Ann. Acad. Rom. Sci.
Ser. Math. Appl.
ISSN 2066-6594 Vol. 12, No. 1-2/2020

ON CAUCHY'S TYPE BOUND FOR ZEROS OF A POLYNOMIAL*

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DOI https://doi.org/10.56082/annalsarscimath.2020.1-2.117

Dedicated to Dr. Vasile Drăgan on the occasion of his 70^{th} anniversary

Abstract

Let $p(z)$ be a polynomial of degree n with real or complex coefficients. Using the Lacunary type polynomial, Gugenheimer generalized the Cauchy bound concerning the moduli of zeros of a polynomial $p(z)$. Jain further improved the Gugenheimer bound. In the present paper an attempt to investigate and extend the previous results were made. In many cases we found that the new bounds are much better than some of the other well-known bounds.
MSC: 30C15, 30C10, 26C10.

keywords: Zeros, Cauchy bound, circular region, Lacunary type polynomial.

1 Introduction and statement of results

The problem of computing the bounds of polynomial zeros has a long history. In recent years, numerous papers (see [1, 3, 4, 7, 9, 10, 12, 15, 16]) and comprehensive books (see also [8, 11]) have been published to determine the circular region for estimating the bounds of polynomial zeros with real or

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complex coefficients. A classical result obtained by Cauchy [2] (see also [8, Ch. VII, Sect. 27, Them. 27.2]) concerning the bounds for the moduli of the zeros of a polynomial can be stated as follows.

Theorem A. *All the zeros of the polynomial*

$$p(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$$

of degree n, *lie in an open circular region*

$$|z| < 1 + A,$$

where

$$A = \max_{1 \leq j \leq n} |a_j|.$$

Guggenheimer [5] obtained a generalization of this result by using a lacunary type polynomial [8, Ch. VIII, Sect. 34, pp. 156] and proved

Theorem B. *All the zeros of the polynomial*

$$g(z) = z^n + a_p z^{n-p} + \cdots + a_{n-1} z + a_n, \ a_p \neq 0, \ p < n,$$

of degree n, *lie in an open circular region*

$$|z| < \delta,$$

where δ (> 1) *is the unique positive root of the equation*

$$t^p - t^{p-1} = |a_q|, \tag{1}$$

and

$$|a_q| = \max_{p \leq k \leq n} |a_k|.$$

Jain [13] improved Gugenheimer bound and obtained

Theorem C. *All the zeros of the polynomial* $g(z)$ *of degree* n, *lie in an open circular region*

$$|z| < t_0,$$

where t_0 $(< \delta)$ *is the unique positive root greater than* 1 *of the equation*

$$t^p - t^{p-1} = |a_q| \left(1 - \frac{1}{\delta^{n-p+1}}\right), \tag{2}$$

and $\delta, |a_q|$ *are as in Theorem B.*

Jain [14] also improved Gugenheimer bound by using a class of lacunary type polynomial

$$f(z) = z^n + a_p z^{n-p} + a_t z^{n-t} + a_{t+1} z^{n-t-1} + \cdots + a_n,\ a_p \neq 0,\ p < t < n,$$

and proved

Theorem D. *All the zeros of the polynomial $f(z)$ of degree n, lie in an open circular region*

$$|z| < k,$$

where k $(\leq \delta)$ is the unique positive root greater than 1 of the equation

$$x^t - x^{t-1} - |a_p| x^{t-p} + |a_p| x^{t-p-1} - |a_s| = 0, \tag{3}$$

and

$$|a_s| = \max_{t \leq j \leq n} |a_j|,$$

δ is same as in Theorem B.

In this paper we obtain two bounds of Cauchy's type. First result gives better bound that obtained in Theorem D and the second result turned out to be an improvement of Theorem C. Moreover, our first result is best possible. More precisely, we prove

Theorem 1.1. *All the zeros of the polynomial $f(z)$ of degree n, lie in an open circular region*

$$|z| \leq 1 + \lambda_0 R,$$

where λ_0 is the greatest positive root of the equation

$$\zeta(\lambda) \equiv \lambda R \left\{ (1+\lambda R)^n - |a_p| (1+\lambda R)^{n-p} \right\} - |a_s| \left\{ (1+\lambda R)^{n-t+1} - 1 \right\} = 0, \tag{4}$$

in the interval $[0, 1)$,

$$R = k - 1\ \ (> 0),$$

and k, $|a_s|$ are as in Theorem D.

Remark 1.1. The limit in Theorem 1.1 is best possible and attained for the polynomial

$$z^n - |a_p| z^{n-p} - |a_s| \left(1 + z + \cdots + z^{n-t}\right) = 0,\ a_p \neq 0,\ a_t \neq 0,\ p < t < n.$$

Remark 1.2. The bound in Theorem 1.1 is better than the bound obtained from Theorem D, as it can be seen by observing that

$$1 + \lambda_0 R = 1 + \lambda_0 (k-1) < k\ \ (\text{as } \lambda_0 < 1).$$

Theorem 1.2. *All the zeros of the polynomial $f(z)$ of degree n, lie in an open circular region*

$$|z| < k_0,$$

where k_0 (>1) is the unique positive root of the equation

$$F(x) \equiv x^t - x^{t-1} - |a_p| x^{t-p} + |a_p| x^{t-p-1} - |a_s| \left(1 - \frac{1}{k^{n-t+1}}\right) = 0, \quad (5)$$

and $k, |a_s|$ are as in Theorem D.

As $F(k) = \frac{|a_s|}{k^{n-t+1}} > 0$ which implies $k > k_0$. Therefore, the bound in Theorem 1.2 is also an improvement bound as compared to the bound in Theorem D.

Remark 1.3. The bound in Theorem 1.2 is an improvement of Theorem C which can be seen by the following observation:
From the Theorems B, C and D, it is clear that

$$k \leq \delta \text{ and } t_0 < \delta,$$

where δ, t_0, k are the roots of the equations $(1), (2)$ and (3) respectively. So, we have

$$t_0^p - t_0^{p-1} - |a_q| \left(1 - \frac{1}{\delta^{n-p+1}}\right) = 0. \quad (6)$$

Also,

$$|a_p| \leq |a_q| \text{ and } |a_s| \leq |a_q|. \quad (7)$$

Then

$$t_0^{n-t+1} F(t_0) = t_0^{n-t+1} \left(t_0^t - t_0^{t-1} - |a_p| t_0^{t-p} + |a_p| t_0^{t-p-1} - |a_s| + \frac{|a_s|}{k^{n-t+1}}\right)$$

$$= t_0^{n-p+1} \left(t_0^p - t_0^{p-1}\right) - |a_p| t_0^{n-p+1} + |a_p| t_0^{n-p} - |a_s| \left(1 - \frac{1}{k^{n-t+1}}\right) t_0^{n-t+1}$$

$$\geq |a_q| \left(1 - \frac{1}{\delta^{n-p+1}}\right) t_0^{n-p+1} - |a_q| \left(1 - \frac{1}{k^{n-t+1}}\right) t_0^{n-t+1} - |a_p| t_0^{n-p} (t_0 - 1)$$

(by 6 & 7)

$$= \left\{|a_q| t_0^{n-p} \left(t_0 - t_0^{p-t+1}\right) - |a_p| t_0^{n-p} (t_0 - 1)\right\} + \left\{|a_q| \left(\frac{t_0^{n-t+1}}{k^{n-t+1}} - \frac{t_0^{n-p+1}}{\delta^{n-p+1}}\right)\right\}.$$

Clearly,

$$p - t + 1 \leq 0 \,(\text{as } p < t < n).$$

If $p-t+1=0$, then $t_0-t_0^{p-t+1}=t_0-1$, and if $p-t+1<0$, then $t_0-t_0^{p-t+1}>t_0-1$.
Hence

$$|a_q|\,t_0^{n-p}\left(t_0-t_0^{p-t+1}\right)-|a_p|\,t_0^{n-p}\left(t_0-1\right)\geq 0.$$

Also $k\leq\delta$, which implies

$$\begin{aligned}|a_q|\left(\frac{t_0^{n-t+1}}{k^{n-t+1}}-\frac{t_0^{n-p+1}}{\delta^{n-p+1}}\right) &= |a_q|\left(\frac{t_0^{n-t+1}}{k^{n-t+1}}-\frac{t_0^{n-p+1}}{\delta^{n-t+1}\delta^{t-p}}\right)\\ &\geq |a_q|\left(\frac{t_0^{n-t+1}}{k^{n-t+1}}-\frac{t_0^{n-p+1}}{k^{n-t+1}\delta^{t-p}}\right)\\ &= |a_q|\frac{t_0^{n-t+1}}{k^{n-t+1}}\left(1-\left(\frac{t_0}{\delta}\right)^{t-p}\right)>0\,(\text{as } t_0<\delta)\,.\end{aligned}$$

Therefore,

$$\left\{|a_q|\,t_0^{n-p}\left(t_0-t_0^{p-t+1}\right)-|a_p|\,t_0^{n-p}\left(t_0-1\right)\right\}+\left\{|a_q|\left(\frac{t_0^{n-t+1}}{k^{n-t+1}}-\frac{t_0^{n-p+1}}{\delta^{n-p+1}}\right)\right\}>0,$$

i.e., $F\left(t_0\right)>0$, and hence $t_0>k_0$.

Remark 1.4. In many cases, our results give better bound than those given by other results. To illustrate this, we consider the polynomial

$$p\left(z\right)=z^8-z^5+z^2-100z+5.$$

Here

$$n=8,p=3,t=6,a_0=1,a_1=0,a_2=0,$$

$$a_3=-1,a_4=0,a_5=0,a_6=1,a_7=-100,a_8=5.$$

Applying C- programming for estimating the real root, we obtained

$$\begin{aligned}\delta &= 5,t_o=4.999901536,k=2.427073007,\\ R &= 1.427073007,\lambda=.954074805,k_o=2.403822443.\end{aligned}$$

Therefore all the zeros of $p(z)$ lie in the regions

$$
\begin{array}{rrcl}
(i) & |z| & < & 101, \text{ by Theorem A}, \\
(ii) & |z| & < & 5, \text{ by Theorem B}, \\
(iii) & |z| & < & 4.999902, \text{ by Theorem C}, \\
(iv) & |z| & < & 178.5555, \text{ by Boese and Luther [4, } Theorem \text{ 1]}, \\
(v) & |z| & < & 10.512492, \text{ by Mohammad [10, } Theorem \text{ 2]}, \\
(vi) & |z| & < & 100.99995 \text{ by Dehmer[7, } Theorem \text{ 3.2]}, \\
(vii) & |z| & < & 106.99995 \text{ by Dehmer[7, } Theorem \text{ 3.3]}, \\
(viii) & |z| & < & 71.444453, \text{ by Jain [16, } Theorem \text{ 1]}, \\
(ix) & |z| & < & 4.999901, \text{ by Das[12, } Theorem \text{ 1.1]}, \\
(x) & |z| & < & 3.426144, \text{ by Jain[15, } Theorem \text{ 1]}, \\
(xi) & |z| & < & 3.490667, \text{ by Jain[15, } Theorem \text{ 2]}, \\
(xii) & |z| & < & 33.401056, \text{ for } t=1, \text{ by Zargar[3, } Theorem \text{ 2]}, \\
(xiii) & |z| & < & 99.959984, \text{ by Jain and Tewary[17, } Theorem \text{]}, \\
(xiv) & |z| & < & 3.153542, \text{ by Lagrange[6](also see [9, } Theorem \text{ 1.1])}, \\
(xv) & |z| & < & 2.849804, \text{ by Batra, Mignotte and Stefanescu[9, } Theorem \text{ 3.1]}, \\
(xvi) & |z| & < & 6.048342, \text{ by Aziz and Rather[1, } Theorem \text{ 1]},
\end{array}
$$

whereas Theorem 1.1 and 1.2 give

$$|z| < 2.361584$$

and

$$|z| < 2.403823$$

respectively.

2 Lemmas

The following lemmas are needed to prove the Theorem 1.1 and 1.2 respectively.

Lemma 2.1. *Let*

$$\zeta(\lambda) = \lambda R\left\{(1+\lambda R)^n - |a_p|(1+\lambda R)^{n-p}\right\} - |a_s|\left((1+\lambda R)^{n-t+1} - 1\right),$$

where $R, |a_p|, |a_s|, p, t$ and n are same as in Theorem 1.1. Then zero is only one root of the equation $\zeta(\lambda) = 0$ when $1 - |a_p| - |a_s|(n-t+1) \geq 0$.

Moreover, if $1-|a_p|-|a_s|(n-t+1)<0$, *then* $\lambda_0\,(<1)$ *is an additional root of the equation* $\zeta(\lambda)=0$ *in* $[0,\infty)$.

Proof. We can write

$$\begin{aligned}\zeta(\lambda) &= \lambda R\left\{(1+\lambda R)^n-|a_p|(1+\lambda R)^{n-p}\right\}-|a_s|\left((1+\lambda R)^{n-t+1}-1\right)\\ &= \lambda R\Gamma(\lambda),\end{aligned}$$

where

$$\begin{aligned}\Gamma(\lambda) &= (1-|a_p|-|a_s|(n-t+1))\\ &\quad+\sum_{k=2}^{n-t+1}\left\{\binom{n}{k-1}-|a_p|\binom{n-p}{k-1}-|a_s|\binom{n-t+1}{k}\right\}(\lambda R)^{k-1}\\ &\quad+\sum_{k=n-t+1}^{n-p}\left\{\binom{n}{k}-|a_p|\binom{n-p}{k}\right\}(\lambda R)^k+\sum_{k=n-p+1}^{n}\binom{n}{k}(\lambda R)^k. \qquad (8)\end{aligned}$$

Now,

$$\binom{n}{k-1}-|a_p|\binom{n-p}{k-1}-|a_s|\binom{n-t+1}{k}$$

$$\begin{aligned}&=\frac{n!k}{k!\,(n-k+1)!}-|a_p|\frac{(n-p)!k}{k!\,(n-p-k+1)!}-|a_s|\frac{(n-t+1)!}{k!\,(n-k-t+1)!}\\ &=\frac{(n-t+1)!}{k!\,(n-k+1)!}\left\{\begin{array}{c}kn(n-1)\cdots(n-t+2)\\ -|a_p|\frac{(n-p)!k}{(n-t+1)!}(n-k+1)(n-k)\cdots(n-k-p+2)\\ -|a_s|(n-k+1)(n-k)\cdots(n-k-t+2)\end{array}\right\}\\ &=\frac{(n-t+1)!}{k!\,(n-k+1)!}(k\tau_k-\kappa_k)\end{aligned}$$

and

$$\binom{n}{k}-|a_p|\binom{n-p}{k}$$

$$\begin{aligned}&=\frac{(n-p)!}{k!\,(n-k)!}\left\{\begin{array}{c}n(n-1)\cdots(n-p+1)\\ -|a_p|(n-k)(n-k-1)\cdots(n-k-p+1)\end{array}\right\}\\ &=\frac{(n-p)!}{k!\,(n-k)!}\nu_k,\end{aligned}$$

where

$$\begin{aligned}
\tau_k &= \prod_{i=0}^{t-2}(n-i) - |a_p| \frac{(n-p)!}{(n-t+1)!} \prod_{j=0}^{p-1}(n-k+1-j)\,, \\
\kappa_k &= |a_s| \prod_{l=0}^{t-1}(n-k+1-l)\,, \\
\nu_k &= \prod_{i=0}^{p-1}(n-i) - |a_p| \prod_{j=0}^{p-1}(n-k-j)\,.
\end{aligned}$$

Using above, (8) becomes

$$\begin{aligned}
\Gamma(\lambda) = {} & (1-|a_p|-|a_s|(n-t+1)) \\
& + \sum_{k=2}^{n-t+1} \frac{(n-t+1)!\,(k\tau_k-\kappa_k)}{k!\,(n-k+1)!}(\lambda R)^{k-1} \\
& + \sum_{k=n-t+1}^{n-p} \frac{(n-p)!\nu_k}{k!\,(n-k)!}(\lambda R)^k + \sum_{k=n-p+1}^{n} \binom{n}{k}(\lambda R)^k\,.
\end{aligned}$$

It is clear that the expressions

$$\begin{aligned}
k\tau_k - \kappa_k = k & \left\{ \prod_{i=0}^{t-2}(n-i) - |a_p| \frac{(n-p)!}{(n-t+1)!} \prod_{j=0}^{p-1}(n-k+1-j) \right\} \\
& - |a_s| \prod_{l=0}^{t-1}(n-k+1-l)
\end{aligned}$$

and

$$\nu_k = \prod_{i=0}^{p-1}(n-i) - |a_p| \prod_{j=0}^{p-1}(n-k-j)$$

are monotonically increasing for $2 \le k \le n-t+1$ and $n-t+1 \le k \le n-p$ respectively.

Now there are two possibilities.

Case 1: If $1-|a_p|-|a_s|(n-t+1) \ge 0$, then the coefficient of λR in $\Gamma(\lambda)$

is

$$
\begin{aligned}
& \frac{(n-t+1)!\,(2\tau_2-\kappa_2)}{2!\,(n-1)!} \\
= \;& n-|a_p|\,(n-p)-|a_s|\,\frac{(n-t+1)\,(n-t)}{2} \\
\geq \;& n-|a_s|\,\frac{(n-t+1)\,(n-t)}{2}-(n-p)\,\{1-|a_s|\,(n-t+1)\} \\
& (\text{since } |a_p|\leq 1-|a_s|\,(n-t+1)) \\
= \;& \frac{1}{2}\,[2p+|a_s|\,(n-t+1)\,\{2\,(n-p)-(n-t)\}]>0 \;\; (\text{as } p<t<n)\,.
\end{aligned}
$$

Also the coefficient of $(\lambda R)^{n-t+1}$ in $\Gamma(\lambda)$ is

$$
\begin{aligned}
& \frac{(n-p)!\nu_{n-t+1}}{(n-t+1)!\,(t-1)!} \\
= \;& \binom{n}{n-t+1}-|a_p|\binom{n-p}{n-t+1} \;\; (\text{by } 8) \\
= \;& \binom{n}{n-t+1}-|a_p|\,\frac{(n-p)!}{(n-t+1)!\,(t-p-1)!} \\
= \;& \binom{n}{n-t+1}-|a_p|\,\frac{n!}{(n-t+1)!\,(t-1)!}\,\frac{(t-1!)\,(n-p)!}{n!\,(t-p-1)!} \\
= \;& \binom{n}{n-t+1}-|a_p|\binom{n}{n-t+1}\left(\frac{t-1}{n}.\frac{t-2}{n-1}.\cdots.\frac{t-p}{n-p+1}\right) \\
> \;& \binom{n}{n-t+1}-|a_p|\binom{n}{n-t+1} \;\; (\text{as } t<n) \\
= \;& (1-|a_p|)\binom{n}{n-t+1} \\
\geq \;& |a_s|\,(n-t+1)\binom{n}{n-t+1}>0 \;\; (\text{as } 1-|a_p|-|a_s|\,(n-t+1)\geq 0)\,.
\end{aligned}
$$

As the coefficients of λR and $(\lambda R)^{n-t+1}$ of $\Gamma(\lambda)$ are positive and the expressions $k\tau_k-\kappa_k$ and ν_k are monotonically increasing for $2\leq k\leq n-t+1$ and $n-t+1\leq k\leq n-p$ respectively, therefore all the coefficients of $\Gamma(\lambda)$ are positive. By Descarte's rule of sign for a polynomial we can easily say that $\Gamma(\lambda)=0$ has no positive root. Hence, $\zeta(\lambda)=0$ has exactly one root, say $\lambda_0=0$ in $[0,\infty)$.
Case 2: If $1-|a_p|-|a_s|\,(n-t+1)<0$, then the expressions $k\tau_k-\kappa_k$ and ν_k are monotonically increasing for $2\leq k\leq n-t+1$ and $n-t+1\leq k\leq n-p$

respectively, and in particular

$$\begin{aligned}
&\nu_{n-t+1}+\kappa_{n-t+1}-(n-t+1)\,\tau_{n-t+1}\\
=&\left\{\begin{array}{c}\prod_{i=0}^{p-1}(n-i)-|a_p|\prod_{j=0}^{p-1}(t-j-1)+|a_s|\prod_{l=0}^{t-1}(t-l)\\ -(n-t+1)\left\{\prod_{i=0}^{t-2}(n-i)-|a_p|\frac{(n-p)!}{(n-t+1)!}\prod_{j=0}^{p-1}(t-j)\right\}\end{array}\right\}\\
=&\left\{\begin{array}{c}\left\{\prod_{i=0}^{p-1}(n-i)-\prod_{i=0}^{t-1}(n-i)\right\}+|a_s|\prod_{l=0}^{t-1}(t-l)\\ +|a_p|\left\{\frac{(n-p)!}{(n-t)!}\prod_{j=0}^{p-1}(t-j)-\prod_{j=0}^{p-1}(t-j-1)\right\}\end{array}\right\}\\
=&\left\{\begin{array}{c}\{(n-t)\,(n-t-1)\cdots(n-p+1)-1\}\prod_{i=0}^{t-1}(n-i)\\ +|a_s|\prod_{l=0}^{t-1}(t-l)+|a_p|\frac{(n-p)!(t-p)}{(n-t)!}\prod_{j=0}^{p-2}(t-j)\end{array}\right\}>0
\end{aligned}$$

$$i.e.,\ \nu_{n-t+1} > (n-t+1)\,\tau_{n-t+1}-\kappa_{n-t+1},$$

so the maximum number of changes of sign in the coefficients of $\Gamma(\lambda)$ is one. By Descarte's rule of sign, $\Gamma(\lambda)=0$ has exactly one zero in $(0,\infty)$.
Now

$$\Gamma(0)=1-|a_p|-|a_s|\,(n-t+1)<0,$$

and

$$\Gamma(1)=\frac{\zeta(1)}{R}=\frac{|a_s|}{R}>0.$$

Hence $\Gamma(\lambda)=0$ has exactly one root, say λ_0, lies in $(0,1)$. This leads us to the desired result. □

Lemma 2.2. *Let*

$$\psi(\lambda)=\lambda R\left\{(1+\lambda R)^{t-1}-|a_p|\,(1+\lambda R)^{t-p-1}\right\}-|a_s|,$$

where $R, |a_p|, |a_s|, p, t$ *and* n *are same as in Theorem* 1.1. *Then* 1 *is exactly one positive root of the equation* $\psi(\lambda)=0$ *in* $[0,\infty)$, *and hence*

$$\psi(\lambda)<0 \text{ for } 0<\lambda<1.$$

Proof. We have

$$\begin{aligned}
\psi(\lambda) &= \lambda R\left\{(1+\lambda R)^{t-1} - |a_p|\,(1+\lambda R)^{t-p-1}\right\} - |a_s| \\
&= -|a_s| + \sum_{k=0}^{t-p-1}\left\{\binom{t-1}{k} - |a_p|\binom{t-p-1}{k}\right\}(\lambda R)^{k+1} \\
&\quad + \sum_{k=t-p}^{t-1}\binom{t-1}{k}(\lambda R)^{k+1} \\
&= -|a_s| + \sum_{k=0}^{t-p-1}\frac{(t-p-1)!\,\omega_k}{k!\,(t-k-1)!}(\lambda R)^{k+1} + \sum_{k=t-p}^{t-1}\binom{t-1}{k}(\lambda R)^{k+1},
\end{aligned}$$

where

$$\omega_k = (t-1)(t-2)\cdots(t-p) - |a_p|\,(t-k-1)(t-k-2)\cdots(t-k-p)$$

Clearly, ω_k is monotonically increasing for $0 \le k \le t-p-1$. Using Descarte's rule of sign we can say that the equation $\psi(\lambda) = 0$ has exactly one positive root in $[0,\infty)$.
Now

$$\begin{aligned}
\psi(1) &= R\left\{(1+R)^{t-1} - |a_p|\,(1+R)^{t-p-1}\right\} - |a_s| \\
&= (1+R)^t - (1+R)^{t-1} - |a_p|\,(1+R)^{t-p} + |a_p|\,(1+R)^{t-p-1} - |a_s| \\
&= 0\,(\text{by } (3)\ \&\ 1+R=k)\,.
\end{aligned}$$

So $\psi(\lambda) = 0$ has exactly one positive root, namely 1, in $[0,\infty)$, and $\psi(0) = -|a_s| < 0$.
Hence,

$$\psi(\lambda) < 0 \text{ for } 0 < \lambda < 1.$$

□

3 Proof of theorems

Proof of Theorem 1.1. If $1 - |a_p| - |a_s|\,(n-t+1) \ge 0$, then by lemma 2.1 $\lambda_0 = 0$.
On $|z| = s > 1$,

$$\begin{aligned}
|f(z)| &\ge z^n - |a_p|\,z^{n-p} - |a_s|\left(1 + |z| + \cdots + |z|^{n-t}\right) \\
&> s^n\left\{1 - \frac{|a_p|}{s^p} - \frac{|a_s|}{s^t}(n-t+1)\right\} \\
&> s^n\left\{1 - |a_p| - |a_s|\,(n-t+1)\right\} \ge 0,
\end{aligned}$$

and consequently, all the zeros of $f(z)$ lie in $|z| \leq 1 + \lambda_0 R = 1$.
If $1 - |a_p| - |a_s|(n-t+1) < 0$, then using lemma 2.1

$$\zeta(\lambda) > 0 \text{ for } \lambda > \lambda_0 \, (0 < \lambda_0 < 1).$$

On $|z| = 1 + \lambda R$,

$$\begin{aligned} |f(z)| &\geq (1+\lambda R)^n - |a_p|(1+\lambda R)^{n-p} - |a_s| \frac{(1+\lambda R)^{n-t+1} - 1}{\lambda R} \\ &= \frac{1}{\lambda R}\zeta(\lambda) > 0 \text{ for } \lambda > \lambda_0. \end{aligned}$$

So $f(z)$ has all the zeros lying in the disc

$$|z| \leq 1 + \lambda_0 R$$

and this proves the desired result. □

Proof of Theorem 1.2. It is clear that

$$\zeta(\lambda) = (1+\lambda R)^{n-t+1} \psi(\lambda) + |a_s|,$$

where

$$\psi(\lambda) = \lambda R \left\{ (1+\lambda R)^{t-1} - |a_p|(1+\lambda R)^{t-p-1} \right\} - |a_s|.$$

Now, for $0 < \lambda < 1$,

$$(1+\lambda R)^{n-t+1} < (1+R)^{n-t+1}.$$

Applying Lemma 2.2, we have

$$(1+\lambda R)^{n-t+1} \psi(\lambda) > (1+R)^{n-t+1} \psi(\lambda) \text{ when } 0 < \lambda < 1,$$

which implies

$$\begin{aligned} \zeta(\lambda) &> (1+R)^{n-t+1} \left[\lambda R \left\{ (1+\lambda R)^{t-1} - |a_p|(1+\lambda R)^{t-p-1} \right\} - |a_s| \right] + |a_s| \\ &= (1+R)^{n-t+1} F(1+\lambda R) \text{ whenever } 0 < \lambda < 1. \end{aligned}$$

It is obvious from (5)

$$F(1+\lambda R) \geq 0 \text{ iff } 1 + \lambda R \geq k_0$$

$$\text{i.e., } \lambda \geq \frac{k_0 - 1}{R} = \frac{d_1}{R}, \text{ say.}$$

Also

$$\begin{aligned} F(k) &= k^t - k^{t-1} - |a_p| k^{t-p} + |a_p| k^{t-p-1} - |a_s| \left(1 - \frac{1}{k^{n-t+1}}\right) \\ &= \frac{|a_s|}{k^{n-t+1}} > 0, \end{aligned}$$

where k is a root of (3).
It implies $k > k_0$ i.e., $1 + R > k_0$ (mentioned in Theorem 1.1).
So we have

$$0 < \frac{d_1}{R} = \frac{k_0 - 1}{R} < 1.$$

Therefore

$$\zeta(\lambda) > 0 \text{ for } \frac{d_1}{R} \leq \lambda < 1.$$

Let us consider two possibilities:
(i) If $1 - |a_p| - |a_s|(n - t + 1) \geq 0$, then by lemma 2.1 $\zeta(\lambda) = 0$ has exactly one root $\lambda_0 = 0$ in $[0, \infty)$.
Hence,

$$\zeta(\lambda) > 0 \text{ for } \lambda > \lambda_0 = 0.$$

(ii) If $1 - |a_p| - |a_s|(n - t + 1) < 0$, then by lemma 2.1 $\zeta(\lambda) = 0$ has exactly two roots, namely zero and λ_0 in $(0, \infty)$.
Hence,

$$\zeta(\lambda) > 0 \text{ for } \lambda > \lambda_0$$

and

$$\zeta(\lambda) < 0 \text{ for } 0 < \lambda < \lambda_0.$$

Combining both possibilities, we get

$$\zeta(\lambda) > 0 \text{ for } \lambda > \lambda_0$$

and

$$\zeta(\lambda) \leq 0 \text{ for } 0 \leq \lambda \leq \lambda_0 < 1.$$

As

$$\zeta(\lambda) > 0 \text{ for } \frac{d_1}{R} \leq \lambda < 1$$

and

$$\zeta(\lambda) > 0 \text{ for } \lambda > \lambda_0,$$

which imply

$$\lambda_0 < \frac{d_1}{R} = \frac{k_0 - 1}{R} < 1.$$

By Theorem 1.1, all the zeros $f(z)$ lie in

$$\begin{aligned} |z| &\leq 1+\lambda_0 R, \\ i.e.\ |z| &< 1+d_1, \\ i.e.\ |z| &< k_0 \end{aligned}$$

and this completes the proof. □

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