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TOEPLITZ AND HANKEL OPERATORS ON WEIGHTED BERGMAN SPACES*

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Dedicated to Dr. Vasile Drăgan on the occasion of his 70^{th} anniversary

Abstract

In this paper we have shown that if $S \in \mathcal{L}(L_a^2(dA_\alpha))$ and $\Theta_S^{(\alpha)}(x,\overline{y})\Theta_T^{(\alpha)}(x,\overline{y})(K^{(\alpha)}(x,\overline{y}))^2 \approx \Theta_{ST}^{(\alpha)}(x,\overline{y})(K^{(\alpha)}(x,\overline{y}))^2$ for all $x, y \in \mathbb{D}$ and for all $T \in \mathcal{L}(L_a^2(dA_\alpha))$, then $S = T_\phi^{(\alpha)}$ for some $\phi \in H^\infty(\mathbb{D})$ and the matrix of S is lower triangular, where $\Theta_S^{(\alpha)}(x,\overline{y})$ for $S \in \mathcal{L}(L_a^2(dA_\alpha))$ is a function on $\mathbb{D} \times \mathbb{D}$ meromorphic in x and conjugate meromorphic in y. Further, we show that if $\psi, \phi \in L^\infty(\mathbb{D})$, $R^{(\alpha)} \in \mathcal{L}(L_a^2(dA_\alpha))$, then $\Theta^{(\alpha)}_{T_\phi^{(\alpha)}}(x,\overline{y})\Theta^{(\alpha)}_{S_\psi^{(\alpha)}}(x,\overline{y})(K^{(\alpha)}(x,\overline{y}))^2 \approx \Theta^{(\alpha)}_{R^{(\alpha)}}(x,\overline{y}) \cdot(K^{(\alpha)}(x,\overline{y}))^2$ holds for all $x, y \in \mathbb{D}$ if and only if there exists $\beta \in \mathbb{C}$ such that $\phi \equiv \beta$ and $R^{(\alpha)} = S_{\beta\psi}^{(\alpha)}$.
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1 Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$ and let $dA(z) = \frac{1}{\pi} dxdy = \frac{1}{\pi} r dr d\theta$ be the area measure on $\mathbb{D}$ normalized so that the area of $\mathbb{D}$ is 1. For any $\alpha > -1$, let dA_α be the measure on $\mathbb{D}$ defined by

$$dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z).$$

The measure dA_α is a probability measure on $\mathbb{D}$. Let $L^2(\mathbb{D}, dA_\alpha)$ be the space of all Lebesgue measurable functions on $\mathbb{D}$ that are absolutely square-integrable with respect to the measure dA_α. This space $L^2(\mathbb{D}, dA_\alpha)$ is a Hilbert space with respect to the inner product defined by

$$\langle f, g \rangle = \int_{\mathbb{D}} f(z)\overline{g(z)} dA_\alpha(z), \quad f, g \in L^2(\mathbb{D}, dA_\alpha).$$

Let $L^2_a(dA_\alpha)$ be the subspace of $L^2(\mathbb{D}, dA_\alpha)$ consisting of all analytic functions. This space $L^2_a(dA_\alpha)$ is referred to as the weighted Bergman space of the disk $\mathbb{D}$. It is not difficult to see [1], [9] that $L^2_a(dA_\alpha)$ is a closed subspace of $L^2(\mathbb{D}, dA_\alpha)$. The weighted Bergman space $L^2_a(dA_\alpha)$ is a reproducing kernel Hilbert space and the reproducing kernel is given by

$$K^{(\alpha)}(z, \overline{w}) = \frac{1}{(1 - z\overline{w})^{\alpha+2}}, \; z, w \in \mathbb{D}.$$

The orthogonal projection P_α from the Hilbert space $L^2(\mathbb{D}, dA_\alpha)$ onto the closed subspace $L^2_a(dA_\alpha)$ is given by

$$P_\alpha f(z) = \int_{\mathbb{D}} K^{(\alpha)}(z, \overline{w}) f(w) dA_\alpha(w).$$

The unweighted Bergman space ($\alpha = 0$) is denoted by $L^2_a(\mathbb{D})$. The functions $K(z, \overline{w}) = \overline{K_z(w)} = \frac{1}{(1 - z\overline{w})^2}$ are the reproducing kernel of $L^2_a(\mathbb{D})$ and the normalized reproducing kernel is given by $k_z(w) = \frac{1-|z|^2}{(1-w\bar{z})^2}$. The functions

$$k_z^{1+\frac{\alpha}{2}} = \frac{K^{(\alpha)}(z, \overline{w})}{\sqrt{K^{(\alpha)}(w, \overline{w})}} = \frac{(1 - |w|^2)^{1+\frac{\alpha}{2}}}{(1 - \overline{w}z)^{2+\alpha}}$$

are the normalized reproducing kernels of $L^2_a(dA_\alpha)$ and

$$K^{(\alpha)}(z, \overline{w}) = (K(z, \overline{w}))^{1+\frac{\alpha}{2}} = \overline{(K_z(w))^{1+\frac{\alpha}{2}}} = \overline{K_z^{(\alpha)}(w)}.$$

The sequence of functions $\{e_n^{(\alpha)}\} = \left\{\frac{z^n}{\gamma_{n,\alpha}}\right\}$ form as an orthonormal basis [9] for $L_a^2(dA_\alpha)$ where

$$\gamma_{n,\alpha}^2 = \|z^n\|^2 = (\alpha+1)\int_{\mathbb{D}} |z|^{2n}(1-|z|^2)^\alpha dA(z) = \frac{\Gamma(n+1)\Gamma(\alpha+1)}{\Gamma(n+\alpha+2)} \sim (n+1)^{-\alpha-1}.$$

Let $L^\infty(\mathbb{D}, dA)$ be the space of all essentially bounded Lebesgue measurable functions on $\mathbb{D}$. For $\phi \in L^\infty(\mathbb{D})$, define $\|\phi\|_\infty = ess\ sup\,\{|\phi(z)| : z \in \mathbb{D}\} < \infty$. The space $L^\infty(\mathbb{D})$ is a Banach space with respect to the $\|\cdot\|_\infty$. Let $H^\infty(\mathbb{D})$ be the space of all bounded analytic functions on $\mathbb{D}$ and $h^\infty(\mathbb{D})$ be the space of all bounded harmonic functions on $\mathbb{D}$. Given a function $\phi \in L^\infty(\mathbb{D})$, we define an operator $T_\phi^{(\alpha)}$ on $L_a^2(dA_\alpha)$ by $T_\phi^{(\alpha)} f = P_\alpha(\phi f),\ f \in L_a^2(dA_\alpha)$. The operator $T_\phi^{(\alpha)}$ is called the Toeplitz operator on the weighted Bergman space with symbol ϕ. Since the projection P_α has norm 1, we have $\|T_\phi^{(\alpha)}\| \leq \|\phi\|_\infty$. We can write $T_\phi^{(\alpha)}$ as an integral operator as follows :

$$T_\phi^{(\alpha)} f(z) = \int_{\mathbb{D}} \phi(w) K^{(\alpha)}(z,\overline{w}) f(w) dA_\alpha(w) = \int_{\mathbb{D}} \frac{\phi(w)f(w)}{(1-z\overline{w})^{\alpha+2}} dA_\alpha(w).$$

A Toeplitz operator $T_\phi^{(\alpha)}$ is an analytic (co-analytic) Toeplitz operator if the symbol ϕ belongs to $H^\infty(\mathbb{D})$ $\left(\overline{H^\infty(\mathbb{D})}\right)$. Let $\overline{L_a^2(dA_\alpha)} = \left\{\overline{f} : f \in L_a^2(dA_\alpha)\right\}$. The space $\overline{L_a^2(dA_\alpha)}$ is a closed subspace of $L^2(\mathbb{D}, dA_\alpha)$. The little Hankel operator $h_\phi^{(\alpha)}$ with symbol ϕ is defined by $h_\phi^{(\alpha)} f = \overline{P_\alpha}(\phi f), f \in L_a^2(dA_\alpha)$ where $\overline{P_\alpha}$ is the orthogonal projection from the Hilbert space $L^2(\mathbb{D}, dA_\alpha)$ onto $\overline{L_a^2(dA_\alpha)}$. Clearly, $\|h_\phi^{(\alpha)}\| \leq \|\phi\|_\infty$. For $\phi \in L^\infty(\mathbb{D})$, we define the Multiplication operator $M_\phi^{(\alpha)}$ from $L^2(\mathbb{D}, dA_\alpha)$ into itself by $M_\phi^{(\alpha)} f = \phi f$. It is easy to see that $\|M_\phi^{(\alpha)}\| \leq \|\phi\|_\infty$. Define J_α from $L^2(\mathbb{D}, dA_\alpha)$ into itself by $(J_\alpha f)(z) = f(\overline{z}),\ z \in \mathbb{D}$. The operator J_α is a unitary operator. For $\phi \in L^\infty(\mathbb{D})$, define $S_\phi^{(\alpha)}$ from $L_a^2(dA_\alpha)$ into itself by $S_\phi^{(\alpha)} f = P_\alpha(J_\alpha(\phi f))$. The operator $S_\phi^{(\alpha)}$ is a linear operator and $\|S_\phi^{(\alpha)}\| \leq \|\phi\|_\infty$. It is not difficult to verify that $h_\phi^{(\alpha)} = J_\alpha S_\phi^{(\alpha)}$. Thus we shall refer in the sequel, both the operators $h_\phi^{(\alpha)}$ and $S_\phi^{(\alpha)}$ as little Hankel operators on $L_a^2(dA_\alpha)$. Let $\mathcal{L}(H)$ be the space of all bounded linear operators from the Hilbert space H into itself. An operator $T \in \mathcal{L}(H)$ is said to be normal if $T^*T = TT^*$ where T^* is the adjoint of T. For $T \in \mathcal{L}(H)$, the numerical range $W(T)$ of T is defined by $W(T) = \{\langle Tx, x\rangle : x \in H, \|x\| = 1\}$. For $z \in \mathbb{D}$, define φ_z on $\mathbb{D}$ by

$$\varphi_z(w) = \frac{z-w}{1-\overline{z}w},\ w \in \mathbb{D}.$$

Define $\rho_\alpha : \mathcal{L}(L^2_a(dA_\alpha)) \longrightarrow L^\infty(\mathbb{D})$ by $\rho_\alpha(T)(z) = \left\langle Tk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}} \right\rangle$, $z \in \mathbb{D}$. Since $|\rho_\alpha(T)(z)| = \left|\left\langle Tk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}} \right\rangle\right| \leq \|T\| \left\|k_z^{1+\frac{\alpha}{2}}\right\|^2 = \|T\|$, hence $\|\rho_\alpha(T)\| \leq \|T\|$. The map ρ_α is linear and one-one. For details see [4].

2 Preliminaries

Lemma 2.1. *Let $S \in \mathcal{L}\left(L^2_a(dA_\alpha)\right)$ be a normal operator. If $\sigma(S)$ (the spectrum of S) lies on the right half place $\mathbb{C}_+ = \{x + iy \in \mathbb{C} : x > 0\}$, then the numerical range $W(S)$ is also contained in the right half plane $\mathbb{C}_+$.*

Proof. Let S be a normal operator and $S = \displaystyle\int_{\sigma(S)} \lambda dE_\lambda$ be the spectral representation for S. If $\beta \in W(S)$, then there exists $f \in L^2_a(dA_\alpha)$ with $\|f\| = 1$ such that

$$\beta = \langle Sf, f\rangle = \left\langle \int_{\sigma(S)} \lambda dE_\lambda f, f \right\rangle = \int_{\sigma(S)} \lambda d\|E_\lambda f\|^2$$

where $\displaystyle\int_{\sigma(S)} d\|E_\lambda f\|^2 = 1$. This implies that β is contained in the convex closure of $\sigma(S)$. Hence $W(S)$ is contained in the right half plane as the $\sigma(S)$ lies in the right half plane. □

Lemma 2.2. *Let $\phi \in L^\infty(\mathbb{D})$. If $\phi f \in L^2_a(dA_\alpha)$ for all $f \in L^2_a(dA_\alpha)$, then $\phi \in H^\infty(\mathbb{D})$.*

Proof. Since $\phi f \in L^2_a(dA_\alpha)$ for all $f \in L^2_a(dA_\alpha)$, hence $T^{(\alpha)}_\phi f = \phi f$ for all $f \in L^2_a(dA_\alpha)$. This implies $\phi(z) = \frac{T^{(\alpha)}_\phi f(z)}{f(z)}$. Hence ϕ is analytic on $\mathbb{D} - \{\text{zeros of } f\}$. Therefore each isolated singularity of ϕ in $\mathbb{D}$ is removable as ϕ is assumed to be bounded. Thus ϕ is analytic on $\mathbb{D}$. Since $\phi \in L^\infty(\mathbb{D})$, hence we obtain $\phi \in H^\infty(\mathbb{D})$. □

Lemma 2.3. *Let $\phi \in L^\infty(\mathbb{D})$ and $S^{(\alpha)}_\phi$ be the little Hankel operator on $L^2_a(dA_\alpha)$ with symbol ϕ. Then there exists no non-zero constant $\lambda \in \mathbb{C}$ such that $S^{(\alpha)}_\phi = \lambda I_{\mathcal{L}(L^2_a(dA_\alpha))}$.*

Proof. The operator $S^{(\alpha)}_\phi$ satisfies the [3] operator equation $\left(T^{(\alpha)}_z\right)^* S^{(\alpha)}_\phi = S^{(\alpha)}_\phi T^{(\alpha)}_z$ where $T^{(\alpha)}_z$ is the Bergman shift operator defined on $L^2_a(dA_\alpha)$ by

$T_z^{(\alpha)} f = zf$. If $S_\phi^{(\alpha)} = \lambda I_{\mathcal{L}(L_a^2(dA_\alpha))}$ for some $\phi \in L^\infty(\mathbb{D})$ and $\lambda \in \mathbb{C}$, then this implies $\left(T_z^{(\alpha)}\right)^* = T_z^{(\alpha)}$. But clearly this is false. □

A function $g(x, \overline{y})$ on $\mathbb{D}\times\mathbb{D}$ is called of positive type (or positive definite), written as $g \gg 0$, if $\sum_{j,k=1}^{n} c_j \overline{c_k} g(x_j, \overline{x_k}) \geq 0$ for any n-tuple of complex numbers $c_1, ..., c_n$ and points $x_1, ..., x_n \in \mathbb{D}$. We write $g \gg h$ if $g - h \gg 0$. If $g \gg h$ and $h \gg g$ then we write $g \approx h$. Let $S \in \mathcal{L}\left(L_a^2(dA_\alpha)\right)$, define $\Theta_S^{(\alpha)}(x, \overline{y}) = \frac{\langle SK_y^{(\alpha)}, K_x^{(\alpha)}\rangle}{\langle K_y^{(\alpha)}, K_x^{(\alpha)}\rangle}$ where $K_x^{(\alpha)} = K^{(\alpha)}(\cdot, \overline{x})$ is the unnormalized weighted reproducing kernel at x. Then $\Theta_S^{(\alpha)}(x, \overline{y})$ is a function on $\mathbb{D} \times \mathbb{D}$ meromorphic in x and conjugate meromorphic in y. Let $\phi_\alpha(z) = \Theta_S^{(\alpha)}(z, \overline{z})$, then $\phi_\alpha(z) = \langle Sk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}}\rangle$ and $\phi_\alpha \in L^\infty(\mathbb{D})$ as S is bounded on $L_a^2(dA_\alpha)$.

Lemma 2.4. *Let S, T and $R \in \mathcal{L}\left(L_a^2(dA_\alpha)\right)$ for fix $\alpha > -1$. Suppose either S or T is normal. If*

$$\Theta_S^{(\alpha)}(x, \overline{y}) K^{(\alpha)}(x, \overline{y}) \approx \Theta_R^{(\alpha)}(x, \overline{y}) K^{(\alpha)}(x, \overline{y}) \quad \text{and} \tag{1}$$

$$\Theta_T^{(\alpha)}(x, \overline{y}) K^{(\alpha)}(x, \overline{y}) \approx K^{(\alpha)}(x, \overline{y}) \tag{2}$$

for all $x, y \in \mathbb{D}$, then either $\rho_\alpha(S)$ or $\rho_\alpha(T)$ is a constant function in $L^\infty(\mathbb{D})$. Further if either $\rho_\alpha(S)$ or $\rho_\alpha(T)$ is the constant function β and (1) *holds then either $\beta T = R$ or $\beta S = R$.*

Proof. Notice that

$$\Theta_S^{(\alpha)}(x, \overline{y}) K^{(\alpha)}(x, \overline{y}) \gg \Theta_R^{(\alpha)}(x, \overline{y}) K^{(\alpha)}(x, \overline{y})$$

and

$$\Theta_T^{(\alpha)}(x, \overline{y}) K^{(\alpha)}(x, \overline{y}) \gg K^{(\alpha)}(x, \overline{y})$$

for all $x, y \in \mathbb{D}$ implies $\frac{\langle SK_y^{(\alpha)}, K_x^{(\alpha)}\rangle}{\langle K_y^{(\alpha)}, K_x^{(\alpha)}\rangle} K^{(\alpha)}(x, \overline{y}) \gg \frac{\langle RK_y^{(\alpha)}, K_x^{(\alpha)}\rangle}{\langle K_y^{(\alpha)}, K_x^{(\alpha)}\rangle} K^{(\alpha)}(x, \overline{y})$ and $\frac{\langle TK_y^{(\alpha)}, K_x^{(\alpha)}\rangle}{\langle K_y^{(\alpha)}, K_x^{(\alpha)}\rangle} K^{(\alpha)}(x, \overline{y}) \gg K^{(\alpha)}(x, \overline{y})$ for all $x, y \in \mathbb{D}$. Hence $\langle SK_y^{(\alpha)}, K_x^{(\alpha)}\rangle \gg \langle RK_y^{(\alpha)}, K_x^{(\alpha)}\rangle$ and $\langle TK_y^{(\alpha)}, K_x^{(\alpha)}\rangle \gg \langle K_y^{(\alpha)}, K_x^{(\alpha)}\rangle$ for all $x, y \in \mathbb{D}$. That is,

$$\sum_{j,k=1}^{n} (c_j \overline{c_k}) \langle SK_{x_j}^{(\alpha)}, K_{x_k}^{(\alpha)}\rangle \geq \sum_{j,k=1}^{n} (c_j \overline{c_k}) \langle RK_{x_j}^{(\alpha)}, K_{x_k}^{(\alpha)}\rangle$$

and

$$\sum_{j,k=1}^{n}(c_j\overline{c_k})\langle TK_{x_j}^{(\alpha)}, K_{x_k}^{(\alpha)}\rangle \geq \sum_{j,k=1}^{n}(c_j\overline{c_k})\langle K_{x_j}^{(\alpha)}, K_{x_k}^{(\alpha)}\rangle$$

for $x_j, x_k \in \mathbb{D},\ c_j, c_k \in \mathbb{C},\ j,k = 1,...,n$. It follows therefore that

$$\left\langle S\left(\sum_{j=1}^{n} c_j K_{x_j}^{(\alpha)}\right), \left(\sum_{j=1}^{n} c_j K_{x_j}^{(\alpha)}\right)\right\rangle \geq \left\langle R\left(\sum_{j=1}^{n} c_j K_{x_j}^{(\alpha)}\right), \left(\sum_{j=1}^{n} c_j K_{x_j}^{(\alpha)}\right)\right\rangle$$

and

$$\left\langle T\left(\sum_{j=1}^{n} c_j K_{x_j}^{(\alpha)}\right), \left(\sum_{j=1}^{n} c_j K_{x_j}^{(\alpha)}\right)\right\rangle \geq \left\langle \left(\sum_{j=1}^{n} c_j K_{x_j}^{(\alpha)}\right), \left(\sum_{j=1}^{n} c_j K_{x_j}^{(\alpha)}\right)\right\rangle.$$

Since the set of vectors $\left\{\sum_{j=1}^{n} c_j K_{x_j}^{(\alpha)} : x_j \in \mathbb{D}, j = 1,...,n\right\}$ is dense [2] in $L_a^2(dA_\alpha)$, we obtain $\langle Sg, g\rangle \geq \langle Rg, g\rangle$ and $\langle Tg, g\rangle \geq \langle g, g\rangle$ for all $g \in L_a^2(dA_\alpha)$. This implies that $\langle Sf, f\rangle\langle Tf, f\rangle \geq \langle Rf, f\rangle$ for all $f \in L_a^2(dA_\alpha)$ with $\|f\| = 1$. Similarly, on simplification of the expressions

$$\begin{aligned}&\Theta_R^{(\alpha)}(x,\overline{y})K^{(\alpha)}(x,\overline{y}) \gg \Theta_S^{(\alpha)}(x,\overline{y})K^{(\alpha)}(x,\overline{y}) \text{ and}\\ &K^{(\alpha)}(x,\overline{y}) \gg \Theta_T^{(\alpha)}(x,\overline{y})K^{(\alpha)}(x,\overline{y})\end{aligned} \tag{3}$$

for all $x, y \in \mathbb{D}$ we obtain $\langle Rf, f\rangle \geq \langle Sf, f\rangle$ and $\langle Tf, f\rangle \leq 1$ for all $f \in L_a^2(dA_\alpha)$ with $\|f\| = 1$. Thus, it follows that $\langle Sf, f\rangle\langle Tf, f\rangle = \langle Rf, f\rangle$ for all $f \in L_a^2(dA_\alpha)$ with $\|f\| = 1$. Let T be a normal operator. If $T = \gamma I$ for some $\gamma \in \mathbb{C}$, then there is nothing to prove. So assume $T \neq \gamma I$ for all $\gamma \in \mathbb{C}$. Assume that T has the spectral representation $T = \int_{\sigma(T)} \lambda dE_\lambda$. Since T is not a scalar multiple of the identity, hence there exist complex numbers γ_1 and γ_2 in $\sigma(T)$ such that $\gamma_1 \neq \gamma_2$. Now since $\langle Sf, f\rangle\langle Tf, f\rangle = \langle Rf, f\rangle$ for all $f \in L_a^2(dA_\alpha)$ with $\|f\| = 1$, we obtain

$$\langle Sf, f\rangle\langle (e^{i\theta}T - \eta I)f, f\rangle = \langle (e^{i\theta}R - \eta S)f, f\rangle,\ f \in L_a^2(dA_\alpha),\ \|f\| = 1 \tag{4}$$

for $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ and $\eta \in \mathbb{C}$. Hence we can assume that there exists a positive number $\beta > 0$, such that $\gamma_1 = \beta$ and $\gamma_2 = -\beta$. Let

$$\Gamma_1 = \sigma(T) \cap \{x + iy : x > 0\} \text{ and } \Gamma_2 = \sigma(T) \cap \{x + iy : x \leq 0\}.$$

It is clear that Γ_1 and Γ_2 are disjoint Borel sets. Let $K_i = E(\Gamma_i)L_a^2(dA_\alpha); i = 1, 2$. Then K_1 and K_2 are reducing subspaces of T and $L_a^2(dA_\alpha) = K_1 \oplus K_2$. Let Q be the orthogonal projection of $L_a^2(dA_\alpha)$ onto K_1. We shall show that QSQ is a scalar multiple of the identity on K_1. Let the operator matrices of S, T and R be given by $S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$, $T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$, $R = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}$. Here T_1, T_2 are normal operators and $T_i = \int_{\Gamma_i} T dE_\lambda$, $i = 1, 2$. Since $W(T_i) =$ convex hull of $\{x : x \in \Gamma_i\}, i = 1, 2$, hence $W(T_1) \cap W(T_2) = \phi$, by Lemma 1. Let $f_1 \in K_1, f_2 \in K_2$ with $\|f_1\| = 1, \|f_2\| = 1$. If $0 \le \vartheta \le 1$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, then $f = \vartheta f_1 + e^{i\theta}(1-\vartheta^2)^{\frac{1}{2}} f_2 \in K_1 \oplus K_2 = L_a^2(dA_\alpha)$ has norm 1 where ϑ and θ are independent of each other. An easy calculation shows that

$$\begin{aligned}
\langle Sf, f\rangle &= \vartheta^2 \langle S_{11} f_1, f_1\rangle + (1-\vartheta^2)\langle S_{22} f_2, f_2\rangle \\
&\quad + \vartheta(1-\vartheta^2)^{\frac{1}{2}} \left(\langle S_{12} f_2, f_1\rangle e^{i\theta} + \langle S_{21} f_1, f_2\rangle e^{-i\theta}\right), \\
\langle Rf, f\rangle &= \vartheta^2 \langle R_{11} f_1, f_1\rangle + (1-\vartheta^2)\langle R_{22} f_2, f_2\rangle \\
&\quad + \vartheta(1-\vartheta^2)^{\frac{1}{2}} \left(\langle R_{12} f_2, f_1\rangle e^{i\theta} + \langle R_{21} f_1, f_2\rangle e^{-i\theta}\right), \\
\langle Tf, f\rangle &= \vartheta^2 \langle T_1 f_1, f_1\rangle + (1-\vartheta^2)\langle T_2 f_2, f_2\rangle \\
&= \vartheta^2 \left(\langle T_1 f_1, f_1\rangle - \langle T_2 f_2, f_2\rangle\right) + \langle T_2 f_2, f_2\rangle,
\end{aligned}$$

where $\langle T_1 f_1, f_1\rangle \ne \langle T_2 f_2, f_2\rangle$. Thus

$$\begin{aligned}
&\langle Sf, f\rangle\langle Tf, f\rangle = \vartheta^4 \langle S_{11} f_1, f_1\rangle \left(\langle T_1 f_1, f_1\rangle - \langle T_2 f_2, f_2\rangle\right) \\
&+ \vartheta^2(1-\vartheta^2)\langle S_{22} f_2, f_2\rangle \left(\langle T_1 f_1, f_1\rangle - \langle T_2 f_2, f_2\rangle\right) \\
&+ \vartheta^3\sqrt{1-\vartheta^2} \left(\langle T_1 f_1, f_1\rangle - \langle T_2 f_2, f_2\rangle\right) \left(e^{i\theta}\langle S_{12} f_2, f_1\rangle + e^{-i\theta}\langle S_{21} f_1, f_2\rangle\right) \\
&+ \vartheta^2 \langle S_{11} f_1, f_1\rangle\langle T_2 f_2, f_2\rangle + (1-\vartheta^2)\langle S_{22} f_2, f_2\rangle\langle T_2 f_2, f_2\rangle \\
&+ \vartheta\sqrt{1-\vartheta^2} \left(e^{i\theta}\langle S_{12} f_2, f_1\rangle + e^{-i\theta}\langle S_{21} f_1, f_2\rangle\right)\langle T_2 f_2, f_2\rangle.
\end{aligned}$$

Comparing the coefficients of $e^{i\theta}$ on both sides of $\langle Sf, f\rangle\langle Tf, f\rangle = \langle Rf, f\rangle$ where $\|f\| = 1$, we obtain

$$\left[\left(\langle T_1 f_1, f_1\rangle - \langle T_2 f_2, f_2\rangle\right)\vartheta^2 + \langle T_2 f_2, f_2\rangle\right]\vartheta\sqrt{1-\vartheta^2}\langle S_{12} f_2, f_1\rangle = \vartheta\sqrt{1-\vartheta^2}\langle R_{12} f_2, f_1\rangle.$$

Thus

$$\left[\left(\langle T_1 f_1, f_1\rangle - \langle T_2 f_2, f_2\rangle\right)\vartheta^2 + \langle T_2 f_2, f_2\rangle\right]\langle S_{12} f_2, f_1\rangle = \langle R_{12} f_2, f_1\rangle. \tag{5}$$

Notice that $\langle T_1 f_1, f_1\rangle \neq \langle T_2 f_2, f_2\rangle$ and $\vartheta \in [0,1]$ is arbitrary. Hence $\langle S_{12} f_2, f_1\rangle = 0$ for any pair of vectors $f_1 \in K_1$ and $f_2 \in K_2$ with $\|f_1\| = \|f_2\| = 1$. Hence $S_{12} = 0$. From (5), it is clear that $R_{12} = 0$. Similarly, considering the coefficients of $e^{-i\theta}$ in both sides of $\langle Sf, f\rangle\langle Tf, f\rangle = \langle Rf, f\rangle$, we obtain $S_{21} = 0$ and $R_{21} = 0$. Thus

$$\begin{aligned}\langle Sf, f\rangle\langle Tf, f\rangle &= \left[\vartheta^2\langle S_{11} f_1, f_1\rangle + (1-\vartheta^2)\langle S_{22} f_2, f_2\rangle\right] \\ &\quad \left[(\langle T_1 f_1, f_1\rangle - \langle T_2 f_2, f_2\rangle)\,\vartheta^2 + \langle T_2 f_2, f_2\rangle\right] \\ &= \left[\langle T_1 f_1, f_1\rangle - \langle T_2 f_2, f_2\rangle\right]\left[\langle S_{11} f_1, f_1\rangle - \langle S_{22} f_2, f_2\rangle\right]\vartheta^4 \\ &\quad + \left[\langle T_1 f_1, f_1\rangle\left(\langle S_{11} f_1, f_1\rangle + \langle S_{22} f_2, f_2\rangle\right)\right. \\ &\quad \left. - 2\langle T_2 f_2, f_2\rangle\langle S_{22} f_2, f_2\rangle\right]\vartheta^2 + \langle T_2 f_2, f_2\rangle\langle S_{22} f_2, f_2\rangle, \quad \text{and} \\ \langle Rf, f\rangle &= \vartheta^2\langle R_{11} f_1, f_1\rangle + (1-\vartheta^2)\langle R_{22} f_2, f_2\rangle \\ &= \left[\langle R_{11} f_1, f_1\rangle - \langle R_{22} f_2, f_2\rangle\right]\vartheta^2 + \langle R_{22} f_2, f_2\rangle.\end{aligned}$$

Comparing the coefficients of ϑ^4 in both sides of $\langle Sf, f\rangle\langle Tf, f\rangle = \langle Rf, f\rangle$ with $\|f\| = 1$; we get

$$\left[\langle T_1 f_1, f_1\rangle - \langle T_2 f_2, f_2\rangle\right]\left[\langle S_{11} f_1, f_1\rangle - \langle S_{22} f_2, f_2\rangle\right] = 0.$$

Since $\langle T_1 f_1, f_1\rangle \neq \langle T_2 f_2, f_2\rangle$, it follows that $\langle S_{11} f_1, f_1\rangle = \langle S_{22} f_2, f_2\rangle$ for $f_1 \in K_1$ and $f_2 \in K_2$ with $\|f_1\| = \|f_2\| = 1$. Hence $S_{11} = \lambda I_1$ and $S_2 = \lambda I_2$ and therefore $S = \lambda I$ and $\rho_\alpha(S)$ is a constant function. Conversely, suppose $\rho_\alpha(S)$ or $\rho_\alpha(T)$ is a constant function. Assume $\rho_\alpha(S) = \beta$, a constant. Then $\langle Sk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}}\rangle = \beta$ for all $z \in \mathbb{D}$. Now if (1) holds, then $\langle Sf, f\rangle\langle Tf, f\rangle = \langle Rf, f\rangle$ for all $f \in L_a^2(dA_\alpha)$ with $\|f\| = 1$. Hence in particular $\langle Sk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}}\rangle\langle Tk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}}\rangle = \langle Rk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}}\rangle$ for all $z \in \mathbb{D}$. This implies $\langle \beta Tk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}}\rangle = \langle Rk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}}\rangle$ for all $z \in \mathbb{D}$. From [4], it follows that $R = \beta T$. □

Lemma 2.5. *If $\phi \in h^\infty(\mathbb{D})$, then $\rho_\alpha\left(M_\phi^{(\alpha)}\right)(z) = \phi(z)$.*

Proof. For all $z \in \mathbb{D}$, $\rho_\alpha\left(M_\phi^{(\alpha)}\right)(z) = \left\langle M_\phi^{(\alpha)} k_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}}\right\rangle = \left\langle \phi k_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}}\right\rangle$ $= \int_{\mathbb{D}} \phi(w)|k_z^{1+\frac{\alpha}{2}}(w)|^2 dA_\alpha(w) = \phi(z)$. The last equality follows from [7] and [6]. The result follows. □

Lemma 2.6. *Let G_α be a finite-dimensional subspace of $L_a^2(dA_\alpha)$ for fix $\alpha > -1$. Let $E^{(\alpha)} \in \mathcal{L}(G_\alpha)$ and let $\{v_1^{(\alpha)}, v_2^{(\alpha)}, ..., v_m^{(\alpha)},\}$ be an orthonormal basis for G_α. Then there exists $\psi \in L^\infty(\mathbb{D})$ such that $T_\psi^{(\alpha)}\big|_{G_\alpha} = E^{(\alpha)}$.*

Proof. We shall consider the operator $X^{(\alpha)} : L^\infty(\mathbb{D}) \to \mathbb{C}^{m\times m}$, defined by $\left(X^{(\alpha)}\psi\right)_{ij} = \int_{\mathbb{D}} \psi(z) v_i^{(\alpha)}(z)\overline{v_j^{(\alpha)}(z)} dA_\alpha(z) = \langle T_\psi^{(\alpha)} v_i^{(\alpha)}, v_j^{(\alpha)}\rangle$. Suppose $r \in \mathbb{C}^{m\times m}$ is orthogonal to $RangeX$. Then $\sum_{i=1}^{m}\sum_{j=1}^{m}\left(X^{(\alpha)}\psi\right)_{ij}\overline{r_{ij}} = 0$ for all $\psi \in L^\infty(\mathbb{D})$. This implies

$$\int_{\mathbb{D}} \psi(z) \sum_{i=1}^{m}\sum_{j=1}^{m} \overline{r_{ij}} v_i^{(\alpha)}(z)\overline{v_j^{(\alpha)}(z)} dA_\alpha(z) = 0$$

for all $\psi \in L^\infty(\mathbb{D})$. Hence

$$\sum_{i=1}^{m}\sum_{j=1}^{m} \overline{r_{ij}} v_i^{(\alpha)}(z)\overline{v_j^{(\alpha)}(z)} = 0 \tag{6}$$

almost everywhere in $\mathbb{D}$. The left hand side of (6) is continuous on $\mathbb{D}$. Hence the equality in (6) holds on the whole of $\mathbb{D}$. Therefore, the function $H(x, y) = \sum_{i=1}^{m}\sum_{j=1}^{m} \overline{r_{ij}} v_i^{(\alpha)}(x)\overline{v_j^{(\alpha)}(\overline{y})}$ which is analytic in $\mathbb{D}\times\mathbb{D}$, is equal to zero whenever $x = \overline{y}$. By the uniqueness theorem [5], we obtain $H \equiv 0$ on $\mathbb{D}\times\mathbb{D}$. The functions $v_i^{(\alpha)}, i = 1, ..., m$ are linearly independent. Thus we get $\sum_{j=1}^{m} r_{ij} v_j^{(\alpha)}(\overline{y}) = 0$ for all $y \in \mathbb{D}, i = 1, ..., m$. Now again since $v_j^{(\alpha)}, j = 1, ..., m$ are linearly independent, we obtain $r_{ij} = 0$ for all $i, j \in \{1, ..., m\}$. Thus $r \equiv 0$. This implies $Range\ E^{(\alpha)} = \mathbb{C}^{m\times m}$. Hence $\langle T_\psi^{(\alpha)} v_i^{(\alpha)}, v_j^{(\alpha)}\rangle = \langle E^{(\alpha)} v_i^{(\alpha)}, v_j^{(\alpha)}\rangle$ for all $i, j \in \{1, ..., m\}$ and therefore it follows that $T_\psi^{(\alpha)}\big|_{G_\alpha} = E^{(\alpha)}$. □

3 On finite dimensional subspaces of $L_a^2(dA_\alpha)$

Theorem 3.1. *Let G_α be a subspace of $L_a^2(dA_\alpha)$ such that dim $G_\alpha < \infty$. If $S, T, R \in \mathcal{L}(G_\alpha)$ and*

$$\langle Sg, g\rangle\langle Tg, g\rangle = \langle Rg, g\rangle\langle g, g\rangle \text{ for all } g \in G_\alpha, \tag{7}$$

then there exists $\beta \in \mathbb{C}$ such that either $S = \beta I_{\mathcal{L}(G_\alpha)}$ or $T = \beta I_{\mathcal{L}(G_\alpha)}$. In case R, T, S are self-adjoint in $\mathcal{L}(G_\alpha)$, then there exists $\psi \in L^\infty(\mathbb{D})$ and a real number β such that $R = T_{\beta\psi}^{(\alpha)}\big|_{G_\alpha}$.

Proof. Suppose the first part of the theorem holds. To prove the second part, suppose R, T, S are self-adjoint operators in $\mathcal{L}(G_\alpha)$ and let $S = \beta I_{\mathcal{L}(G_\alpha)}$. Then $\beta \in \mathbb{R}$ as $\langle Sg, g\rangle = \langle \beta I_{\mathcal{L}(G_\alpha)} g, g\rangle = \beta\langle g, g\rangle$ is real for all $g \in G_\alpha$. Thus $\langle Rg, g\rangle = \beta\langle Tg, g\rangle$ for all $g \in G_\alpha, \beta \in \mathbb{R}$. From Lemma 2.6, it follows that $T = T_\psi^{(\alpha)}|_{G_\alpha}$ for some $\psi \in L^\infty(\mathbb{D})$ as $dim\ G_\alpha < \infty$. Thus we obtain $\langle Rg, g\rangle = \langle T_{\beta\psi}^{(\alpha)} g, g\rangle$ for all $g \in G_\alpha$. This implies $\left\langle \left(R - T_{\beta\psi}^{(\alpha)}\right) g, g\right\rangle = 0$ for all $g \in G_\alpha$. Thus $\left\| R - T_{\beta\psi}^{(\alpha)} \right\|_{G_\alpha} = \sup_{g\in G_\alpha \|g\|=1} \left| \left\langle \left(R - T_{\beta\psi}^{(\alpha)}\right) g, g\right\rangle \right| = 0.$ Hence $R = T_{\beta\psi}^{(\alpha)}|_{G_\alpha}$. Notice that G_α is finite dimensional and hence it is a closed subspace of $L_a^2(dA_\alpha)$. We shall prove the theorem using mathematical induction. From (7), it follows that $\langle Sf, f\rangle\langle Tf, f\rangle = \langle Rf, f\rangle$ for all $f \in G_\alpha$ with $\|f\| = 1$. Hence

$$\langle VSV^* f, f\rangle\langle VTV^* f, f\rangle = \langle VRV^* f, f\rangle \quad \text{and}$$

$$\langle (S - rI)f, f\rangle\langle (T - sI)f, f\rangle = \langle (R - rT - sS + rsI)f, f\rangle$$

for all unitary operators $V \in \mathcal{L}(G_\alpha)$, $r, s \in \mathbb{C}$ and $f \in G_\alpha$ with $\|f\| = 1$. Assume $dim\ G_\alpha = 2$ and S, T and R belong to $\mathcal{L}(G_\alpha)$. Then G_α is isomorphic to $\mathbb{C}^2$. Thus without loss of generality, we shall assume $S = \begin{pmatrix} s_{11} & s_{12} \\ 0 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 0 & t_{12} \\ t_{21} & t_{22} \end{pmatrix}$ and $R = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}$. If S is normal, then $s_{12} = 0$ and $\langle Sf, f\rangle\langle Tf, f\rangle = \langle Rf, f\rangle$ with $\|f\| = 1$ implies $s_{11} = 0$. This implies $S = 0 = 0 \cdot I_{\mathcal{L}(G_\alpha)}$. Now assume S is not normal. We shall show that T is a scalar multiple of the identity. Since S is not normal, we have $s_{12} \neq 0$. Observe that if $\langle Sf, f\rangle\langle Tf, f\rangle = \langle Rf, f\rangle$ for all $f \in G_\alpha$ with $\|f\| = 1$ then

$$\langle Sf, f\rangle\langle Tf, f\rangle = \langle QSQf, f\rangle\langle QTQf, f\rangle = \langle QRQf, f\rangle = \langle Rf, f\rangle$$

for all $f \in G_\alpha^1$ with $\|f\| = 1$ if $G = G_\alpha^1 \oplus G_\alpha^2$ and Q is the orthogonal projection from G_α onto G_α^1. Thus it follows that $r_{22} = 0$ and $r_{11} = 0$. Hence, R has the matrix representation $R = \begin{pmatrix} 0 & r_{12} \\ r_{21} & 0 \end{pmatrix}$. Choosing the unit vectors $f = \begin{pmatrix} x \\ e^{i\theta}\sqrt{1 - x^2} \end{pmatrix}$, $0 \leq x \leq 1$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, we obtain $\langle Sf, f\rangle = s_{11}x^2 + s_{12}x\sqrt{1 - x^2}e^{i\theta}$ and $\langle Tf, f\rangle = -t_{22}x^2 + t_{12}x\sqrt{1 - x^2}e^{i\theta} + t_{21}x\sqrt{1 - x^2}e^{-i\theta} + t_{22}$ and $\langle Rf, f\rangle = r_{12}x\sqrt{1 - x^2}e^{i\theta} + r_{21}x\sqrt{1 - x^2}e^{-i\theta}$. Thus from (7), it follows that $s_{12}t_{12}x(1 - x^2) = 0$, $-x^2 s_{12}t_{22} + s_{12}t_{22} = r_{12}$ and $x^2(1 - x^2)s_{12}t_{21} = 0$. Thus from all these identities, it follows that $t_{12} = t_{21} = 0$. Moreover as $0 \leq x \leq 1$, we obtain $s_{12}t_{22} = r_{12}$,

$s_{12}t_{22} = 0$ and hence $t_{22} = 0$. Thus, $T \equiv 0$. Now assume dim $G_\alpha = 3$ and (7) holds. Then G_α is isomorphic to $\mathbb{C}^3$. Without loss of generality, we shall assume $G_\alpha = \mathbb{C}^3$, and S is the upper triangular matrix $S = \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ 0 & s_{22} & s_{23} \\ 0 & 0 & s_{33} \end{pmatrix}$, $T = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix}$ and $R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$. We shall show either $S = \beta I_{\mathcal{L}(G_\alpha)}$ or $T = \beta I_{\mathcal{L}(G_\alpha)}$. If S has three 2×2 principal submatrices which are identity I_3, then S is a scalar multiple of the identity I_3. So assume S has just two principal submatrices which are scalar multiples of I_2. Assume $S_{12} = \begin{pmatrix} s_{11} & s_{12} \\ 0 & s_{22} \end{pmatrix}$ and $S_{13} = \begin{pmatrix} s_{11} & s_{13} \\ 0 & s_{33} \end{pmatrix}$ are scalar multiples of identity I_2. This implies $S = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & s_{23} \\ 0 & 0 & \lambda \end{pmatrix}$ for some $\lambda \in \mathbb{C}$. If $s_{23} = 0$ then $S = \lambda I_3$. If $s_{23} \neq 0$, then $T_{23} = \begin{pmatrix} \gamma & 0 \\ 0 & \gamma \end{pmatrix}$ for some $\gamma \in \mathbb{C}$, since S_{23} is not a scalar multiple of identity. By considering elementary matrices that are unitaries, without loss of generality, we shall assume that $S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & s_{23} \\ 0 & 0 & 0 \end{pmatrix}$, $T = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & 0 & 0 \\ t_{31} & 0 & 0 \end{pmatrix}$. Using the identity (7) it follows that $R \equiv 0$. Now consider the unit vector $f = \left(e^{i\theta_1}x_1, e^{i\theta_2}x_2, e^{i\theta_3}\sqrt{1 - x_1^2 - x_2^2}\right)^{\mathsf{T}}$ where $0 \leq x_1, x_2 \leq 1$, $0 \leq \theta_1, \theta_2, \theta_3 \leq 2\pi$ and $x_1^2 + x_2^2 \leq 1$. Then $\langle Sf, f\rangle = s_{23}e^{i(\theta_3-\theta_2)}x_2\sqrt{1 - x_1^2 - x_2^2}$. Since $R \equiv 0$, it follows that $\langle Sf, f\rangle\langle Tf, f\rangle = 0$ and $s_{23}e^{i(\theta_3-\theta_2)} \neq 0$. Hence $x_2\sqrt{1 - x_1^2 - x_2^2}\langle Tf, f\rangle = 0$. This implies $\langle Tf, f\rangle = 0$ for all $f \in \mathbb{C}^3$, $\|f\| = 1$. Hence $T \equiv 0$ as numerical range of T is equal to $\{0\}$. Proceeding similarly, one can show that if G_α is isomorphic to $\mathbb{C}^4$ and (7) holds then either $S = \beta I_{\mathcal{L}(G_\alpha)}$ or $T = \beta I_{\mathcal{L}(G_\alpha)}$ for some $\beta \in \mathbb{C}$. So now suppose $n = \mathit{dim}\ G_\alpha < \infty$, $n \geq 5$ and (7) holds for all $f \in G_\alpha$ with $\|f\| = 1$. Since the matrix representation of S with respect to the orthonormal basis of G_α is an $n \times n$ matrix, hence S have more than four $(n-1) \times (n-1)$ principal submatrices. If (7) holds, then atleast three $(n-1) \times (n-1)$ principal submatrices of S or T are scalar multiples of the identity I_{n-1}. Thus $S = \beta I_n$ or $T = \beta I_n$ for some $\beta \in \mathbb{C}$. Hence $\rho_\alpha(S) = \beta$ or $\rho_\alpha(T) = \beta$. □

4 Main results

Theorem 4.1. *Let* $\psi, \phi \in L^\infty(\mathbb{D})$, $R^{(\alpha)} \in \mathcal{L}(L^2_a(dA_\alpha))$. *Then*

$$\Theta^{(\alpha)}_{T^{(\alpha)}_\phi}(x,\overline{y})\Theta^{(\alpha)}_{S^{(\alpha)}_\psi}(x,\overline{y})(K^{(\alpha)}(x,\overline{y}))^2 \approx \Theta^{(\alpha)}_{R^{(\alpha)}}(x,\overline{y})(K^{(\alpha)}(x,\overline{y}))^2$$

holds for all $x, y \in \mathbb{D}$ *if and only if there exists* $\beta \in \mathbb{C}$ *such that* $\phi \equiv \beta$ *and* $R^{(\alpha)} = S^{(\alpha)}_{\beta\psi}$.

Proof. The sufficient part is obvious. For the necessary part, assume

$$\langle T^{(\alpha)}_\phi f, f\rangle\langle S^{(\alpha)}_\psi f, f\rangle = \langle R^{(\alpha)} f, f\rangle$$

for all $f \in L^2_a(dA_\alpha)$ with $\|f\| = 1$. From Lemma 2.3, it follows that $S^{(\alpha)}_\psi \neq \beta I_{\mathcal{L}(L^2_a(dA_\alpha))}$ for all $\beta \in \mathbb{C}$. Let G_α be a closed subspace of $L^2_a(dA_\alpha)$, $dim\ G_\alpha \geq 1$, $L^2_a(dA_\alpha) = G_\alpha \oplus G_\alpha^\perp$ and with respect to this decomposition the matrix of $S^{(\alpha)}_\psi$ is given by $S^{(\alpha)}_\psi = \begin{pmatrix} 0 & S^{(\alpha)}_{12} \\ S^{(\alpha)}_{21} & S^{(\alpha)}_{22} \end{pmatrix}$. If $G_\alpha = L^2_a(dA_\alpha)$, then $S^{(\alpha)}_\psi \equiv 0$ and $R^{(\alpha)} \equiv 0$ and there is nothing to prove. So, suppose $G_\alpha \neq L^2_a(dA_\alpha)$ and assume $T^{(\alpha)}_\phi$ and $R^{(\alpha)}$ have operator matrices of the forms, $T^{(\alpha)}_\phi = \begin{pmatrix} T^{(\alpha)}_{11} & T^{(\alpha)}_{12} \\ T^{(\alpha)}_{21} & T^{(\alpha)}_{22} \end{pmatrix}$ and $R^{(\alpha)} = \begin{pmatrix} 0 & R^{(\alpha)}_{12} \\ R^{(\alpha)}_{21} & R^{(\alpha)}_{22} \end{pmatrix}$ respectively with respect to the decomposition $L^2_a(dA_\alpha) = G_\alpha \oplus G_\alpha^\perp$. Since $G_\alpha \neq L^2_a(dA_\alpha)$, there exists a [8] unit vector $g_0 \in G_\alpha^\perp$ and denote $G^0_\alpha = \bigvee\{G_\alpha, g_0\}$. Then $dim\ G^0_\alpha \geq 2$. If Q_0 is the orthogonal projection on G^0_α, then $Q_0 S^{(\alpha)}_\psi Q_0 \neq 0$ and $rank(Q_0 S^{(\alpha)}_\psi Q_0) \leq 2$. Let $\left(T^{(\alpha)}_\phi\right)^0 = Q_0 T^{(\alpha)}_\phi Q_0$, $\left(S^{(\alpha)}_\psi\right)^0 = Q_0 S^{(\alpha)}_\psi Q_0$, and $\left(R^{(\alpha)}\right)^0 = Q_0 R^{(\alpha)} Q_0$. Then

$$\left\langle \left(T^{(\alpha)}_\phi\right)^0 f, f\right\rangle \left\langle \left(S^{(\alpha)}_\psi\right)^0 f, f\right\rangle = \left\langle \left(R^{(\alpha)}\right)^0 f, f\right\rangle$$

for all $f \in G^0_\alpha$ with $\|f\| = 1$ and $\left(S^{(\alpha)}_\psi\right)^0 \neq 0$. We shall now show that $\left(T^{(\alpha)}_\phi\right)^0$ is a scalar multiple of I_0, the identity operator from G^0_α into itself. If $\left(S^{(\alpha)}_\psi\right)^0$ is normal, then $\left(S^{(\alpha)}_\psi\right)^0$ is not a scalar multiple of I_0 as $dim\ G_\alpha \geq 1$. Hence $\left(T^{(\alpha)}_\phi\right)^0 = \vartheta_0 I_0$ by Lemma 2.4. Hence $\psi = \vartheta_0$ and $R^{(\alpha)} = \vartheta_0 S^{(\alpha)}_\psi$. We shall now consider the case when $\left(S_\psi^{(\alpha)}\right)^0$ is not normal. If $dim\ G_\alpha^0 < \infty$, then $\left(T_\phi^{(\alpha)}\right)^0 = \vartheta_0 I_0$ by Theorem 3.1 and Lemma 2.4. Suppose $dim\ G_\alpha^0$ is infinite. Let $G_\alpha^1 = \bigvee\left\{Range\left(\left(S_\psi^{(\alpha)}\right)^0\right), Range\left(\left(\left(S_\psi^{(\alpha)}\right)^0\right)^*\right)\right\}$. Then $\left(T_\phi^{(\alpha)}\right)^0$, $\left(S_\psi^{(\alpha)}\right)^0$ and $\left(R^{(\alpha)}\right)^0$ belong to $\mathcal{L}(G_\alpha^0)$ and have operator matrices of the following form with respect to the decomposition $G_\alpha^0 = G_\alpha^1 \oplus (G_\alpha^1)^\perp$: $\left(T_\phi^{(\alpha)}\right)^0 = \begin{pmatrix} \left(T_{11}^{(\alpha)}\right)^0 & \left(T_{12}^{(\alpha)}\right)^0 \\ \left(T_{21}^{(\alpha)}\right)^0 & \left(T_{22}^{(\alpha)}\right)^0 \end{pmatrix}$, $\left(S_\psi^{(\alpha)}\right)^0 = \begin{pmatrix} \left(S_{11}^{(\alpha)}\right)^0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\left(R^{(\alpha)}\right)^0 = \begin{pmatrix} \left(R_{11}^{(\alpha)}\right)^0 & \left(R_{12}^{(\alpha)}\right)^0 \\ \left(R_{21}^{(\alpha)}\right)^0 & 0 \end{pmatrix}$. Since $dim\ G_\alpha^1 \leq 4$ and $\left(S_{11}^{(\alpha)}\right)^0$ is not normal, without loss of generality, we can take $\left(T_{11}^{(\alpha)}\right)^0 = 0$ by Theorem 3.1. So $\left(R_{11}^{(\alpha)}\right)^0 = 0$. If $f_1 \in G_\alpha^1$ and $f_2 \in (G_\alpha^1)^\perp$ and $\|f_1\| = \|f_2\| = 1$. Let $f = \lambda f_1 + e^{i\theta}\sqrt{1-\lambda^2} f_2$ are unit vectors in G_α for all $\lambda \in [0,1]$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Now

$$\left\langle \left(T_\phi^{(\alpha)}\right)^0 f, f \right\rangle = (1-\lambda^2)\left\langle \left(T_{22}^{(\alpha)}\right)^0 f_2, f_2 \right\rangle + \lambda\sqrt{1-\lambda^2}\left[e^{i\theta}\left\langle \left(T_{12}^{(\alpha)}\right)^0 f_2, f_1 \right\rangle + e^{-i\theta}\left\langle \left(T_{21}^{(\alpha)}\right)^0 f_1, f_2 \right\rangle\right],$$

$$\left\langle \left(R^{(\alpha)}\right)^0 f, f \right\rangle = \lambda\sqrt{1-\lambda^2}\left\langle \left(R_{12}^{(\alpha)}\right)^0 f_2, f_1 \right\rangle e^{i\theta} + \lambda\sqrt{1-\lambda^2}\left\langle \left(R_{21}^{(\alpha)}\right)^0 f_1, f_2 \right\rangle e^{-i\theta},$$

$$\left\langle \left(S_\psi^{(\alpha)}\right)^0 f, f \right\rangle = \lambda^2\left\langle \left(S_{11}^{(\alpha)}\right)^0 f_1, f_1 \right\rangle \text{ and}$$

$$\begin{aligned}\left\langle \left(T_\phi^{(\alpha)}\right)^0 f, f \right\rangle\left\langle \left(S_\psi^{(\alpha)}\right)^0 f, f \right\rangle &= \lambda^2(1-\lambda^2)\left\langle \left(T_{22}^{(\alpha)}\right)^0 f_2, f_2 \right\rangle\left\langle \left(S_{11}^{(\alpha)}\right)^0 f_1, f_1 \right\rangle \\ &+ \lambda^3\sqrt{1-\lambda^2}\left\langle \left(T_{12}^{(\alpha)}\right)^0 f_2, f_1 \right\rangle\left\langle \left(S_{11}^{(\alpha)}\right)^0 f_1, f_1 \right\rangle e^{i\theta} \\ &+ \lambda^3\sqrt{1-\lambda^2}\left\langle \left(T_{21}^{(\alpha)}\right)^0 f_1, f_2 \right\rangle\left\langle \left(S_{11}^{(\alpha)}\right)^0 f_1, f_1 \right\rangle e^{-i\theta}.\end{aligned}$$

Since $\left\langle \left(T_{\phi}^{(\alpha)}\right)^0 f, f\right\rangle \left\langle \left(S_{\psi}^{(\alpha)}\right)^0 f, f\right\rangle = \left\langle \left(R^{(\alpha)}\right)^0 f, f\right\rangle$ and $0 \le \lambda \le 1$; by comparing the coefficients of $e^{i\theta}$, $e^{-i\theta}$ and the constant terms both the sides, we obtain

$$\left\langle \left(T_{22}^{(\alpha)}\right)^0 f_2, f_2\right\rangle \left\langle \left(S_{11}^{(\alpha)}\right)^0 f_1, f_1\right\rangle = 0,$$

$$\left\langle \left(T_{12}^{(\alpha)}\right)^0 f_2, f_1\right\rangle \left\langle \left(S_{11}^{(\alpha)}\right)^0 f_1, f_1\right\rangle = 0,$$

$$\begin{aligned} &\left\langle \left(R_{12}^{(\alpha)}\right)^0 f_2, f_1\right\rangle = 0, \text{and} \\ &\left\langle \left(T_{21}^{(\alpha)}\right)^0 f_1, f_2\right\rangle \left\langle \left(S_{11}^{(\alpha)}\right)^0 f_1, f_1\right\rangle = 0, \ \left\langle \left(R_{21}^{(\alpha)}\right)^0 f_1, f_2\right\rangle = 0. \end{aligned} \tag{8}$$

Thus we get $\left\langle \left(T_{22}^{(\alpha)}\right)^0 f_2, f_2\right\rangle = 0$ for $f_2 \in (G_\alpha^1)^\perp$. Hence $\left(T_{22}^{(\alpha)}\right)^0 = 0$, since $\left(S_{11}^{(\alpha)}\right)^0 \neq 0$. We shall now show that $\left(T_{21}^{(\alpha)}\right)^0 = 0$. Assume the opposite, that is, $\left(T_{21}^{(\alpha)}\right)^0 \neq 0$. This implies there exists a unit vector $f_0 \in (G_\alpha^1)^\perp$, such that $\left(\left(T_{21}^{(\alpha)}\right)^0\right)^* f_0 = g \neq 0$. Let $g_0 = \frac{g}{\|g\|}$ and take a unit vector $g_1 \in (G_\alpha^1)^\perp$ with $g_1 \perp g_0$. Let $f = rg_0 + e^{i\eta}\sqrt{1-r^2}g_1$. Then $\|f\| = 1$. From the last identity in (8), we obtain $\left\langle \left(S_{11}^{(\alpha)}\right)^0 f, f\right\rangle = 0$. Moreover, $\left\langle \left(S_{11}^{(\alpha)}\right)^0 g_0, g_0\right\rangle = 0$, $\left\langle \left(S_{11}^{(\alpha)}\right)^0 g_1, g_0\right\rangle = 0$, $\left\langle \left(S_{11}^{(\alpha)}\right)^0 g_0, g_1\right\rangle = 0$, $\left\langle \left(S_{11}^{(\alpha)}\right)^0 g_1, g_1\right\rangle = 0$. Notice that g_1 is an arbitrary vector in G_α^1 with $g_0 \perp g_1$. The operator $\left(S_{11}^{(\alpha)}\right)^0$ has the block matrix $\left(S_{11}^{(\alpha)}\right)^0 = \begin{pmatrix} 0 & 0 \\ 0 & \left(S_{11}^{(\alpha)}\right)^{01} \end{pmatrix}$ with respect to the decomposition $G_\alpha^1 = \bigvee\{g_0\} \oplus \{g_0\}^\perp$. This implies

$$\begin{aligned} &dim\ G_\alpha^1 = dim \bigvee \left\{ Range\left(\left(S_{\psi}^{(\alpha)}\right)^0\right), Range\left(\left(\left(S_{\psi}^{(\alpha)}\right)^0\right)^*\right)\right\} \\ &= dim \bigvee \left\{ Range\left(\left(S_{11}^{(\alpha)}\right)^0\right),\ Range\left(\left(\left(S_{11}^{(\alpha)}\right)^0\right)^*\right)\right\} \le dim\ (G_\alpha^1) - 1. \end{aligned}$$

This is a contradiction. So $\left(T_{21}^{(\alpha)}\right)^0 = 0$. Similarly, one can show that $\left(T_{12}^{(\alpha)}\right)^0 = 0$. Thus we get $\left(T_{\phi}^{(\alpha)}\right)^0 = 0$. If $G_\alpha^0 = L_a^2(dA_\alpha)$, then $T_\phi^{(\alpha)} = 0 = 0 \cdot I_{\mathcal{L}(L_a^2(dA_\alpha))}$. Hence $\phi \equiv 0$. If $T_\phi^{(\alpha)} \neq 0$, then there exists a unit vector $u \in (G_\alpha^0)^\perp$, such that $\left(T_\phi^{(\alpha)}\right)^u = Q_u T_\phi^{(\alpha)} Q_u \neq 0$, where Q_u is the orthogonal on $\bigvee\{G_\alpha^0, u\}$. Denote $\left(S_\psi^{(\alpha)}\right)^u = Q_u S_\psi^{(\alpha)} Q_u$ and $R_u^{(\alpha)} = Q_u R^{(\alpha)} Q_u$. Then $\left(S_\psi^{(\alpha)}\right)^u \neq 0$. Notice that $Rank\left(\left(T_\phi^{(\alpha)}\right)^u\right) \leq 2$ and $Rank\left(\left(S_\psi^{(\alpha)}\right)^u\right) \leq 4$. Since $\left(T_\phi^{(\alpha)}\right)^u, \left(S_\psi^{(\alpha)}\right)^u$ are finite rank, we have $\left(T_\phi^{(\alpha)}\right)^u = 0$ or $\left(S_\psi^{(\alpha)}\right)^u = 0$ by Theorem 3.1. But if $\left(T_\phi^{(\alpha)}\right)^u \neq 0$, we have $\left(S_\psi^{(\alpha)}\right)^u = 0$. This contradicts the assumption on G_α. The proof is complete. □

Theorem 4.2. *Let $S \in \mathcal{L}(L_a^2(dA_\alpha))$. If*

$$\begin{aligned} &\Theta_S^{(\alpha)}(x,\overline{y})K^{(\alpha)}(x,\overline{y}) \approx \Theta_{ST}^{(\alpha)}(x,\overline{y})K^{(\alpha)}(x,\overline{y}) \text{ and} \\ &\Theta_T^{(\alpha)}(x,\overline{y})K^{(\alpha)}(x,\overline{y}) \approx K^{(\alpha)}(x,\overline{y}) \end{aligned} \tag{9}$$

for all $x, y \in \mathbb{D}$ and for all $T \in \mathcal{L}(L_a^2(dA_\alpha))$, then $S = T_\phi^{(\alpha)}$ for some $\phi \in H^\infty(\mathbb{D})$ and the matrix of S is lower triangular.

Proof. The identity

$$\Theta_S^{(\alpha)}(x,\overline{y})K^{(\alpha)}(x,\overline{y}) \approx \Theta_{ST}^{(\alpha)}(x,\overline{y})K^{(\alpha)}(x,\overline{y})$$

and

$$\Theta_T^{(\alpha)}(x,\overline{y})K^{(\alpha)}(x,\overline{y}) \approx K^{(\alpha)}(x,\overline{y})$$

for all $x, y \in \mathbb{D}$ implies that

$$\left\langle S\left(\sum_{j=1}^n c_j K_{x_j}^{(\alpha)}\right), \left(\sum_{j=1}^n c_j K_{x_j}^{(\alpha)}\right)\right\rangle = \left\langle ST\left(\sum_{j=1}^n c_j K_{x_j}^{(\alpha)}\right), \left(\sum_{j=1}^n c_j K_{x_j}^{(\alpha)}\right)\right\rangle$$

and

$$\left\langle T\left(\sum_{j=1}^n c_j K_{x_j}^{(\alpha)}\right), \left(\sum_{j=1}^n c_j K_{x_j}^{(\alpha)}\right)\right\rangle = \left\langle \left(\sum_{j=1}^n c_j K_{x_j}^{(\alpha)}\right), \left(\sum_{j=1}^n c_j K_{x_j}^{(\alpha)}\right)\right\rangle$$

where c_j's are constants and $x_1, ..., x_n \in \mathbb{D}$. Since the set of vectors

$$\left\{\sum_{j=1}^n c_j K_{x_j}^{(\alpha)}, x_j \in \mathbb{D}, j = 1, ..., n,\ c_1, ..., c_n \text{ constants}\right\}$$

is dense [2] in $L^2_a(dA_\alpha)$, we obtain $\langle STg, g\rangle\langle g, g\rangle = \langle Sg, g\rangle\langle Tg, g\rangle$ for all $g \in L^2_a(dA_\alpha)$. That is, $\langle STf, f\rangle = \langle Sf, f\rangle\langle Tf, f\rangle$ for all $f \in L^2_a(dA_\alpha)$ with $\|f\| = 1$. This implies

$$\langle STk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}}\rangle = \langle Sk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}}\rangle\langle Tk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}}\rangle \tag{10}$$

for all $z \in \mathbb{D}$. So assume (10) holds for all $T \in L^2_a(dA_\alpha)$ and for all $z \in \mathbb{D}$. We shall first show that if $f, g \in L^2_a(dA_\alpha)$, then $\rho_\alpha(f \otimes g)(z) = \frac{\overline{g(z)}}{\|K_z^{1+\frac{\alpha}{2}}\|^2} f(z)$, $z \in \mathbb{D}$. This can be verified as follows: Let $f, g \in L^2_a(dA_\alpha)$ and $z \in \mathbb{D}$. Then $\rho_\alpha(f \otimes g)(z) = \left\langle (f \otimes g)\frac{K_z^{1+\frac{\alpha}{2}}}{\|K_z^{1+\frac{\alpha}{2}}\|}, \frac{K_z^{1+\frac{\alpha}{2}}}{\|K_z^{1+\frac{\alpha}{2}}\|}\right\rangle = \frac{1}{\|K_z^{1+\frac{\alpha}{2}}\|^2}\langle K_z^{1+\frac{\alpha}{2}}, g\rangle\langle f, K_z^{1+\frac{\alpha}{2}}\rangle = \frac{\overline{g(z)}}{\|K_z^{1+\frac{\alpha}{2}}\|^2} f(z)$. Now if (9) holds for all $T \in L^2_a(dA_\alpha)$ and $z \in \mathbb{D}$, then let $T = f \otimes g$ for $f, g \in L^2_a(dA_\alpha)$. Then $\langle STk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}}\rangle = (S\rho_\alpha(f \otimes g))(z) = \frac{\overline{g(z)}}{\|K_z^{1+\frac{\alpha}{2}}\|^2}(Sf)(z)$. From (9), it follows that

$$\frac{\overline{g(z)}}{\|K_z^{1+\frac{\alpha}{2}}\|^2}(Sf)(z) = \frac{\overline{g(z)}}{\|K_z^{1+\frac{\alpha}{2}}\|^2}\rho_\alpha(S)(z)f(z).$$

This implies $(Sf)(z) = \rho_\alpha(S)(z)f(z)$ for all $f \in L^2_a(dA_\alpha)$. From Lemma 3 and Lemma 2.5, it follows that $S = T_{\rho_\alpha(S)}$ where $\rho_\alpha(S) \in H^\infty(\mathbb{D})$. Now S is a Toeplitz operator with bounded analytic symbol. It is not difficult to verify that an operator $C \in \mathcal{L}(L^2_a(dA_\alpha))$ is an analytic Toeplitz operator if and only if

$$\left\langle Ce_{j+1}^{(\alpha)}, e_{i+1}^{(\alpha)}\right\rangle = \begin{cases} \sqrt{\frac{\gamma_{i+1,\alpha}}{\gamma_{j+1,\alpha}}}\sqrt{\frac{\gamma_{j,\alpha}}{\gamma_{i,\alpha}}}\langle Ce_j^{(\alpha)}, e_i^{(\alpha)}\rangle & \text{if } i \geq j\ ; \\ 0, & \text{otherwise}\ . \end{cases}$$

Thus the matrix of S is lower triangular. □

Corollary 4.1. *Let $S \in \mathcal{L}(L^2_a(dA_\alpha))$. Then $\rho_\alpha(ST) = \rho_\alpha(S)\rho_\alpha(T)$ for all $T \in \mathcal{L}(L^2_a(dA_\alpha))$ if and only if $S = T_\phi^\alpha$ for some $\phi \in H^\infty(\mathbb{D})$.*

Proof. The necessary part follows from Theorem 4.2. For the sufficient part, let $S = M_\phi^{(\alpha)} = T_\phi^{(\alpha)}$, $\phi \in H^\infty(\mathbb{D})$. Then

$$\rho_\alpha(T_\phi^{(\alpha)}T)(z) = \langle T_\phi^{(\alpha)}Tk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}}\rangle = \phi(z)\frac{(Tk_z^{1+\frac{\alpha}{2}})(z)}{\left\|K_z^{1+\frac{\alpha}{2}}\right\|}$$

for all $T \in \mathcal{L}(L_a^2(dA_\alpha))$. Thus from [7], [6], it follows that

$$\begin{aligned}\rho_\alpha(T_\phi^{(\alpha)}T)(z) &= \phi(z)\langle Tk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}}\rangle = \phi(z)\rho_\alpha(T)(z)\\ &= \langle T_\phi^{(\alpha)}k_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}}\rangle\rho_\alpha(T)(z) = \rho_\alpha(T_\phi^{(\alpha)})(z)\rho_\alpha(T)(z),\end{aligned}$$

for all $T \in \mathcal{L}(L_a^2(dA_\alpha))$. □

Corollary 4.2. *Let $T \in \mathcal{L}(L_a^2(dA_\alpha))$. Then $\rho_\alpha(ST)(z) = \rho_\alpha(S)(z)\rho_\alpha(T)(z)$ for all $S \in \mathcal{L}(L_a^2(dA_\alpha))$ if and only if $T = T_{\overline{\psi}}^{(\alpha)}$ where $\psi \in H^\infty(\mathbb{D})$.*

Proof. Notice that $\rho_\alpha(A^*)(z) = \overline{\rho_\alpha(A)}(z)$ for all $A \in \mathcal{L}(L_a^2(dA_\alpha))$. The result follows from Theorem 4.2. □

Corollary 4.3. *Let $S \in \mathcal{L}(L_a^2(dA_\alpha))$ be invertible. If $\langle S^{-1}f, g\rangle\langle Sf, g\rangle = \langle f, g\rangle^2$ for all $f, g \in L_a^2(dA_\alpha)$, then there exists a constant $\beta \in \mathbb{C}$ such that $S = \beta I$ where $I \in \mathcal{L}(L_a^2(dA_\alpha))$ is the identity operator.*

Proof. Let $f \in L_a^2(dA_\alpha)$ be such that $f \neq 0$. Then for every $g \in (sp\{f\})^\perp \subset L_a^2(dA_\alpha)$, we have $\langle S^{-1}f, g\rangle = 0$ or $\langle Sf, g\rangle = 0$ because $\langle S^{-1}f, g\rangle\langle Sf, g\rangle = \langle f, g\rangle^2 = 0$. Let $K_f = \{g \in (sp\{f\})^\perp \mid \langle Sf, g\rangle = 0\}$ and $L_f = \{g \in (sp\{f\})^\perp \mid \langle S^{-1}f, g\rangle = 0\}$. Then $K_f \cup L_f = (sp\{f\})^\perp$. Since $(sp\{f\})^\perp$, K_f and L_f are closed subspaces, we obtain $K_f \subseteq L_f = (sp\{f\})^\perp$ or $L_f \subseteq K_f = (sp\{f\})^\perp$. If $L_f = (sp\{f\})^\perp$, then $S^{-1}f \in sp\{f\}$. So there exists a $\beta_f \in \mathbb{C}$ such that $S^{-1}f = \beta_f f \neq 0$, that is, $Sf = \frac{1}{\beta_f}f$. If $K_f = (sp\{f\})^\perp$, then $Sf \in sp\{f\}$, that is, $Sf = \beta_f f$ for some scalar β_f. Since f is arbitrary, we see that, for every $f \in L_a^2(dA_\alpha)$, there is a scalar β_f such that $Sf = \beta_f f$. This implies that there exists a $\beta \in \mathbb{C}$ such that $S = \beta I$. □

Corollary 4.4. *Let $\phi \in L^\infty(\mathbb{D})$ be such that $S_\phi^{(\alpha)}$ is invertible. Then there exist $f, g \in L_a^2(dA_\alpha)$, such that $\left\langle \left(S_\phi^{(\alpha)}\right)^{-1} f, g\right\rangle \left\langle S_\phi^{(\alpha)} f, g\right\rangle \neq \langle f, g\rangle^2$.*

Proof. Suppose $\left\langle \left(S_\phi^{(\alpha)}\right)^{-1} f, g\right\rangle \left\langle S_\phi^{(\alpha)} f, g\right\rangle = \langle f, g\rangle^2$ for all $f, g \in L_a^2(dA_\alpha)$. Then by Corollary 4.3, we obtain $S_\phi^{(\alpha)} = \beta I$ for some constant $\beta \in \mathbb{C}$. This is not possible by Lemma 2.3. □

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