \sqrt{2k}/n + 1/n).$$

By similar reasonings we get

$$0 \le x - t_1 = \frac{-(k+1)/n + x + 2x/n + \sqrt{[x+(k+1)/n]^2 - 4x(k/n)[(n+1)/n]}}{2(1+1/n)}$$

$$\le \frac{1}{2} \cdot (x - k/n + (2x-1)/n + |x - k/n| + \sqrt{2x}/\sqrt{n} + \sqrt{2k}/n + 1/n),$$

which proves the corollary. □

Theorem 3.4 *For all $f \in B_+[0,1]$, $n \in \mathbb{N}$ and $x \in [0,1]$, we have*

$$|D_n(f)(x) - f(x)| \le 2\omega_1(f; D_n(\varphi_x))_{[0,1]},$$

where $\omega_1(f;\delta)_{[0,1]} = \sup\{|f(t) - f(x)|; t, x \in [0,1], |t - x| \le \delta\}$ and $\varphi_x(t) = |t - x|$.

Proof. Since by the above Lemma 3.1 in this paper, D_n is positively homogeneous, sublinear and monotonically increasing, by Lemma 3.1 and its proof in [10], we immediately get $|D_n(f) - D_n(g)| \le D_n(|f - g|)$, for all $f, g \in B_+[0,1]$.

Denoting $e_0(t) = 1$ for all $t \in [0,1]$, since obviously $D_n(e_0)(x) = 1$ for all $x \in [0,1]$, by the above mentioned properties of D_n, for any fixed x we obtain

$$|D_n(f)(x) - f(x)| = |D_n(f(t))(x) - D_n(f(x))(x)| \le D_n(|f(t) - f(x)|)(x). \tag{3}$$

But taking into account the properties of the modulus of continuity, for all $t, x \in [0,1]$ and $\delta > 0$, we get

$$|f(t) - f(x)| \le \omega_1(f; \|t - x\|)_I \le \left[\frac{1}{\delta}\|t - x\| + 1\right] \omega_1(f;\delta)_I. \tag{4}$$

Now, from (3) and applying D_n to (4), by the properties of D_n, we immediately get

$$|D_n(f)(x) - f(x)| \le \left[\frac{1}{\delta} D_n(\varphi_x)(x) + 1\right] \omega_1(f;\delta)_{[0,1]}.$$

Choosing here $\delta = D_n(\varphi_x)(x)$, we arrive at the desired estimate. □

Corollary 3.5. *For all $f \in B_+[0,1]$, $n \in \mathbb{N}$ and $x \in [0,1]$, we have*

$$|D_n(f)(x) - f(x)| \le 2\omega_1\left(f; \frac{\sqrt{x(1-x)} + 2\sqrt{2x} + 1}{\sqrt{n}}\right)_{[0,1]}.$$

Proof. Since $\frac{(t_2-x)t_2^k(1-t_2)^{n-k}}{k^k n^{-n}(n-k)^{n-k}} \leq 1$ and $\frac{(t_1-x)t_1^k(1-t_1)^{n-k}}{k^k n^{-n}(n-k)^{n-k}} \leq 1$ and since by Lemma 3.2 and Corollary 3.3 , we get

$$\frac{A_{n,k}(x)}{k^k n^{-n}(n-k)^{n-k}} \leq (t_2-x)\cdot\frac{t_2^k(1-t_2)^{n-k}}{k^k n^{-n}(n-k)^{n-k}} + (x-t_1)\cdot\frac{t_1^k(1-t_1)^{n-k}}{k^k n^{-n}(n-k)^{n-k}}$$

$$\leq \frac{1}{2}\cdot(k/n - x + 2(1-x)/n + |x-k/n| + \sqrt{2x}/\sqrt{n} + \sqrt{2k}/n)$$

$$+\frac{1}{2}\cdot(x - k/n + 2x/n + |x-k/n| + \sqrt{2x}/\sqrt{n} + \sqrt{2k}/n),$$

this immediately implies

$$D_n(\varphi_x)(x) \leq \frac{1}{2}\sum_{k=0}^{n} p_{n,k}(x)(k/n - x + 2(1-x)/n + |x-k/n| + \sqrt{2x}/\sqrt{n} + \sqrt{2k}/n)$$

$$+\frac{1}{2}\sum_{k=0}^{n} p_{n,k}(x)(x - k/n + 2x/n + |x-k/n| + \sqrt{2x}/\sqrt{n} + \sqrt{2k}/n)$$

$$= \frac{1}{n} + \frac{\sqrt{2x}}{\sqrt{n}} + \sum_{k=0}^{n} p_{n,k}(x)|x-k/n| + \frac{\sqrt{2}}{\sqrt{n}}\cdot\sum_{k=0}^{n} p_{n,k}(x)\cdot\sqrt{\frac{k}{n}}$$

$$\leq \frac{1}{n} + \frac{\sqrt{2x}}{\sqrt{n}} + \frac{\sqrt{x(1-x)}}{\sqrt{n}} + \frac{\sqrt{2}}{\sqrt{n}}\cdot\sqrt{x}.$$

Above we have used the well-known estimate $\sum_{k=0}^{n} p_{n,k}(x)|x-k/n| \leq \frac{\sqrt{x(1-x)}}{\sqrt{n}}$ and the inequality $B_n(f)(x) \leq \sqrt{B_n(f^2)(x)}$ (Cauchy-Schwarz inequality for Bernstein polynomials), applied in the case of $f(t) = \sqrt{t}$. Applying now Theorem 3.4, the corollary is immediate. □

Remark 3.6. If we denote by $C_+[0,1]$ the space of all positive and continuous functions on $[0,1]$, then from Corollary 3.5 and using the properties of the modulus of continuity, it is immediate that for $f \in C_+[0,1]$, the sequence $D_n(f)$, $n \in \mathbb{N}$, of possibilistic Durrmeyer polynomial operators converges uniformly to f on $[0,1]$.

In what follows, we deal with the possibilistic Kantorovich operators, introduced by the formula (2), for $\lambda_{n,k}(t) = 1$, for all $t \in [0,1]$ and $n \in \mathbb{N}$,and $k \in \{0,...,n\}$ in (2), that is

$$K_n(f)(x) = \sum_{k=0}^{n} p_{n,k}(x)\cdot\sup\{f(t); t \in [k/(n+1),(k+1)/(n+1)]\}. \quad (5)$$

It is immediate that $K_n(f)(x)$ on $B_+[0,1]$ has the following properties :

$$K_n(af)(x) = aK_n(f)(x),\ K_n(f+g)(x) \leq K_n(f)(x) + K_n(g)(x),$$

$f(x) \leq g(x)$ for all $x \in [0,1]$ implies $K_n(f)(x) \leq K_n(g)(x)$, for all $x \in [0,1]$ and finally, $|K_n(f)(x) - K_n(g)(x)| \leq K_n(|f-g|)(x)$, for all $x \in [0,1]$.

Also, similar reasoning with those for $D_n(f)(x)$ immediately leads to the general quantitative estimate

$$|K_n(f)(x) - f(x)| \leq 2\omega_1(f; K_n(\varphi_x))_{[0,1]},$$

where $\omega_1(f;\delta)_{[0,1]} = \sup\{|f(t) - f(x)|; t, x \in [0,1], |t-x| \leq \delta\}$ and $\varphi_x(t) = |t-x|$.

We are now in position to state the following result.

Theorem 3.7. *For all $f \in B_+[0,1]$, $n \in \mathbb{N}$ and $x \in [0,1]$, for $K_n(f)$ given by (5), we have*

$$|K_n(f)(x) - f(x)| \leq 2\omega_1\left(f; \frac{\sqrt{x(1-x)}}{\sqrt{n}} + \frac{2}{n+1}\right)_{[0,1]}.$$

Proof. From the above estimate, it is good enough if we find an upper estimate for $K_n(\varphi_x)$.

We get

$$K_n(\varphi_x)(x) = \sum_{k=0}^{n} p_{n,k}(x) \cdot \sup\{|x-t|; t \in [k/(n+1), (k+1)/(n+1)]\},$$

where simple reasonings leads to the estimate

$$\sup\{|x-t|; t \in [k/(n+1), (k+1)/(n+1)]\}$$

$$\leq \max\{\frac{1}{n+1}, |x - k/(n+1)|, |x - (k+1)/(n+1)|\}$$

$$\leq \max\{\frac{1}{n+1}, |x-k/(n+1)|, \frac{1}{n+1} + |x-k/(n+1)|\} = \frac{1}{n+1} + |x-k/(n+1)|$$

$$\leq \frac{1}{n+1} + |x - k/n| + |k/n - k/(n+1)| \leq \frac{2}{n+1} + |x - k/n|.$$

Therefore, it easily follows

$$K_n(\varphi_x)(x) \leq \frac{2}{n+1} + B_n(\varphi_x)(x) \leq \frac{2}{n+1} + \frac{\sqrt{x(1-x)}}{\sqrt{n}},$$

where we have denoted by B_n the classical Bernstein polynomials of degree n, fact which easily leads to the required conclusion. $\square$

Remark 3.8. From Theorem 3.7 it follows that for $f \in C_+[0,1]$, the sequence $K_n(f)$, $n \in \mathbb{N}$, of possibilistic Kantorovich polynomial operators converges uniformly to f on $[0,1]$.

Remark 3.9. With respect to the classical Kantorovich polynomials denoted here by

$$\tilde{K}_n(f)(x) = \sum_{k=0}^{n} p_{n,k}(x) \cdot (n+1) \cdot \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)dt$$

$$= \sum_{k=0}^{n} p_{n,k}(x) \cdot \int_0^1 f\left(\frac{k+t}{n+1}\right) dt,$$

the possibilistic Kantorovich polynomials $K_n(f)(x)$ considered in this paper, have the advantage that for some particular classes of functions f, they can be written explicitly, while the classical Kantorovich polynomials $\tilde{K}_n(f)(x)$ cannot be written explicitly.

For example, for $f(x) = e^{x^2}$, the classical Kantorovich polynomials $\tilde{K}_n(f)(x)$ cannot be written in an explicit form, because the exact value of the integral $\int_{k/(n+1)}^{(k+1)/(n+1)} e^{t^2} dt$ cannot be described in terms of elementary functions.

On the other hand, since e^{x^2} is strictly increasing on $[0,1]$, for the possibilisitic Kantorovich polynomials given by (5), we immediately get the explicit/exact form

$$K_n(f)(x) = \sum_{k=0}^{n} p_{n,k}(x) e^{(k+1)^2/(n+1)^2}.$$

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