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QUASI-EXACT SOLVABILITY OF THE D-DIMENSIONAL SEXTIC POTENTIAL IN TERMS OF TRUNCATED BI-CONFLUENT HEUN FUNCTIONS*

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Dedicated to Dr. Vasile Drăgan on the occasion of his 70^{th} anniversary

Abstract

The D-dimensional Schrödinger equation for an isotropic sextic potential is brought to a form compatible with the canonical bi-confluent Heun differential equation. The quasi-exactly solvable properties of the model are recovered by considering polynomial solutions for the bi-confluent Heun equation which constrains the potential parameters in terms of rotation quantum number, space dimension and order of the exact solvability. It is shown that the state independence of the potential can be maintained by using a see-saw adjustment between the rotation quantum number and the exact solvability order. An analysis on the exactly solvable instances of the sextic potential is presented in correlation with the extended set of exactly solvable states.
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1 Introduction

The study of Heun's differential equation [1] and its confluent forms is very important in mathematics [2, 3, 4, 5, 6, 7, 8, 9] due to its many valuable physics applications [10, 11, 12, 13, 14, 15, 16, 17]. Indeed, the special cases of the confluent Heun equation include well known mathematical physics equations, such as the Spheroidal, Generalized Spheroidal, Whittaker-Hill, Razavy, Mathieu and many other equations. In particular, the confluent Heun equation was consistently used in quantum mechanics for the purpose of finding new categories of solvable potentials [18, 19, 20]. As the Schrödinger equation is the cornerstone of quantum mechanical treatment of physical systems, the information related to it is essential. Exact solutions of the Schrödinger equation determined in a fully algebraic manner, are directly related to the symmetry properties of the model. The study of these solutions reveals real or hidden and unexpected properties of the modeled physical systems and provides guidelines to construct consistent perturbation approaches for quantitative calculations of relevant quantities for more complex potentials. The limited number of exactly solvable potentials include the Coulomb, Kratzer, harmonic oscillator, Davidson, Morse, Pöschl-Teller, Scarf, Rosen-Morse, Eckart, Nathanzon and a few others. A bridge between exact models and the exactly non-solvable potentials is offered by the notion of quasi-exact solvability [22], which is understood as the property of the model to have only a finite number of exact and explicit analytical solutions for certain parametrizations of the considered potential. All these quasi-exactly solvable models arise as special cases of the confluent Heun equation with polynomial solutions [14, 8]. In this sense, especially useful for physical phenomena are few double-well potentials, whose low-lying eigenstates are related to the finite polynomial solutions of the confluent Heun equation. In many cases, extension to multiple dimensions is possible, however with particular changes in physical implications [14].

Here, the case of the quasi-exactly solvable multidimensional isotropic sextic potential will be considered in terms of the polynomial solutions of the bi-confluent Heun equation. The aim of the study is to obtain the restriction of the potential parameters in terms of the dimension of the coordinate space and using this condition to extend and optimize the number of exactly solvable states for a certain set of potential parameters. Additionally, the quasi-exactly solvable form of the potential is analyzed in what concerns the number of exhibiting critical points.

The paper is structured as follows. In the next section, the general canonical form of the bi-confluent Heun differential equation will be presented together with the conditions which accommodate solutions of the polynomial type. Section 3 will be devoted to the relation between the quasi-exactly solvable D-dimensional Schrödinger equation for an isotropic sextic potential and the bi-confluent Heun equation with polynomial solutions. In Section 4, an example calculation will be presented for the three-dimensional case. The final conclusions will be drawn in the last section.

2 Polynomial solutions of the bi-confluent Heun differential equation

The canonical form of the bi-confluent Heun differential equation is [2, 3, 4, 5, 6]:

$$\begin{aligned} &yh''(y) + \left(1 + \alpha - \beta y - 2y^2\right) h'(y) \\ &+ \left\{ (\gamma - \alpha - 2)y - \frac{1}{2}\left[\delta + \beta(1+\alpha)\right] \right\} h(y) = 0. \end{aligned} \tag{1}$$

If $\alpha > 0$, it admits solutions of the power series form [4]:

$$h(y) = \sum_{p=0}^{\infty} \frac{A_p}{(1+\alpha)_p p!} y^p, \tag{2}$$

where $A_0 = 1$, and

$$(x)_p = \frac{\Gamma(x+p)}{\Gamma(x)} = x(x+1)...(x+p-1). \tag{3}$$

is a Pochhammer symbol [21]. The coefficients A_p must then satisfy the three-term recurrence relation

$$\begin{aligned} &A_{p+2} - A_{p+1} \left\{ (p+1)\beta + \frac{1}{2}\left[\delta + \beta(1+\alpha)\right] \right\} \\ &+ A_p(\gamma - 2 - \alpha - 2p)(p+1)(p+\alpha+1) = 0, \end{aligned} \tag{4}$$

which is obtained from Eq.(1) when inserting (2) into it. In order to have polynomial solutions, the power series (2) must be firstly bounded below, which results in the initial condition $A_{-1} = 0$ for the recurrence relation. The truncation from above of the power series is conditioned by

$$\gamma - 2 - \alpha = 2n, \;\; n = 0, 1, 2, .., \tag{5}$$

and

$$A_{n+1} = 0. \tag{6}$$

Applying these conditions to the recurrence relation (4), one can easily see that all coefficients A_p with $p > n$ vanish and the series (2) is indeed truncated to a polynomial of degree n. The last condition actually represents a set of linear equations for the non-vanishing coefficients A_n:

$$-\frac{1}{2}\left[\delta + \beta(1+\alpha)\right] A_0 + A_1 = 0, \tag{7}$$

$$2n(1+\alpha)A_0 - \left\{\beta + \frac{1}{2}\left[\delta + \beta(1+\alpha)\right]\right\} A_1 + A_2 = 0,$$

$$4(n-1)(2+\alpha)A_1 - \left\{2\beta + \frac{1}{2}\left[\delta + \beta(1+\alpha)\right]\right\} A_2 + A_3 = 0,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$2n(n+\alpha)A_{n-1} + \left\{n\beta + \frac{1}{2}\left[\delta + \beta(1+\alpha)\right]\right\} A_n = 0. \tag{8}$$

The system of linear equations can be written in a matrix form as:

$$\mathbf{M}A = \begin{pmatrix} \Delta_0 & 1 & 0 & 0 & : & 0 & 0 \\ \Gamma_1 & \Delta_1 & 1 & 0 & : & 0 & 0 \\ 0 & \Gamma_2 & \Delta_2 & 1 & : & 0 & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & 0 & : & \Delta_{n-1} & 1 \\ 0 & 0 & 0 & 0 & : & \Gamma_{n-1} & \Delta_n \end{pmatrix} \begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ .. \\ A_{n-1} \\ A_n \end{pmatrix} = 0, \tag{9}$$

where

$$\Delta_k = -\frac{1}{2}\left[\delta + \beta(1+\alpha)\right] - k\beta, \tag{10}$$

$$\Gamma_k = 2k(n-k+1)(k+\alpha). \tag{11}$$

Finally, one can see that the second restriction (6) amounts to the compatibility condition

$$det\mathbf{M} = 0. \tag{12}$$

The truncation of the power series (2) infer that the associated model is quasi-exactly solvable [22], that is only a limited set of its states can be explicitly determined in an algebraic manner. As a matter of fact, quasi-exact solvability is directly connected to polynomial solutions of the general Heun equation [14, 8, 23]. The quasi-exact solvability of Schrödinger equations which can be brought to the bi-confluent Heun equation form is of two types. If the energy is contained explicitly in the first condition (5), then it is said that the model's quasi-exact solvability is of type two, else the energy is determined from the compatibility condition (6) and the quasi-exact solvability is of type one [14]. Note however, that in the first case the compatibility condition will be used to determine the other parameters involved in the first condition (5) and consequently defining the energy. For example, oscillator-like potentials lead to quasi-exactly solvable problems of first type, while Coulomb-like potentials are of the second type.

3 Sextic potential in D dimensions

For a particle moving in an isotropic potential in D dimensions, the hyper-radial equation has the form:

$$\left[-\frac{d^2}{dr^2} - \frac{D-1}{r}\frac{d}{dr} + \frac{l(l+D-2)}{r^2} + V(r) - E_{nl}\right]\Psi_{n_r l}(r) = 0. \tag{13}$$

The energy units are such that $\hbar = 2m = 1$, while l is the quantum number associated to the orthogonal group of rotations in D dimensions $SO(D)$. The index n_r denotes distinct solutions of the equation for fixed l. The above equation is written in a convenient Schrödinger canonical form

$$\left[-\frac{d^2}{dr^2} + \frac{\lambda(\lambda+1)}{r^2} + V(r) - E_{nl}\right]\Phi_{n_r l}(r) = 0, \tag{14}$$

with the help of the change of function $\Phi_{n_r l}(r) = r^{(D-1)/2}\Psi_{n_r l}(r)$ and using the notation $\lambda = l + (D-3)/2$. In what follows one will consider a sextic potential of the following form:

$$V(r) = Ar^2 + Br^4 + Cr^6. \tag{15}$$

It is easy to verify that the energy eigenvalue of the Schrödinger equation for such a potential satisfies the scaling property:

$$E(A, B, C) = C^{-\frac{1}{4}} E(AC^{-\frac{1}{2}}, BC^{-\frac{3}{4}}, 1). \tag{16}$$

Other two relationships can be found such that to obtain parameter free factor for the harmonic (r^2) or the quartic (r^4) term. The choice made here, will become useful in comparing the results with the quasi-exactly solvable model of Ref.[22]. Leaving aside the scale dependence, all information can be obtained by solving just the sextic potential:

$$V(r) = ar^2 + br^4 + r^6. \tag{17}$$

In order to solve the Schrödinger equation for this potential by means of Heun functions, one first make the change of variable $y = r^2/\sqrt{2}$. The new differential equation then reads as

$$\left[y\frac{d^2}{dy^2} + \frac{1}{2}\frac{d}{dy} - \frac{\lambda(\lambda+1)}{4y} - \frac{1}{4}\left(ay + 2by^2 + 4y^3 - E\right)\right]\tilde{\Phi}_{n_r l}(y) = 0, \quad (18)$$

where $\tilde{\Phi}_{n_r l}(y) = \Phi_{n_r l}(\sqrt{y\sqrt{2}})$.

Making now the change of function $\tilde{\Phi}_{n_r l}(y) = y^{\frac{\lambda+1}{2}} e^{-\frac{x}{4}(\sqrt{2}b+2x)} h(y)$, one arrives at the following equation

$$\begin{aligned} & yh''(y) + \left(\lambda + \frac{3}{2} - \frac{b}{\sqrt{2}}y - 2y^2\right)h'(y) \\ & + \left\{\left(\frac{b^2}{8} - \lambda - \frac{5}{2} - \frac{a}{2}\right)y + \frac{\sqrt{2}}{4}\left[E - \frac{b}{2}(2\lambda+3)\right]\right\}h(y) = 0. \quad (19) \end{aligned}$$

Comparing it with (1), the following correspondences can be made:

$$\alpha = \lambda + \frac{1}{2},\ \beta = \frac{b}{\sqrt{2}},\ \delta = -\frac{E}{\sqrt{2}},\ \gamma = \frac{1}{2}\left(\frac{b^2}{4} - a\right), \quad (20)$$

and

$$\begin{aligned} \Delta_k &= \frac{\sqrt{2}}{4}\left[E - b\left(\lambda + \frac{3}{2} + 2k\right)\right], \quad (21) \\ \Gamma_k &= k(n-k+1)(2k+2\lambda+1). \quad (22) \end{aligned}$$

In order to have finite polynomial solutions, the first condition (5) becomes a relation between the potential parameters, the rotation quantum number λ and the truncation order n:

$$a = \frac{b^2}{4} - 2\lambda - 5 - 4n = \frac{b^2}{4} - 2l - D - 4n - 2. \quad (23)$$

In order to have a state-independent potential, the coefficients a and b must be invariant with the change of rotation quantum number l and the truncation order n. This is realized, if the following condition is fulfilled:

$$l + 2n = K = const. \quad (24)$$

A see-saw variation of l and n can work within this restriction [24, 25, 26, 27, 28]. Indeed, increasing l with two units, will trigger the decrease of n with a single unit. Setting a maximum value n_{Max} for n, one can exactly determine only the odd-l or even-l states, with l-dependent number of solutions for the r variable.

Let us turn to the general form of the sextic potential, whose critical points are $r = 0$ and

$$r_{\pm} = \sqrt{\frac{1}{3}\left(-b \pm \sqrt{b^2 - 3a}\right)}. \tag{25}$$

Plugging in the above equation the identities (23) and (24), one obtains the quasi-exactly solvable form of the sextic potential

$$V_{QE}(r) = \left(\frac{b^2}{4} - 2(K+1) - D\right) r^2 + br^4 + r^6, \tag{26}$$

whose non-zero critical points are given by

$$r_{\pm} = \sqrt{\frac{1}{3}\left(-b \pm \sqrt{\frac{b^2}{4} + 6(K+1) + 3D}\right)}. \tag{27}$$

Judging by the number of critical points, there are three different cases:

1) For $b > 2\sqrt{2(K+1)+D}$, the potential (26) has a single minimum in $r = 0$. The domain of existence for this case decreases with the increase of the dimensions number D and the number of exactly solvable states involved in the quantity K.

2) The critical point $r = 0$ of the potential (26) becomes a maximum, and an additional minimum appears at r_+ if $-2\sqrt{2(K+1)+D} < b < 2\sqrt{2(K+1)+D}$. The increase of $2(K+1)+D$ quantity causes the increase of the existence interval, the lowering in energy of the potential minimum and the displacement of the minimum position to higher r values. The effect of b variation is opposite.

3) Finally, potential (26) can have simultaneously minima in $r = 0$ and r_+, separated by a maximum in r_- when $b < -2\sqrt{2(K+1)+D}$. In this case, increasing $2(K+1)+D$ leads to an energy lowering for the maximum and non-zero minimum of the potential, and to their shifting to low and respectively larger values of r. While larger values of b correspond to a simultaneously increased maximum and decreased minimum of the potential, both being displaced to higher r values.

The total wave function corresponding to the D-dimensional Schrödinger equation for a quasi-exactly solvable sextic potential can be written as follows:

$$\Psi_{n_r l}(r) = N_{n_r l} r^l e^{-\frac{r^4}{4} - \frac{br^2}{4}} \sum_{p=0}^{n_{Max} - \frac{l+\tau}{2}} \frac{A_p^{n_r}}{\left(l + \frac{D}{2}\right)_p p!} \left(\frac{r^2}{\sqrt{2}}\right)^p, \tag{28}$$

where $\tau = 0$ for even l states, and $\tau = 1$ for odd l states, while n_r denotes the order of the solution for the secular equation involving the coefficients A_p. The norm $N_{n_r l}$ can be determined in terms of hypergeometric functions of the first kind [21].

At this point, it is instructive to compare this formalism with the one-dimensional quasi-exactly solvable model of Ref.[22], whose differential operator is

$$-\frac{d^2}{dx^2} + \frac{\left(2s - \frac{3}{2}\right)\left(2s - \frac{1}{2}\right)}{x^2} + \left[b'^2 - 4a'\left(s + \frac{1}{2} + n\right)\right]x^2 + 2a'b'x^4 + a'^2x^6. \tag{29}$$

The above equation can be easily recovered from the formulas (13) and (26) by matching the involved parameters as:

$$a' = 1,\ \ b' = \frac{b}{2},\ \ 2s = l + \frac{D}{2}, \tag{30}$$

and with integer n having the same significance of exact solvability order.

4 Three-dimensional case

In order to clarify the procedure, one will treat here the case of $D = 3$ for a maximal truncation order $n_{Max} = 1$ and consider only the even l states. This choice amounts to $K = 2$ and to the following state-independent form of the quasi-exactly solvable sextic potential

$$V_{QE}(r) = \left(\frac{b^2}{4} - 9\right)r^2 + br^4 + r^6. \tag{31}$$

As can be seen, the results will have a parametric dependence only on b. For $l = 0$, the truncation order is $n = 1$ and the compatibility condition for the non-vanishing coefficients A_0 and A_1 reads:

$$det\begin{vmatrix} \Delta_0 & 1 \\ \Gamma_1 & \Delta_1 \end{vmatrix} = 0. \tag{32}$$

It can be expanded into a quadratic equation for the energy

$$\left[E - b\left(l + \frac{5}{2}\right)\right]^2 - b^2 - 8(2l+3) = 0. \tag{33}$$

The two solution for the energy are then

$$
\begin{aligned}
E_{00} &= b\left(l+\frac{5}{2}\right)-\sqrt{b^2+16\left(l+\frac{3}{2}\right)}, & (34)\\
E_{10} &= b\left(l+\frac{5}{2}\right)+\sqrt{b^2+16\left(l+\frac{3}{2}\right)}, & (35)
\end{aligned}
$$

with the corresponding wave-functions:

$$
\begin{aligned}
\Psi_{00}(r) &= N_{00}e^{-\frac{r^4}{4}-\frac{br^2}{4}}\left[1+\frac{b-\sqrt{b^2+16\left(l+\frac{3}{2}\right)}}{4\left(l+\frac{3}{2}\right)}r^2\right], & (36)\\
\Psi_{10}(r) &= N_{10}e^{-\frac{r^4}{4}-\frac{br^2}{4}}\left[1+\frac{b+\sqrt{b^2+16\left(l+\frac{3}{2}\right)}}{4\left(l+\frac{3}{2}\right)}r^2\right], & (37)
\end{aligned}
$$

where coefficients $A_0 = 1$ and A_1 were determined from the two-dimensional system of linear equations with compatibility condition (32).

Now, for $l = 2$ and $n = 0$, there is a single solution which is simply

$$
E_{02} = b\left(l+\frac{3}{2}\right), \quad \Psi_{02}(r) = N_{02}r^2e^{-\frac{r^4}{4}-\frac{br^2}{4}}. \qquad (38)
$$

5 Conclusions

The emergence of polynomial solutions for the bi-confluent Heun differential equation was discussed in connection to the notion of quasi-exact solvability. The formalism was used to investigate the quasi-exact solvability of the D-dimensional Schrödinger equation for an isotropic sextic potential. This was done by bringing the corresponding Schrödinger equation, through a change of variable and function, to a canonical bi-confluent Heun form. The conditions for polynomial solutions of the bi-confluent Heun equation are transposed into constrains on the potential parameters in terms of rotation quantum number, space dimension and order of the exact solvability. The properties of the sextic potential within these constraints suggest distinct responses to the variation of the involved quantities associated with to three well defined phases, where the potential have specific critical point characteristics. A mathematical artifice is used to adapt the formalism to state independent potentials with extended number of exactly solvable states in what concerns both radial and rotational quantum numbers. An illustrative example of the method was presented for the three-dimensional case. In conclusion, the present study provides a complete description of the quasi-exact solvability property of the isotropic sextic potential in connection to the polynomial solutions of the bi-confluent Heun equation, which is easily transposable to concrete applications. Also, it was shown that the mechanism assuring the potential's state independence leads to an extension of the quasi-exact solvability's utility, reflected in a greater number of exactly solvable states associated the considered potential.

References

[1] K. Heun. Zur theorie der Riemann'schen functionen zweiter ordnung mit vier verzweigungspunkten. *Math. Ann.* 33:161, 1888.

[2] P. Maroni. Equations differentielle. Sur le forme bi-confluente de l'equation de Heun. *C. R. Acad. Sci. Paris Sér. I Math.* 264:503-505, 1967.

[3] A. Decarreau, M. Cl. Dumont-Lepage, P. Maroni, A. Robert, A. Ronveaux. Formes canoniques des equations confluentes de l'equation de Heun. *Ann. Soc. Sci. Bruxelles Sér. I* 92:53-78, 1978.

[4] A. Decarreau, P. Maroni, A. Robert. Sur les equations confluentes de l'equation de Heun. *Ann. Soc. Sci. Bruxelles Sér. I* 92:151-189, 1978.

[5] A. Ronveaux. *Heun's Differential Equations*, Oxford University Press, Oxford, 1995.

[6] S. Y. Slavyanov, W. Lay. *Special Functions: A Unified Theory Based on Singularities*, Oxford University Press, Oxford, 2000.

[7] E. R. Arriola, A. Zarzo, J. S. Dehesa. Spectral properties of the biconfluent Heun differential equation. *J. Comp. and App. Math.* 37:161-169, 1991.

[8] M. A. Gonzalez Leon, J. Mateos Guilarte, A. Moreno Mosquera, M. de la Torre Mayado. On the quasi-exact solvability of the confluent heun equation. arXiv:1406.2643v2.

[9] H. Karayer, D. Demirhan, F. Büyükkiliç. Extension of Nikiforov-Uvarov method for the solution of Heun equation. *J. Math. Phys.* 56:063504, 2015.

[10] H. Ciftci, R. L. Hall, N. Saad, E. Dogu. Physical applications of second-order linear differential equations that admit polynomial solutions. *J. Phys. A: Math. Theor.* 43:415206, 2010.

[11] Y.-H. Zhang. Exact polynomial solutions of second order differential equations and their applications. *J. Phys. A: Math. Theor.* 45:065206, 2012.

[12] Bei-Hua Chen Yan Wu, Qiong-Tao Xie. Heun functions and quasi-exactly solvable double-well potentials. *J. Phys. A: Math. Theor.* 46:035301, 2013.

[13] F. Caruso, J. Martins, V. Oguri. Solving a two-electron quantum dot model in terms of polynomial solutions of a Biconfluent Heun equation. *Ann. Phys.* 347:130140, 2014.

[14] A. V. Turbiner. One-dimensional quasi-exactly solvable Schrödinger equations. *Phys. Rep.* 642:1-71, 2016.

[15] H. Karayer, D. Demirhan, F. Büyükkiliç. Solution of Schrödinger equation for two different potentials using extended Nikiforov-Uvarov method and polynomial solutions of biconfluent Heun equation. *J. Math. Phys.* 59:053501, 2018.

[16] Q. Xie. QAnalytical results for periodically-driven two-level models in relation to Heun functions. *Pramana J. Phys.* 91:19, 2018.

[17] M. Hortaçsu. Heun functions and some of their applications in physics. *Adv. High Energy Phys.* 2018:8621573, 2018.

[18] A. K. Bose. Solvable potentials. *Phys. Lett.* 7:4-7, 1963.

[19] A. Lemieux, A. K. Bose. Construction de potentiels pour lesquelles lequation de Schrödinger est soluble. *Ann. Inst. H Pointcare* A10:259-270, 1969.

[20] F. T. Hioe, D. MacMillan. E. W. Montroll, Quantum theory of anharmonic oscillators. *J. Math. Phys.* 17:1320-1327, 1976.

[21] M. Abramowitz, I. A. Stegun. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, 9th printing. New York: Dover, 1972.

[22] A. G. Ushveridze, *Quasi-Exactly Solvable Models in Quantum Mechanics* (Institute of Physics Publishing, Bristol, 1994).

[23] G. Lévai, A. M. Ishkhanyan. Exact solutions of the sextic oscillator from the bi-confluent Heun equation. *Mod. Phys. Lett. A* 34:1950134, 2019.

[24] G. Lévai, J. M. Arias. The sextic oscillator as a γ-independent potential. *Phys. Rev. C* 69:014304, 2004.

[25] G. Lévai, J. M. Arias. Search for critical-point nuclei in terms of the sextic oscillator. *Phys. Rev. C* 81:044304, 2010.

[26] P. Buganu, R. Budaca. Analytical solution for the Davydov-Chaban Hamiltonian with a sextic potential for $\gamma = 30°$. *Phys. Rev. C* 91:014306, 2015.

[27] P. Buganu, R. Budaca. Sextic potential for γ-rigid prolate nuclei. *J. Phys. G: Nucl. Part. Phys.* 42:105106, 2015.

[28] R. Budaca, P. Buganu, M. Chabab, A. Lahbas, M. Oulne. Extended study on a quasi-exact solution of the Bohr Hamiltonian. *Ann. Phys. (NY)* 375:65-90, 2016.

[29] B. Leaute, G. Marcilhacy. On the Schrodinger equations of rotating harmonic, three-dimensional and doubly anharmonic oscillators and a class of confinement potentials in connection with the biconfluent Heun differential equation. *J. Phys. A: Math. Gen.* 19:3527, 1986.