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INPUT–TO–STATE STABILITY FINITE–TIME LYAPUNOV FUNCTIONS FOR CONTINUOUS–TIME SYSTEMS*

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Dedicated to Dr. Vasile Drăgan on the occasion of his 70$^{\text{th}}$ anniversary

Abstract

In this paper we propose an input-to-state stability (ISS) criterion for continuous–time systems based on a finite–time decrease condition for a positive definite function of the norm of the state. This yields a so–called ISS finite–time Lyapunov function, which allows for easier choice of candidate functions compared to standard ISS Lyapunov functions. An alternative converse ISS theorem in terms of ISS finite–time Lyapunov functions is also provided. Moreover, we prove that ISS finite–time Lyapunov functions are equivalent with standard ISS Lyapunov functions using a Massera–type construction. The developed ISS framework can be utilized in combination with Sontag's "universal" stabilisation formula to develop input–to–state stabilizing control laws for continuous–time nonlinear systems that are affine in the control and disturbance inputs, respectively.

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1 Introduction

Stabilization of nonlinear dynamical systems, see for example, [1], or the earlier work [2], typically relies on exact models of the system dynamics. However, the estimated model parameters are subject to errors, the developed models are simplified and proposed control inputs based on feedback stabilization can be affected by perturbations or inaccuracies in measurements. Hence, it is of interest to study the stability and stabilization problems in the presence of disturbance inputs. Stability under perturbations has already been studied in early works such as [3] and [4]. Later on, the property of input–to–state stability (ISS), first introduced in [5], proved to be very useful for studying stability in the presence of disturbances. Indeed, in [6], equivalences between the ISS property and a range of other concepts such as robustness and robust Lyapunov–like functions, called ISS Lyapunov functions, were introduced. The ISS concept was further extended for stability with respect to compact sets in [7] and [8].

Since it was shown that the ISS property is equivalent with existence of an ISS LF, the problem of constructing such functions gained attention from researchers. In [9], a Zubov approach is proposed to compute ISS LFs, followed by [10] where an alternative Zubov type approach was derived. Both papers rely on the idea that ISS LFs can be obtained by computing robust LFs for suitably designed auxiliary systems. The robust LF is computed as the numerical solution of a Zubov type of equation, thus yielding a numerical approximation of an ISS LF.

Recently, in [11] and [12], a linear programming based algorithm for computing continuous, piecewise affine ISS LFs for locally ISS systems was developed. The ISS LF computed therein is a viscosity subsolution of a Hamilton-Jacobi-Bellman partial differential equation.

In the case of discrete–time systems, for the ISS concept formulation, we refer to [13]. As for computing ISS LFs in the discrete–time setting, in [14] for example, an auxiliary system was also used to compute ISS LFs via a set oriented approach.

Recently, a finite–time decrease condition for a $\mathcal{K}_\infty$ function of the norm of the state was introduced to characterize $\mathcal{KL}$-stability and to compute Lyapunov functions for continuous–time systems in [15]. This condition yields so–called finite–time Lyapunov functions, which are non–monotonic and allow for more freedom of selecting candidate functions, compared to standard Lyapunov functions. Therefore, it is of interest to extend finite–time Lyapunov functions to continuous–time systems affected by disturbances, in order to gain more freedom in selecting/constructing candidate ISS–type of functions.

In this paper, ISS finite–time Lyapunov functions for continuous–time nonlinear systems are defined. We show that existence of such a function implies ISS for systems which satisfy a mild, $\mathcal{K}$–boundedness condition. Under suitable assumptions, we also establish a converse finite–time ISS result, i.e., we show that ISS systems admit any $\mathcal{K}_\infty$–function of the norm of the state as a ISS finite–time LF function. A Massera–type construction is then developed to compute a standard ISS Lyapunov function from a finite–time ISS LF. Lastly, for the problem of designing a input-to-state stabilizing state feedback control law, we show how the constructed ISS LF can be used in combination with Sontag's "universal" formula for ISS stabilization.

Remark 1.1. *Finite–time type of non–monotonic decrease conditions were used previously for computing ISS LFs for discrete–time nonlinear systems in [16]. Therein it was shown that for sufficiently regular dynamics inherent global ISS can be established via finite–step LFs (correspondent of finite–time LFs for discrete–time systems). In [17] the concept of dissipative finite–step ISS LFs was introduced, where similarly as in [16], the candidate ISS LF function is required to decrease after a finite number of time steps, rather than at each time step. Moreover, therein an equivalent characterization of ISS in terms or existence of a dissipative finite–step ISS Lyapunov functions was shown for discrete–time systems.*

2 Preliminaries

Consider systems of the form

$$\dot{x}(t) = f(x(t), v(t)), \quad t \in \mathbb{R}_{\geq 0}, \tag{1}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a locally Lipschitz function and $v(\cdot) : [0, \infty) \to \mathbb{R}^m$ is a measurable locally essentially bounded map that represents the disturbance input. We denote the solution of the system (1) with initial value $x(0)$ at time $t = 0$, under disturbance input $v(t)$, by $x(t)$ and we assume that $x(t)$ exists and it is unique for all $t \in \mathbb{R}_{\geq 0}$ (see [18, Chapter 3] for sufficient smoothness conditions on f). The locally Lipschitz assumption on $f(x, v)$ implies that $x(t)$ is a continuous function of $x(0)$ and $v(0)$ [19, Chapter III]. Furthermore, $f(0, 0) = 0$. For brevity we omit t as an argument where its presence is obvious.

We say that a set $S \subseteq \mathbb{R}^n$ is proper if it contains the origin in its interior and it is compact.

Definition 2.1. *A proper set $S \subseteq \mathbb{R}^n$ is called an invariant set for the system* (1) *with $v(t) = 0$ if for any $x(0) \in S$, for the corresponding solution it holds that $x(t) \in S$, for all $t \in \mathbb{R}_{\geq 0}$.*

Definition 2.2. *Given a positive, real scalar d, a proper set $S \subseteq \mathbb{R}^n$ is called a d–invariant set for the system* (1) *with $v(t) = 0$ if for any $t \in \mathbb{R}_{\geq 0}$, if $x(t) \in S$, then it holds that $x(t + d) \in S$.*

Note that the d–invariance property does not imply that $x(t) \in S$ for all $t \in \mathbb{R}_{\geq 0}$ if $x(0) \in S$. See [15] for illustrations of d–invariant sets versus standard invariant sets for continuous-time nonlinear systems.

The following result was introduced in [19, Definition 24.3] to relate positive definite functions and $\mathcal{K}$–functions. A proof was proposed in [18, Lemma 4.3].

Lemma 2.1. *Consider a function $W : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ with $W(0) = 0$.*

1. *If $W(x)$ is continuous and positive definite in some neighborhood around the origin, $\mathcal{N}(0)$, then there exist two functions $\hat{\alpha}_1, \hat{\alpha}_2 \in \mathcal{K}$ such that*

$$\hat{\alpha}_1(\|x\|) \leq W(x) \leq \hat{\alpha}_2(\|x\|), \quad \forall x \in \mathcal{N}(0). \tag{2}$$

2. *If $W(x)$ is continuous and positive definite in $\mathbb{R}^n$ and additionally, $W(x) \to \infty$, when $x \to \infty$ then* (2) *holds with $\hat{\alpha}_1, \hat{\alpha}_2 \in \mathcal{K}_\infty$ and for all $x \in \mathbb{R}^n$.*

We will use the following notation for the disturbance signal v:

$$|v| := \sup_{s \geq 0} \|v(s)\|, \ |v_{[t_1,t_2]}| := \sup_{t_1 \leq s \leq t_2} \|v(s)\|, \ t_1, t_2 \in \mathbb{R}_{\geq 0},$$

with $v(s) \in \mathbb{R}^m$, for all s in the corresponding interval. Furthermore, $\|v\|$ denotes the standard vector Hölder norm. Next, we proceed by recalling the input–to–state stability definition as introduced in [6] and the definition of an ISS Lyapunov function.

Definition 2.3. *Given a proper set $\mathcal{S} \subseteq \mathbb{R}^n$, the system* (1) *is said to be input–to–state stable (ISS) in $\mathcal{S}$ with respect to v if there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for all $x(0) \in \mathcal{S}$, and every v, the corresponding solution of* (1) *satisfies*

$$\|x(t)\| \leq \beta(\|x(0)\|, t) + \gamma(|v|), \quad \forall t \in \mathbb{R}_{\geq 0}. \tag{3}$$

If the set $\mathcal{S} = \mathbb{R}^n$, then we call the ISS property global ISS.

The characterization in (3) can be equivalently stated [6] in terms of suitably modified functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ as:

$$\|x(t)\| \leq \max\{\beta(\|x(0)\|, t), \gamma(|v|)\}, \quad \forall t \in \mathbb{R}_{\geq 0}. \tag{4}$$

Definition 2.4. *A continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, for which there exist functions $\alpha_1, \alpha_2, \alpha, \chi \in \mathcal{K}_\infty$ such that for all $v \in \mathbb{R}^m$ it holds that*

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \quad \forall x \in \mathbb{R}^n \tag{5}$$

$$\dot{V}(x) \leq -\alpha(\|x\|) + \chi(|v|), \quad \forall x \in \mathcal{S}, \tag{6}$$

with $\mathcal{S} \subseteq \mathbb{R}^n$ a proper invariant set, is called an ISS Lyapunov function in $\mathcal{S}$ for the system (1)*.*

Observe that the condition (6) can be equivalently restated [6, Remark 2.4.] in terms of the existence of a function $\theta \in \mathcal{K}_\infty$ such that

$$\dot{V}(x) \leq -\theta(\|x\|), \quad \forall x \in \mathcal{S} \tag{7}$$

and for any $v \in \mathbb{R}^m$ such that $\|x\| \geq \hat{\chi}(|v|)$, $\hat{\chi}(\cdot) := (2\alpha^{-1} \circ \chi)(\cdot) \in \mathcal{K}_\infty$. Note also that in order to establish ISS in $\mathcal{S}$ from an ISS LF in the case that $\mathcal{S}$ is a strict subset of $\mathbb{R}^n$, it is additionally required that $\mathcal{S}$ is an invariant set. Otherwise the ISS property will hold within the largest proper invariant subset of $\mathcal{S}$.

3 ISS finite–time Lyapunov functions

In this section we present the concept of a ISS finite–time Lyapunov function in a continuous–time setting, in which the bound on the function's derivative is replaced by a non–monotonic, periodic decrease condition.

Let there be a continuous function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, and a real scalar $d > 0$ for which the proper set $\mathcal{S} \subseteq \mathbb{R}^n$ is d–invariant and the conditions

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \forall x \in \mathbb{R}^n, \tag{8}$$

$$V(x(t+d)) - V(x(t)) \leq -\alpha(\|x(t)\|) + \chi(|v|), \tag{9}$$

are satisfied for all $t \in \mathbb{R}_{\geq 0}$ and α_1, $\alpha_2, \alpha, \chi \in \mathcal{K}_\infty$, and for all $x(t)$, with $x(0) \in \mathcal{S}$ and $v(s) \in \mathbb{R}^m$, for any $s \in \mathbb{R}_{\geq 0}$. By similar arguments as in [6, Remark 2.4.], an equivalent form of (9) is

$$V(x(t+d)) - V(x(t)) \leq -\theta(\|x(t)\|), \tag{10}$$

whenever $\|x(t)\| \geq \hat{\chi}(|v|)$, with $\hat{\chi}(\cdot) := (2\alpha^{-1} \circ \chi)(\cdot) \in \mathcal{K}_\infty$, for a suitable function $\theta \in \mathcal{K}_\infty$.

The function V which satisfies (8) and (9) (or (8) and (10)) is called an *ISS finite–time Lyapunov function* (ISS–FTLF) for the system (1).

In order for condition (9) to be well–defined, additionally to the locally Lipschitz property of the map $f(x)$, it is assumed that there exists no finite escape time in each interval $[t, t+d]$, for all $t \in \mathbb{R}_{\geq 0}$. However, as it will be shown later, it is sufficient to require that there is no finite escape time in the time interval $[0, d]$, if the set $\mathcal{S}$ is d–invariant.

Remark 3.1. *When $\mathcal{S} = \mathbb{R}^n$, a sufficient condition for existence of the solution for all $t \in \mathbb{R}_{\geq 0}$ (additional to continuity of f) is that the map $f(x)$ is Lipschitz bounded [19, Chapter III.16]. Furthermore, note that existence of a finite escape time for initial conditions in a given set in $\mathbb{R}^n$ implies that the origin is unstable in that set [19, Chapter III.16].*

The following result relates inequality (9) with another known type of decrease condition, which will be instrumental.

Lemma 3.1. *The decrease condition* (9) *on V is equivalent with*

$$V(x(t+d)) \leq \rho(V(x(t))) + \chi(|v|), \quad \forall t \in \mathbb{R}_{\geq 0}, \tag{11}$$

for all $x(t)$ with $x(0) \in \mathcal{S}$, where ρ is a positive definite, continuous function such that $(\mathrm{id} - \rho) \in \mathcal{K}_\infty$.

Proof. The proof follows similarly as in [17, Remark 3.7]. We provide it below for completeness.
(9)⇒(11):

$$\begin{aligned} 0 \leq V(x(t+d)) &\leq V(x(t)) - \alpha(\|x(t)\|) + \chi(|v|) \\ &\leq V(x(t)) - \alpha(\alpha_2^{-1}(V(x(t)))) + \chi(|v|) \\ &= (\mathrm{id} - \alpha \circ \alpha_2^{-1})(V(x(t))) + \chi(|v|) \\ &= \rho(V(x(t))) + \chi(|v|), \end{aligned}$$

where $\rho = \mathrm{id} - \alpha \circ \alpha_2^{-1}$ can be assumed to be positive, since one can always take $\alpha_2(s) \geq 2\alpha(s) > \alpha(s)$, for all $s > 0$, thus $\rho = \mathrm{id} - \alpha \circ \alpha_2^{-1} > 0$. Additionally, $\mathrm{id} - \rho = \alpha \circ \alpha_2^{-1} \in \mathcal{K}_\infty$.

(11)⇒(9):

$$\begin{aligned}V(x(t+d)) - V(x(t)) &\leq \rho(V(x(t))) - V(x(t)) + \chi(|v|)\\ &= -(-\rho + \mathrm{id})(V(x(t))) + \chi(|v|)\\ &= -\hat{\alpha}(V(x(t))) + \chi(|v|)\\ &\leq -\hat{\alpha}(\alpha_1(\|x(t)\|)) + \chi(|v|)\\ &= -\alpha(\|x(t)\|) + \chi(|v|),\end{aligned}$$

where $\hat{\alpha} = \mathrm{id} - \rho \in \mathcal{K}_\infty$, by hypothesis and $\alpha = \hat{\alpha} \circ \alpha_1 \in \mathcal{K}_\infty$. □

Assumption 3.1. *The function $f(\cdot, \cdot)$ in* (1) *satisfies*

$$\|f(x, v)\| \leq \mu_1(\|x\|) + \mu_2(\|v\|) \tag{12}$$

for all $x \in \mathbb{R}^n$, $v \in \mathbb{R}^m$, and some $\mu_1, \mu_2 \in \mathcal{K}$.

The above assumption is a natural consequence of the Lipschzitz continuity requirement for f at the origin. In fact, it implies $\mathcal{K}$–boundedness with respect to each argument of f.

In the next remark we use the above assumption to establish an instrumental upper bound on $\|x(t)\|$ for all $t \in \mathbb{R}_{\geq 0}$.

Remark 3.2. *For some fixed $t > 0$, the solution of the system* (1) *is given by*

$$x(t) = x(0) + \int_0^t f(x(s), v(s))\mathrm{d}s.$$

Let the locally Lipschitz condition be

$$\|f(x, v) - f(y, u)\| \leq L(\|x - y\| + \|v - u\|), \quad L > 0$$

for $x, y \in \mathcal{N}(0, r)$, where $\mathcal{N}(0, r)$ denotes a neighborhood around the origin of radius r and $v \in \mathbb{R}^m$ and $u \in \mathbb{U}$, $\mathbb{U}$ a compact subset of $\mathbb{R}^m$. Then,

$$\begin{aligned}\|x(t) - x(0)\| &\leq \int_0^t (\|f(x(s), v(s)) - f(x(0), v(0)) + f(x(0), v(0)))\|\mathrm{d}s\\ &\leq L\int_0^t \|x(s) - x(0)\|\mathrm{d}s + L\int_0^t \|v(s) - v(0)\|\mathrm{d}s + \int_0^t \|f(x(0), v(0)\|\mathrm{d}s\\ &\underset{(12)}{\leq} L\int_0^t \|x(s) - x(0)\|\mathrm{d}s + L\int_0^t \|v(s) - v(0)\|\mathrm{d}s + \int_0^t \mu_1(\|x(0)\|)\mathrm{d}s +\\ &\quad \int_0^t \mu_2(\|v(0)\|)\mathrm{d}s.\end{aligned}$$

By applying the Gronwall Lemma above, we obtain that

$$\begin{aligned}
&\|x(t) - x(0)\| \\
&\quad \leq \big(L \int_0^t \|v(s) - v(0)\| \mathrm{d}s + \int_0^t \mu_2(\|v(0)\|)\mathrm{d}s + \int_0^t \mu_1(\|x(0)\|)\mathrm{d}s\big)e^{Lt} \\
&\quad \leq G_t(|v_{[0,t]}|) + \int_0^t \mu_1(\|x(0)\|)\mathrm{d}s e^{Lt},
\end{aligned}$$

and further,

$$\begin{aligned}
\|x(t)\| &\leq G_t(|v_{[0,t]}|) + \|x(0)\| + \int_0^t \mu_1(\|x(0)\|)\mathrm{d}s e^{Lt} \\
&= G_t(|v_{[0,t]}|) + F_t(\|x(0)\|).
\end{aligned}$$

From the inequalities above and the standing assumptions on f *we know that* $F_t(\|x(0)\|)$ *and* $G_t(|v_{[0,t]}|)$ *are continuous with respect to* $x(0)$ *and* $v(t)$*, respectively. Furthermore,* $F_t(0) = 0$*,* $G_t(0) = 0$ *and* $F_t(\|x(0)\|), G_t(|v_{[0,t]}|)$ *are positive definite and continuous as* $\mu_1, \mu_2 \in \mathcal{K}$*.*

Remark 3.3. *In [17, Theorem 4.1], it is shown that the set*

$$\mathcal{S}_v := \{x \in \mathbb{R}^n \mid V(x) \leq (\mathrm{id} - \rho)^{-1} \circ \nu^{-1} \circ \chi(|v|)\}, \tag{13}$$

where $\nu \in \mathcal{K}_\infty$ *and* $\mathrm{id} - \nu \in \mathcal{K}_\infty$ *is d–invariant for the discrete–time correspondent of system* (1) *and for all* $v(s) \in \mathbb{R}^m$*,* $s \geq 0$*. This still holds true for the continuous–time system* (1) *under the given assumptions on* V*. Following from the proof* (9)⇒(11) *of Lemma 3.1, we have that* $\rho = \mathrm{id} - \alpha \circ \alpha_2^{-1}$*. If one takes* ν *in* (13) *to be such that* $\alpha = \nu^{-1}$*, we obtain that*

$$\begin{aligned}
(\mathrm{id} - \rho)^{-1} \circ \nu^{-1} \circ \chi(|v|) &= (\alpha \circ \alpha_2^{-1})^{-1} \circ \alpha \circ \chi(|v|) \\
&= \alpha_2 \circ \alpha^{-1} \circ \alpha \circ \chi(|v|) \\
&= \alpha_2 \circ \chi(|v|).
\end{aligned}$$

The set $\mathcal{S}_v$ *defined in* (13) *can be written equivalently (with respect to condition* (9)*) as*

$$\mathcal{S}_v := \{x \in \mathbb{R}^n \mid V(x) \leq \alpha_2 \circ \chi(|v|)\}, \tag{14}$$

which corresponds to the construction with classical LFs in [6, Lemma 2.14], where $\mathcal{S}_v$ *is an invariant set when* V *is a classical LF.*

Theorem 3.1. *If there exist a function* V *and a proper d-invariant set* $\mathcal{S}$ *that satisfy* (8) *and* (9) *for system* (1)*, and Assumption 3.1 holds, then the system* (1) *is ISS in* $\mathcal{S}$*.*

Proof. Consider the set $\mathcal{S}_v$ defined as in (13). Let

$$t_1 = \inf\{t \geq 0 \,|\, x(j) \in \mathcal{S}_v, j \in [t, t+d)\}.$$

Then, when $\underline{t \geq t_1}$, $x(t) \in \mathcal{S}_v$ and

$$V(x(t)) \leq (\mathrm{id} - \rho)^{-1} \circ \nu^{-1} \circ \chi(|v|) =: \hat{\gamma}(|v|)$$

implies that

$$\|x(t)\| \leq \alpha_1^{-1} \circ \hat{\gamma}(|v|) =: \tilde{\gamma}(|v|).$$

When $\underline{t < t_1}$, it is possible that $x(t) \in \mathcal{S}_v$ which implies that $\|x(t)\| \leq \tilde{\gamma}(|v(t)|)$, and that $x(t) \notin \mathcal{S}_v$. For the latter case, let $t = Nd + j$, with $N \in \mathbb{N}$ and $0 \leq j < d$. The d–invariance of $\mathcal{S}_v$ implies that for $x(j) \in \mathcal{S}_v$, $x(d+j) \in \mathcal{S}_v$ and $x(Nd+j) \in \mathcal{S}_v$. Thus, $x(Nd+j) \notin \mathcal{S}_v$ implies that $x(j) \notin \mathcal{S}_v$. Hence,

$$\begin{aligned} V(x(j)) &> (\mathrm{id} - \rho)^{-1} \circ \nu^{-1} \circ \chi(|v|) \\ (\mathrm{id} - \rho)V(x(j)) &> \nu^{-1} \circ \chi(|v|) \\ \nu \circ (\mathrm{id} - \rho)V(x(j)) &> \chi(|v|), \end{aligned}$$

and

$$\begin{aligned} V(x(j+d)) &\leq \rho(V(x(j))) + \chi(|v|) \\ &< \rho(V(x(j))) + \nu \circ (\mathrm{id} - \rho) \circ V(x(j)) \\ &= (\rho + \nu \circ (\mathrm{id} - \rho)) \circ V(x(j)) \\ &=: \hat{\rho}(V(x(j))), \end{aligned}$$

where $\hat{\rho} = \rho + \nu \circ (\mathrm{id} - \rho)$ satisfies $\mathrm{id} - \hat{\rho} = \mathrm{id} - \rho - \nu \circ (\mathrm{id} - \rho) = (\mathrm{id} - \nu) \circ (\mathrm{id} - \rho)$, thus we can write $\hat{\rho} = \mathrm{id} - (\mathrm{id} - \nu) \circ (\mathrm{id} - \rho)$. $\hat{\rho} < \mathrm{id}$ since $(\mathrm{id} - \nu), (\mathrm{id} - \rho) \in \mathcal{K}_\infty$, thus $\mathrm{id} - \hat{\rho} \in \mathcal{K}_\infty$, which implies that $\mathrm{id}(s) - \hat{\rho}(s) > 0$ for $s \neq 0$. Furthermore, $\hat{\rho}$ is positive definite, continuous and $\hat{\rho}(0) = 0$.

Let $N^* := \sup\{N \in \mathbb{N} \,|\, V(x(Nd+j)) \notin \mathcal{S}_v\}$. Then for all $\underline{N \leq N^*}$ we have:

$$\begin{aligned} V(x(t)) &= V(x(Nd+j)) \\ &= V(x(((N-1)d + j) + d)) \\ &\leq \hat{\rho}(V(x((N-1)d + j))) \\ &= \hat{\rho}(V(x(((N-2)d + j) + d))) \\ &\leq \hat{\rho}^2(V(x((N-2)d + j))) \\ &\cdots \\ &\leq \hat{\rho}^N(V(x(j))) \\ &\leq \hat{\rho}^N(\alpha_2(\|x(j)\|)), \end{aligned} \tag{15}$$

where $\hat{\rho}^N$ denotes the N–times composition of $\hat{\rho}$. Following from Remark 3.2, by applying Lemma 2.1 we obtain that there exist functions $\omega_1, \omega_2 \in \mathcal{K}_\infty$, such that $F_j(\|x(0)\|) \leq \omega_1(\|x(0)\|)$ and $G_j(|v_{[0,j]}|) \leq \hat{\omega}_2(|v_{[0,j]}|)$ and consequently,

$$\|x(j)\| \leq \omega_1(\|x(0)\|) + \omega_2(|v_{[0,j]}|),$$

for all $0 \leq j < d$. Thus

$$\begin{aligned}
V(x(t)) &\leq \hat{\rho}^N(\alpha_2(\omega_1(\|x(0)\|) + \omega_2(|v_{[0,j]}|))) \\
&\leq \hat{\rho}^N(\alpha_2(2\omega_1(\|x(0)\|)) + \alpha_2(2\omega_2(|v_{[0,j]}|))) \\
&= \hat{\rho}^N(\sigma_1(\|x(0)\|) + \sigma_2(|v_{[0,j]}|)) \\
&\leq \hat{\rho}^N(2\sigma_1(\|x(0)\|)) + \hat{\rho}^N(2\sigma_2(|v_{[0,j]}|)) \\
&\leq \hat{\rho}^N(2\sigma_1(\|x(0)\|)) + \hat{\rho}^N(2\sigma_2(|v_{[0,j]}|)), \quad \forall t \geq j \\
&= \hat{\rho}^N(\hat{\sigma}_1(\|x(0)\|)) + \hat{\rho}^N(\hat{\sigma}_2(|v_{[0,j]}|)) \\
&= \hat{\rho}^{\frac{t-j}{d}}(\hat{\sigma}_1(\|x(0)\|)) + \tilde{\gamma}_1(|v_{[0,j]}|) \\
&\leq \hat{\rho}^{\lfloor \frac{t}{d} \rfloor - 1} \circ \hat{\sigma}_1(\|x(0)\|) + \tilde{\gamma}_1(|v_{[0,j]}|) \\
&= \hat{\rho}^{\lfloor \frac{t}{d} \rfloor} \circ \hat{\rho}^{-1} \circ \hat{\sigma}_1(\|x(0)\|) + \tilde{\gamma}_1(|v_{[0,j]}|) \\
&\leq \hat{\rho}^{\lfloor \frac{t}{d} \rfloor} \circ \tilde{\rho} \circ \hat{\sigma}_1(\|x(0)\|) + \tilde{\gamma}_1(|v_{[0,j]}|), \quad \tilde{\rho} \in \mathcal{K}_\infty \\
&=: \hat{\beta}(\|x(0)\|, t) + \tilde{\gamma}_1(|v_{[0,j]}|) \\
&\leq \hat{\beta}(\|x(0)\|, t) + \tilde{\gamma}_1(|v|).
\end{aligned}$$

Without loss of generality we can assume that $\hat{\rho}$ is a one–to–one (injective) and onto (surjective) function, thus invertible. Furthermore, since $\hat{\rho}$ is continuous, then by [20, Theorem 3.16], $\hat{\rho}^{-1}$ is continuous. Additionally, $\hat{\rho}^{-1}(0) = \hat{\rho}^{-1}(\rho(0)) = 0$. Thus, there exists a function $\tilde{\rho} \in \mathcal{K}_\infty$, such that $\hat{\rho}^{-1} \leq \tilde{\rho}$, as follows from Lemma 2.1. We can conclude that $\hat{\beta} \in \mathcal{KL}$ since $\tilde{\rho} \circ \hat{\sigma}_1(s) \in \mathcal{K}_\infty$ and $\hat{\rho}^{\lfloor \frac{t}{d} \rfloor} \in \mathcal{L}$. Next, $\tilde{\gamma}_1 = \hat{\rho}^N \circ 2\alpha_2 \circ 2\omega_2$, thus $\tilde{\gamma}_1 > 0$ is continuous and $\tilde{\gamma}_1(0) = 0$. Therefore, there exists a function $\hat{\gamma}_1 \in \mathcal{K}_\infty$ such that $\tilde{\gamma}_1 < \hat{\gamma}_1$. We obtain

$$\begin{aligned}
\|x(t)\| &\leq \alpha_1^{-1}(2\hat{\beta}(\|x(0)\|, t)) + \alpha_1^{-1}(2\hat{\gamma}_1(|v|)) \\
&:= \beta(\|x(0\|, t) + \tilde{\gamma}_2(|v|),
\end{aligned}$$

with $\beta \in \mathcal{KL}$ and $\tilde{\gamma}_2 \in \mathcal{K}_\infty$.

For $\underline{N > N^*}$ it holds that $V(x(Nd+j)) \in \mathcal{S}_v$, thus $\|x(Nd+j)\| \leq \tilde{\gamma}(|v|)$.

For all $N \in \mathbb{N}^*$ it follows that

$$\|x(t)\| \leq \beta(\|x(0\|, t) + (\tilde{\gamma}_2 + \tilde{\gamma})(|v|),$$

thus, also for all $t \geq t_1$

$$\|x(t)\| \leq \beta(\|x(0\|, t) + (\tilde{\gamma}_2 + 2\tilde{\gamma})(|v|).$$

Hence, for all $t \geq 0$ it holds that

$$\|x(t)\| \leq \beta(\|x(0\|, t) + (\tilde{\gamma}_2 + 2\tilde{\gamma})(|v|),$$

which implies ISS for system (1) in $\mathcal{S}$. □

4 A continuous–time ISS converse theorem

In this section we provide a converse result that enables construction of a standard ISS Lyapunov function from an ISS finite–time Lyapunov function in the continuous–time setting. To this end, the following assumption, also used in [15], is instrumental.

Assumption 4.1. *There exists a pair* $(\beta(\cdot,\cdot), d) \in \mathcal{KL} \times \mathbb{R}_{>0}$ *with* β *satisfying* (3) *for the system* (1) *such that*

$$\beta(s, d) < s \tag{16}$$

for all $s > 0$.

The above assumption ensures a certain uniformity of the convergence prescribed by the $\mathcal{KL}$–function β, e.g., with exponential convergence a particular case. Next, we present a converse ISS theorem in terms of existence of ISS finite–time Lyapunov functions.

Theorem 4.1. *If the system* (1) *is ISS in some* d*–invariant set* $\mathcal{S}$ *with* $d > 0$ *as in* (16) *and Assumption 4.1 is satisfied, then for any function* $\eta \in \mathcal{K}_\infty$ *and for any norm* $\|\cdot\|$*, the function* $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$*, with*

$$V(x) := \eta(\|x\|), \quad \forall x \in \mathbb{R}^n \tag{17}$$

satisfies (8) *and* (9) *with the same* $d > 0$ *as in* (16).

Proof. Let the pair (β, d) be such that Assumption 4.1 holds. Then, from the ISS hypothesis we obtain that for all initial conditions $x(0) \in \mathcal{S}$, it holds that

$$\begin{aligned}\|x(t+d)\| &\leq \max(\beta(\|x(t)\|, d), \gamma(|v|)) \\ &= \beta(\|x(t)\|, d),\end{aligned}$$

whenever $\alpha_d(\|x(t)\|) := \beta(\|x(t)\|, d) \geq \gamma(|v|)$, where $\alpha_d \in \mathcal{K}_\infty$, or equivalently, whenever $\|x(t)\| \geq (\alpha_d^{-1} \circ \gamma)(|v|) = \hat{\gamma}(|v|)$. Consequently,

$$\begin{aligned}\eta(\|x(t+d)\|) &\leq \eta(\beta(\|x(t)\|, d)) \\ &\leq \eta(\beta(\eta^{-1}(V(x(t))), d)) \\ &:= \rho(V(x(t))),\end{aligned}$$

where $\rho = \eta(\beta(\eta^{-1}(\cdot), d))$. Then, via Assumption 4.1, it follows that $\rho < \eta(\eta^{-1}(\cdot)) = \text{id}$ and $\text{id} - \rho \in \mathcal{K}_\infty$. Thus, we get

$$V(x(t+d)) - \rho(V(x(t))) \leq 0, \; \forall x(0) \in \mathcal{S}, \; \|x(t)\| \geq \hat{\gamma}(|v(t)|).$$

Next, this implies

$$\begin{aligned}V(x(t+d)) - V(x(t)) &\leq \rho(V(x(t))) - V(x(t)) \\ &= -(V(x(t)) + \rho(V(x(t))) \\ &= -((\text{id} - \rho)(V(x(t)))) \\ &\leq -((\text{id} - \rho)(\alpha_1(\|x(t)\|)) \\ &= -\alpha(\|x(t)\|), \quad \|x(t)\| \geq \hat{\gamma}(|v|),\end{aligned}$$

with $\alpha = (\text{id} - \rho) \circ \alpha_1$. Since $\text{id} - \rho \in \mathcal{K}_\infty$, then $\alpha \in \mathcal{K}_\infty$ and we have obtained (10) which further yields (9), as shown in Remark 2.4 in [6]. Furthermore, since V is defined by a $\mathcal{K}_\infty$ function, then there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that (8) holds. □

Consider next the function defined as

$$W(x(t)) := \int_t^{t+d} V(x(\tau)) \mathrm{d}\tau, \tag{18}$$

for any V that satisfies (8) and (9).

Lemma 4.1. *There exists a V satisfying* (8) *and* (9) *for some $d > 0$ for* (1) *if and only if the function W as defined in* (18) *with the same $d > 0$ is an ISS Lyapunov function for the system* (1).

Proof. Let there be a function V satisfying (8) and (9). $V(x(t))$ is continuous, thus it is integrable over any closed, bounded interval $[t, t+d]$, $t \geq 0$. By Theorem 5.30 in [20], this implies that $W(x(t))$ is continuous on each interval $[t, t+d]$, for any t. Since V is also positive definite, by integrating over the bounded interval $[t, t+d]$ the resulting function $W(x(t))$ will also be positive definite. Since $W(x(t))$ is continuous, $W(0) = 0$ and it positive definite the result in Lemma 2.1 can be applied. Therefore, there exist two functions $\hat{\alpha}_1, \hat{\alpha}_2 \in \mathcal{K}_\infty$ such that

$$\hat{\alpha}_1(\|x\|) \leq W(x) \leq \hat{\alpha}_2(\|x\|), \quad \forall x \in \mathbb{R}^n, \tag{19}$$

holds. Next, by making use of the general Leibniz integral rule, we get that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}W(x(t)) &= \int_t^{t+d} \underbrace{\frac{\mathrm{d}}{\mathrm{d}t}V(x(\tau))}_{=0} \mathrm{d}\tau + V(x(t+d))(t \dot{+} d) - V(x(t))\dot{t} \\ &= V(x(t+d)) - V(x(t)) \\ &\leq -\alpha(\|x(t)\|) + \chi(|v|).\end{aligned}$$

In $\frac{\mathrm{d}}{\mathrm{d}t}V(x(\tau))$ note that $x(\tau) = x(\tau, v(\tau))$, which implies that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V(x(\tau, v(\tau))) &= \frac{\partial V}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} \\ &= \frac{\partial V}{\partial x}\left(\frac{\partial x}{\partial \tau}\frac{\mathrm{d}\tau}{\mathrm{d}t} + \frac{\partial x}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}t}\right) \\ &= \frac{\partial V}{\partial x}\left(\frac{\partial x}{\partial \tau}\cdot 0 + \frac{\partial x}{\partial v}\cdot 0\right) \\ &= 0.\end{aligned}$$

Thus, W is an ISS Lyapunov function for (1).

Now assume that W is an ISS Lyapunov function for (1), i.e. (5) holds and for some $\hat{\alpha}, \hat{\chi} \in \mathcal{K}_\infty$ it holds that

$$\dot{W}(x(t)) \leq -\hat{\alpha}(\|x(t)\|) + \hat{\chi}(|v|), \quad \forall x(t) \in \mathcal{S}, \forall v \in \mathbb{R}^m.$$

By the same Leibniz rule, we know that $\dot{W}(x(t)) = V(x(t+d)) - V(x(t))$, thus for the difference $V(x(t+d)) - V(x(t))$ inequality (9) holds. Now we have to show that (8) holds.

Assume that there exists an $x \in \mathbb{R}^n$ such that $V(x) < 0$. Then we obtain that $W(x) = \int_t^{t+d} V(x(\tau))\mathrm{d}\tau < 0$. But this is a contradiction since we assumed that W satisfies (19) on $\mathbb{R}^n$, thus $V(x)$ must be positive definite on $\mathbb{R}^n$. By the definition of W, we have that V must be a continuous function, because it needs to be integrable for W to exist. By assumption, W is upper and lower bounded by $\mathcal{K}_\infty$ functions, thus for $x \to \infty$, $W(x) \to \infty$. This can only happen when $V(x) \to \infty$. Thus, using a similar reasoning as above, based on Lemma 2.1, this implies that V is upper and lower bounded by $\mathcal{K}_\infty$ functions, hence (8) holds. □

Note that the above equivalence result between candidate ISS finite–time LFs and candidate ISS LFs holds for any proper set $S \subseteq \mathbb{R}^n$. However, if ISS in $\mathcal{S}$ is to be established and $\mathcal{S}$ is a strict subset of $\mathbb{R}^n$, it is required that $\mathcal{S}$ is an invariant set, according to Definition 2.4.

The next result summarizes the proposed alternative converse theorem for ISS of (1) in $\mathcal{S}$, enabled by the finite–time conditions (8) and (9), which assumes invariance of $\mathcal{S}$ for the reasons stated above.

Corollary 4.1. *If the the system* (1) *is ISS in some invariant set* $\mathcal{S}$ *and Assumption 4.1 holds for* $(\beta, d) \in \mathcal{KL} \times \mathbb{R}_{>0}$*, then by Theorem 4.1 and Lemma 4.1 for any function* $\eta \in \mathcal{K}_\infty$ *and any norm* $\|\cdot\|$*, the function* $W(\cdot)$ *as defined in* (18) *with* $V(x) = \eta(\|x\|)$ *for the same* $d > 0$ *as in Assumption 4.1, is an ISS Lyapunov function for the system* (1)*.*

4.1 Construction of the ISS Lyapunov function W

We have established that if a system is ISS, then a ISS LF can be constructed *via* (18) with $V(x)$ defined by any function $\eta \in \mathcal{K}_\infty$ and any norm, and for a suitable $d > 0$. This constructive method starts with a given candidate d–invariant set $\mathcal{S}$ and a candidate function $V(x) = \eta(\|x\|)$. Due to the d–invariance property of $\mathcal{S}$ verifying condition (9) for the chosen V is reduced to verifying

$$V(x(d)) - V(x(0)) \leq -\alpha(\|x(0)\|) + \chi(|v|), \tag{20}$$

for all $x(0) \in \mathcal{S}$. The difficulty in verifying (20) is given by the need to compute $x(d)$, for all $x(0) \in \mathcal{S}$. However if $x(d)$ is known analytically and the disturbance input v is a known or estimated signal in some analytic form, then it suffices to verify (20) for all initial conditions in a chosen set $\mathcal{S}$. Since V is a continuous function of time and the integral in (18) is defined over a closed time interval, the expression of W becomes

$$W(x(0)) = \int_0^d V(x(\tau))\mathrm{d}\tau. \tag{21}$$

Since v is however not known in general, we propose to compute a value for d when $v = 0$ and rely on inherent ISS. As such, we will make use of the next result, which enables the verification of the finite–time condition (20) for the system (1) with $v = 0$.

Lemma 4.2. *Let* $V(x) = \|x\|$*. If* $V(x)$ *satisfies* (9) *for* $\dot{x} = f(x, 0)$ *and* $v(t) = 0$ *for all* $t \in \mathbb{R}_{\geq 0}$*, then* $V(x)$ *satisfies the condition* (9) *for* $\dot{x} = f(x, v)$*.*

Proof. We shall write the proof for t in the interval $[0, d]$. Let $\bar{x}(d)$ denote the solution of $\dot{x} = f(x, 0)$ for $x(0) \in \mathcal{S}$. It follows from (9) that

$$\|\bar{x}(d)\| - \|x(0)\| \leq -\alpha(\|x(0)\|),$$

for all $x(0) \in \mathcal{S}$, where $\mathcal{S}$ a d–invariant set. Then,

$$\begin{aligned} \|x(d)\| - \|x(0)\| &\leq \|x(d)\| - \|\bar{x}(d)\| - \alpha(\|x(0)\|) \\ &\leq \|x(d) - \bar{x}(d)\| - \alpha(\|x(0)\|), \; \forall x(0) \in \mathcal{S}. \end{aligned}$$

$$x(d) - \bar{x}(d) = \int_0^d f(x(s), v(s))\mathrm{d}s - \int_0^d f(x(s), 0)\mathrm{d}s.$$

This implies, by using the local Lipschtiz condition on f with respect to its both arguments, that

$$\begin{aligned} \|x(d) - \bar{x}(d)\| &\leq \int_0^d \|f(x(s), v(s)) - f(x(s), 0)\|\mathrm{d}s \\ &\leq \int_0^d L\|v(s) - 0\|\mathrm{d}s \end{aligned}$$

where $L > 0$ is the Lipschitz constant. Since $\int_0^d L\|v(s) - 0\|\mathrm{d}s$ is positive definite, zero at zero and continuous then via Lemma 2.1, there exists $\chi \in \mathcal{K}_\infty$ such that

$$\|x(d) - \bar{x}(d)\| \leq \int_0^d L\|v(s)\|\mathrm{d}s \leq \chi(\|v(d)\|) \leq \chi(|v|).$$

As such, we have that

$$\|x(d)\| - \|x(0)\| \leq \chi(|v|) - \alpha(\|x(0)\|), \; \forall x(0) \in \mathcal{S},$$

which recovers (20), and further (9) for $V(x) = \|x\|$. □

The result of Lemma 4.2 allows us to obtain a d for which a candidate function $V(x) = \|x\|$, with any norm, is an ISS FTLF in a much simpler way than verifying (20) directly.

Remark 4.1. *In [21] a scheme for constructing LFs starting from a given LF, which at every iterate provides a less conservative estimate of the DOA of a nonlinear system of the type* (1) *was proposed and it was based on iterative constructions of the type $W_1(x) = W(x + \alpha_1 f(x))$. In [15], the expansion idea in [21] was used to generate a sequence of FTLFs, with the* *purpose to generate a relevant d–invariant set for constructing Massera–type of LFs. Thus the sequence of functions*

$$\begin{aligned} V_1(x) &= V(x+\alpha_1 f(x,0)) \\ V_2(x) &= V_1(x+\alpha_2 f(x,0)) \\ &\vdots \\ V_n(x) &= V_{n-1}(x+\alpha_n f(x,0)), \end{aligned} \tag{22}$$

with $\alpha_i \in \mathbb{R}_{\geq 0}$, $i = 1, 2 \ldots, n$ *yields FTLFs, when* V *is FTLF. From Lemma 4.2, we know that* $V_1(x) = V(x + \alpha_1 f(x, v))$ *for example is an inherent ISS FTLF. This fact will be relevant in the computation of an ISS LF* W*, as in* (21)*, which requires the solution of* (1) *up to time* d*.*

For the particular case of $V(x) = \|x\|$ we propose next a tractable computational procedure to construct $W(x)$. First, we consider the nominal system dynamics, i.e., $\dot{x} = f(x, 0)$. When the analytical solution is not known, or obtaining a numerical approximation is computationally tedious, as it can be the case for higher order nonlinear systems, then we propose to use the approach in [15] starting from the linearized dynamics of (1) with $v = 0$:

$$\|e^{d\left[\frac{\partial f(x,0)}{\partial x}\right]_{x=0}} x(0)\| - \|x(0)\| < 0, \tag{23}$$

for all $x(0)$ in some compact, proper set $\mathcal{S}$. Then, for the computed d, the approach in [15] relies on constructing W as

$$W(x) = \int_0^d V(x + \tau f(x, 0))\mathrm{d}\tau, \quad \forall x \in \mathcal{S}. \tag{24}$$

An ISS Lyapunov function is then obtained as follows. Let $\bar{W}(x) = \int_0^d V(x+\tau(f(x,0))\mathrm{d}\tau$ and $\mathcal{S}(c) = \{x \in \mathbb{R}^n \mid \bar{W}(x) \leq c\}$ with $c > 0$ such that $\mathcal{S}(c) \subseteq \{x \in \mathbb{R}^n \mid \dot{\bar{W}}(x) < 0\}$. Now consider $\dot{\bar{W}}(x) \leq -\alpha(\|x\|)$, for any $x \in \mathcal{S}(c)$. By the Leibniz integral rule, this implies that

$$V(x + df(x, 0)) - V(x) \leq \alpha(\|x\|) \quad \forall x \in \mathcal{S}(c).$$

For $V(x) = \|x\|$, by a similar reasoning as in the proof of Lemma 4.2 it can be seen that

$$V(x + df(x, v)) - V(x) \leq \alpha(\|x\|) + dL|v| \quad \forall x \in \mathcal{S}(c),$$

which implies that

$$W(x) = \int_0^d V(x + \tau f(x, v))\mathrm{d}\tau, \quad \forall x \in \mathcal{S}(c) \tag{25}$$

is an ISS Lyapunov function.

5 Input–to–state stabilizing control laws

In this section we consider nonlinear systems that are affine in the control and disturbance inputs, respectively. The construction of LF candidates *via* finite–time LF candidates will be exploited in combination with Sontag's formula to derive ISS stabilizing controllers.

To this end, consider systems subject to control inputs, of the type

$$\dot{x} = f(x) + g(x)u \tag{26}$$

where $u \in \mathbb{U} \subseteq \mathbb{R}^m$ denotes the control input and $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ with $f(0) = 0$ and $g(0) = 0$, recall the concept of a control Lyapunov function (CLF), as defined in [1].

Definition 5.1. *Let $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a continuously differentiable function with $V(0) = 0$ and the following properties:*

a) $V(x)$ is positive definite and radially unbounded[1]*, i.e. $V(x) \to \infty$ as $\|x\| \to \infty$, for all $x \in \mathbb{R}^n$;*

b) There exists a state–feedback control law $u \in \mathbb{U} \subseteq \mathbb{R}^m$ such that the derivative of $V(x)$ along the trajectories of (26), $\dot{V}(x) = \nabla V \hat{f}(x,u)$, is negative definite, i.e.

$$\nabla V(x) f(x) + \nabla V(x) g(x) u < 0, \quad \forall x \in \mathbb{R}^n,\ x \neq 0. \tag{27}$$

Then V is a CLF for the system (26).

When a CLF W is known, an explicit formula for a state feedback control that makes the system asymptotically stable or $\mathcal{KL}$–stable was provided in [1]. In this paper we will use a slightly modified version for the expression of $k(a, b)$ defined in [22]. Let $u = k(a(x), b(x))$, where $a(x) := \nabla W(x) f(x)$, $b(x) = \nabla W(x) g(x)$, and k is a function $k : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ defined by

$$k(a,b) := \begin{cases} -\frac{a+\sqrt{a^2+\|b\|^4}}{\|b\|^2} b, & \text{if } b \neq 0 \\ 0, & \text{if } b = 0. \end{cases} \tag{28}$$

In the remainder of this paper we consider the case when u is a scalar.

Let there be disturbance inputs acting on (26), as described by

$$\dot{x} = f(x, v) + g(x)u, \tag{29}$$

[1]This condition corresponds to condition (5).

where v, u and $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are defined as in (1) and (26), respectively with $v \in \mathbb{R}^p$ and $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$.

For the problem of disturbance attenuation by choice of feedback in terms of ISS, it is necessary to define the notion of an ISS CLF. We recall the definition from [22].

Definition 5.2. *Let $W : \mathbb{R}^n \to \mathbb{R}$, be a continuously differentiable function for which there exist $\alpha_1, \alpha_2, \alpha, \chi \in \mathcal{K}_\infty$, such that*

$$\alpha_1(\|x\|) \le W(x) \quad \le \alpha_2(\|x\|),\ \forall x \in \mathbb{R}^n \tag{30}$$

$$\nabla W(x)(f(x,v) + g(x)u) \quad \le -\alpha(\|x\|) + \chi(|v|), \tag{31}$$

for all $x \neq 0$, and $v \in \mathbb{R}^p$. Then W is an ISS CLF for the system (29)*.*

Note that the second condition above is equivalent with the statement that

$$\nabla W(x)g(x) = 0 \Rightarrow \nabla W(x)f(x,v) < 0, \quad \forall x \in \mathbb{R}^n.$$

Consider as candidate ISS CLF, the function $W(x)$ defined in (24) computed for (29) when the control input is $u = 0$. In [22], a similar "universal" construction as in (28) was proposed for computing feedback stabilizers for (29). We propose to use $W(x)$ to compute a feedback stabilizer as constructed in [22] for (29) with $f(x,v) = \hat{f}(x) + \hat{g}(x)v$. Thus, let us consider systems described by

$$\dot{x} = \hat{f}(x) + \hat{g}(x)v + g(x)u. \tag{32}$$

Let $a(x)$ be redefined as

$$a(x,v) := \nabla W(x)\hat{f}(x) + \nabla W(x)\hat{g}(x)v = \hat{a}(x) + \hat{b}(x)v.$$

Then W is an ISS CLF for (32) if

$$\hat{a}(x) + \hat{b}(x)v + b(x)u \le -\alpha(\|x\|) + \chi(|v|). \tag{33}$$

In [22], the condition (33) was equivalently written so as not to involve v as

$$\hat{a}(x) + \|\hat{b}(x)\|\varrho^{-1}(\|x\|) + b(x)u \le -\tilde{\alpha}(\|x\|), \tag{34}$$

where $\varrho, \tilde{\alpha} \in \mathcal{K}_\infty$ are such that it holds that $\|x\| \ge \varrho(|v|)$ implies that

$$\hat{a}(x) + \hat{b}(x)v + b(x)u \le \tilde{\alpha}(\|x\|).$$

Then a feedback stabilizer can be computed by using the same formula (28) with

$$a(x) = \hat{a}(x) + \|\hat{b}(x)\|\varrho^{-1}(\|x\|). \tag{35}$$

5.1 Computation of an ISS feedback stabilizer

In this subsection we summarize the steps required for computing an ISS feedback stabilizer. First, we should compute a CLF candidate function W for the uncontrolled dynamics $\dot{x} = \hat{f}(x)$. To this end, one needs to find a suitable value of $d > 0$ and compute W similarly as in (23) and (24), i.e.

$$V(e^{d\left[\frac{\partial \hat{f}(x)}{\partial x}\right]_{x=0}}x) - V(x) \leq -\alpha(\|x\|) \tag{36}$$

and

$$W(x) = \int_0^d V(x + \tau \hat{f}(x))\mathrm{d}\tau, \quad \forall x \in \mathcal{S}. \tag{37}$$

For more details the interested reader is referred to [15], which focuses on construction of candidate LFs for uncontrolled systems.

Once W is computed, a feedback control law $k(x)$ can be obtained *via* (28) with W and $a(x)$ from (35) as follows. Compute $\tilde{\alpha}$ and ϱ such that (34) holds. From (33) it follows that

$$\varrho(|v|) = \alpha^{-1} \circ \chi(|v|)$$

and via the inherent ISS FT LF lemma, Lemma 4.2 we have that $\chi(\cdot)$ can be any $\mathcal{K}_\infty$–function such that $\chi(|v|) \geq \int_0^d L\|v(s)\|\mathrm{d}s$.

6 Illustrative example: Whirling pendulum

We consider the system below, which was studied in [23] with the purpose to compute the domain of attraction (DOA) of the zero equilibrium when the system is autonomous ($u = v = 0$),

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{-k_f}{mb}x_2 + \omega^2 \sin(x_1)\cos(x_1) - \frac{g}{l_p}\sin(x_1) + c(u+v). \end{aligned} \tag{38}$$

Therein a polynomial LF was computed, whose level set rendering a DOA estimate is shown in Figure 1(a) with the black contour. Additionally, in this paper we are interested in constructing regionally (input–to–state) stabilizing control laws u with respect to the zero equilibrium.

It is worth to mention that stability analysis or stabilization with respect to one equilibrium point out of a set of equilibria has been referred to as local stabilization, see, for example, [24]. We use the term regional to denote the fact that the synthesized ISS control law yields a bounded subset of $\mathbb{R}^n$ as the corresponding (closed–loop) domain of attraction.

Following the construction in (36)–(37) (for more details we refer to [15]), a LF W was computed from a quadratic FTLF, $V(x) = x^\top Px$, with $P = \left(\begin{smallmatrix} 3.6831 & 2.3169 \\ 2.3169 & 14.7694 \end{smallmatrix}\right)$ and $d = 1.1$ and $\alpha(\|x\|) = \|x\|$. The level set $C = 3.55$ of $W(x)$ defines an estimate of the true DOA of the origin of (38) and it is shown with blue in Figure 1(a) together with a vector field plot of the system.

Then, we consider the W corresponding to (37) for the above parameter values as a CLF candidate for computing an ISS feedback stabilizer $u = k(x)$. Let $\tilde{\alpha}(\|x\|) = \|x\|$. Next we have to compute ϱ. Since the proposed computational procedure relies on an inherent ISS FTLF, the result in Lemma 4.2 will be used to compute $\chi(|v|)$ which is needed in the expression of ϱ, and consequently in the feedback stabilizer.

Let the disturbance signal be a generated as an uniformly distributed sequence in the interval $(-0.5, 0.5)$. Thus $|v| = \sup_{s\in[0,t]} \|v(s)\| \leq 0.5$ for some $t \in \mathbb{R}_{\geq 0}$. As such we have that

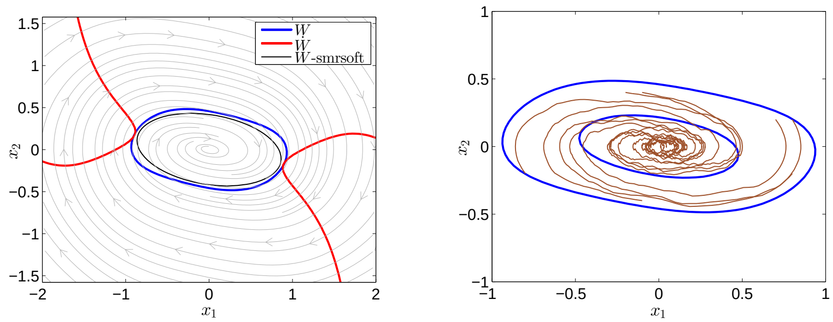
$$L\int_0^d \|v(s)\|\mathrm{d}s \leq L\int_0^d |v_{[0,d]}|\mathrm{d}s \leq Ld|v_{[0,d]}| = Ld\bar{v}, \tag{39}$$

where L is the Lipschitz constant and $\bar{v} = 0.5$ for the considered system and disturbance signal. L can be approximated as suggested in [18], from the fact that $\|\frac{\partial f}{\partial v}\| \leq L$ implies that $\|f(x,v) - f(x,u)\| \leq L\|v-u\|$. In this case $L = c = 0.2$. However, to compute $k(x)$ we need $\varrho^{-1}(\|x\|)$. We know that $\varrho(|v|) = \alpha^{-1} \circ \chi(|v|)$, thus $\varrho^{-1}(s) = \chi^{-1}(s) \circ \alpha = \chi^{-1}(s) \circ \mathrm{id} = \chi^{-1}(s)$. Following from (39) we shall take $\varrho^{-1}(s) = \frac{1}{Ld}s$, thus $\varrho^{-1}(\|x\|) = \frac{1}{Ld}\|x\|$ in (34).

Furthermore, since the computed $W(x)$ is based on a FTLF which is an inherent ISS FTLF, it follows from the equivalence in Lemma 4.1 that $W(x)$ is an (inherent) ISS Lyapunov function. From [6, Lemma 2.14] it follows that the set

$$\mathcal{S}_v = \{x \in \mathbb{R}^n \mid W(x) \leq \alpha_2 \circ \chi(|v|)\}$$

is an invariant set for (38). If we consider $\chi(|v|) = Ld\bar{v}$ and we know that $W(x)$ can be upper bounded by $\varepsilon V(x) = x^\top \varepsilon Px$ such that the level set $V(x) = \frac{1}{\varepsilon}$ is included in the level set $W(x) = C$, then $\alpha_2(\|x\|) = \lambda_1(\varepsilon P)\|x\|$, where λ_1 is the largest eigenvalue of P. Then we obtain $\alpha_2 \circ \chi(|v|) = \lambda_1(\varepsilon P)Ld\bar{v}$. For this example $\varepsilon = 0.4762$ and a plot of the resulting level set is shown in Figure 1(b) together with several trajectories of the system.



(a) (b)

Figure 1: (a)–level set of W for $C = 3.55$ computed for (38) with $v = u = 0$, $d = 1.1$, plotted in blue, its corresponding derivative–red and the level set computed in Chesi with smrsoft; (b)–level set of W for $C = 3.55$ computed for (38) with $v \neq 0$ plotted in blue and simulations from the same $x(0)$ different cases.

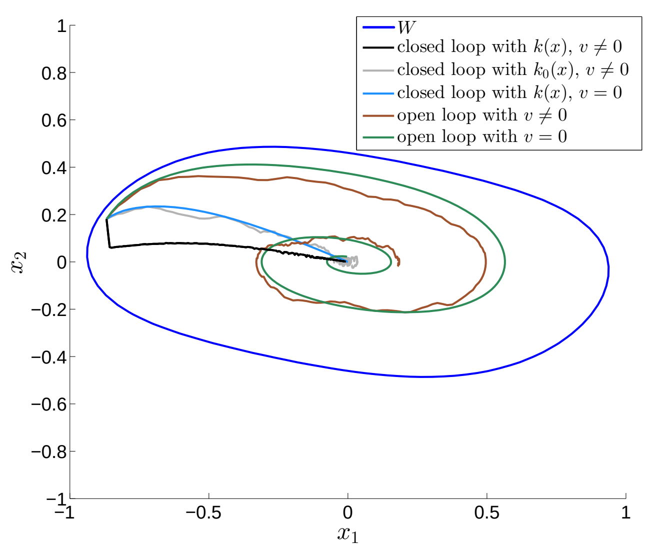


Figure 2: Level set of W for $C = 3.55$ computed for (38) with $v \neq 0$ plotted in blue and simulations from the same $x(0)$ different cases.

The same level set defined by $C = 3.55$ of W is shown in Figures 1(a), 1(b) and 2. Next, we compute $k(x)$ from (28) with $a(x)$ from (35) on basis of

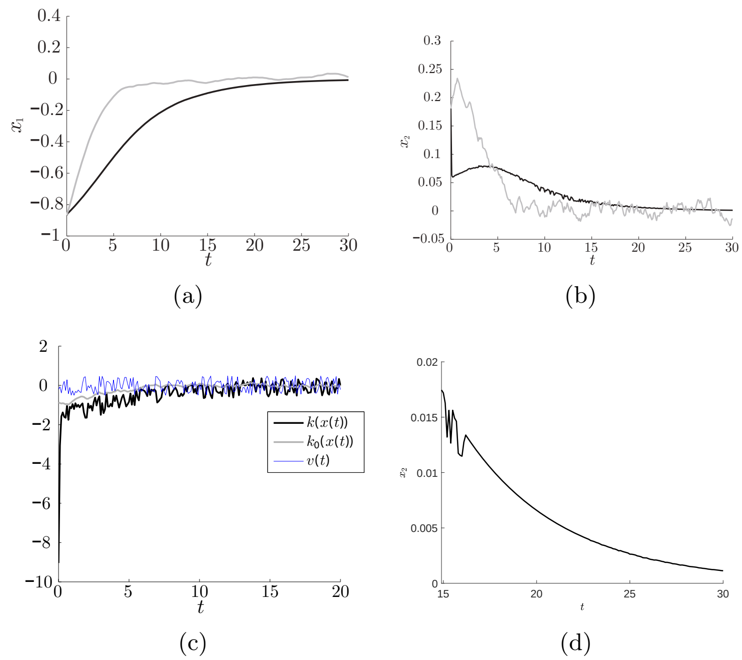


Figure 3: The corresponding states for k computed without considering v–grey and k computed with $v \neq 0$ via the function ϱ ((a)-(b)). In (c): v, $k(x)$ and $k_0(x)$ as functions of the solution of (38) initiated at $x_0 = (-0.8680 \quad 0.1810)^\top$; (d): plot of x_2 corresponding to $u = k(x)$ when the disturbance is set equal to zero after $t = 16s$.

the computed $W(x)$. We provide a plot of $u = k(x)$ for the initial condition $x_0 = (\, -0.8680 \; 0.1810 \,)^\top$ in Figure 3(c). For comparison, in the same figure we also plot $u = k_0(x)$, where $k_0(x)$ is computed without considering v in the dynamics, i.e., in the computation of $a(x)$ in (28).

A trajectory of the closed–loop system is shown in Figure 2 for a particular initial condition, together with trajectories initiated at the same initial condition for different cases: the closed loop system with $u = k(x)$–black, the closed loop system with $u = k_0(x)$ and $v \neq 0$–grey, the closed loop system with $u = k_0(x)$ and $v = 0$–blue, the open loop system with $v \neq 0$–brown and the open loop system with $v = 0$–green.

For the same initial condition, we show each of the states of the closed–loop system for $u = k(x)$–black and for $u = k_0(x)$–grey in Figure 3(a) and Figure 3(b). The corresponding time histories of the control laws $k(x)$ and $k_0(x)$, and of the disturbance v are provided in Figure 3(c). We also illustrate the case when the disturbance becomes zero after some time in Figure 3(d) for the x_2 state. It can be observed that the trajectory corresponding to $u = k(x)$–black after the disturbance becomes zero at $t = 16s$ converges to the origin, as guaranteed by the closed–loop ISS property.

7 Conclusions

In this paper, we have provided an equivalent characterization of the ISS property in terms of existence of an ISS–FTLF for continuous–time systems. For what concerns the converse result, when the ISS property is known, under a certain assumption on the solution estimate of the system subject to disturbance inputs, we showed that any $\mathcal{K}_\infty$–function of the norm of the state is an ISS–FTLF. Furthermore, inspired by classical converse results, we showed how an ISS–LF can be computed from an ISS–FTLF via a construction of the Massera–type.

When the considered finite–time function candidate is the norm of the state, i.e. let $V(x) = \|x\|$, for verification purposes, we show that V is an inherent finite–time ISS LF if it is a FTLF for the system with zero disturbance inputs. Finally, we proposed a procedure to construct an ISS feedback stabilizer by using Sontag's "universal" formula for ISS stabilization and a Massera–type of function constructed for the system without disturbances.

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