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STATIC OUTPUT-FEEDBACK STABILIZATION OF MARKOVIAN JUMP SYSTEMS WITH UNCERTAIN PROBABILITY RATES*

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Dedicated to Dr. Vasile Drăgan on the occasion of his 70th anniversary

Abstract

This paper provides a treatment for the mode-dependent static output-feedback control problem of linear systems subject to random Markovian jumps in its parameters. For this kind of systems, we consider the mean-square stability and we develop a numerical method to find static output-feedback stabilizing control. We show how one can handle the uncertainties that can affect the transition probability matrix. The robust static output-feedback stabilization problem (against unkown or uncertain probability rates) is formulated in terms of the minimization of a scalar product of definite positive matrices under convex constraint (LMIs). Such problem can be solved via a cone complementarity algorithm.
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1 Introduction

In this paper, we consider the following linear system with Markovian jumps

$$\begin{aligned} \frac{dx}{dt} &= A(r(t))x + B(r(t))u \\ y &= C(r(t))x, \end{aligned} \tag{1}$$

where $x \in \mathbf{R}^n$ is the state vector, $u \in \mathbf{R}^{n_u}$ is the control input, $y \in \mathbf{R}^{n_y}$ is the measurement output and the time-varying parameter $r(t)$ satisfies

- $r : \mathbf{R}^+ \to \{1, ..., N\}$ is a stochastic Markovian process, with transition probabilities defined by

$$\text{Prob}\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \pi_{ij}(t)\Delta + o(\Delta) & \text{if } i \neq j, \\ 1 + \pi_{ii}(t)\Delta + o(\Delta) & \text{else,} \end{cases}$$

Such transition probabilities are associated to the *matrix of the transition probability rates* $\Pi(t) := [\pi_{ij}(t)]_{1 \leq i,j \leq N}$ which verifies

$$\pi_{ij}(t) \geq 0, \forall i \neq j, \text{ and } -\pi_{ii}(t) = \sum_{j \neq i}^{N} \pi_{ij}(t).$$

- $A(r(t)) = A_i$, $B(r(t)) = B_i$ and $C(r(t)) = C_i$ when $r(t) = i$,

such that $A_i \in \mathbf{R}^{n \times n}$, $B_i \in \mathbf{R}^{n \times n_u}$ and $C_i \in \mathbf{R}^{n_y \times n}$, $i = 1, \dots N$, are constante matrices.

In reality, the transition rates matrix $\Pi(t)$ can be "*uncertain*", which is only known to belong to a bounded set. In this paper, we assume that this set can be approximated by a polytope $\Pi(t) \in \mathbf{Co}\{\Pi^1, \dots, \Pi^L\}$, where the vertices $\Pi^k = [\pi^k_{ij}]_{1 \leq i,j \leq N}$, $k = 1, \dots, N$ are known transition matrices. Prior to extensively reported literature, such *notion of uncertainty on the transition probabilities* has been introduced by [1, 10] where, in addition, a Linear Matrix Inequality (LMI) framework for the stabilization of jump systems by state feedback was initiated.

System (1) can be viewed as a linear system subject to stochastic abrupt changes in its components and such that its evolution is governed by several "*matrix modes*" (A_i, B_i, C_i). Hence, under the influence of the Marovian process $r(t)$, this system "jumps" from one mode i to the other j, according to the transitions rates π_{ij}. This kind of systems has a wide range of real-world applications, see for instance, [19]. Jump Ito differential systems that extend system (1) under the influence of multiplicative noise can be found in [16, 7, 8, 9].

In this paper, we study the problem of stabilization by static output-feedback controls whose gains take constant values in function of a Markov process $r(t)$. These control laws have the form $u(t) = K(r(t))y(t)$ where y represents a measure on the state of the system.

Note that any dynamic control of order $k < n$ of the form

$$\begin{array}{rcl} \frac{dx_c}{dt}(t) & = & A_c(r(t))x_c + B_c(r(t))y \\ u(t) & = & C_c(r(t))x_c + D_c(r(t))y \end{array}$$

where $x_c(t)$ belongs to $\mathbf{R}^k$ and $x_c(0) = 0$. This dynamic case can be reduced to a static one by a system augmentation technique. This point will be detailed further.

The problem of static output-feedback stabilization (SOFS) for a stationary deterministic system $(A, B, C) \in \mathbf{R}^{n\times n} \times \mathbf{R}^{n\times n_u} \times \mathbf{R}^{n_y\times n}$ with the probability transition matrix $\Pi = 0$), consists in searching for a matrix $K \in \mathbf{R}^{n_u\times n_y}$ and a symmetric positive definite matrix $P \in \mathbf{R}^{n\times n}$ such that

$$(A + BKC)^T P + P(A + BKC) < 0, P > 0$$

Although at the first glance this problem seems to be simple, we emphasize that it is not yet fully resolved and it is one of the important chalenging problems that remains open in automatic control [23, 5]. In contrast, a surprising result in our reported work [2] is that this problem is completely solved for SI or SO positive systems via linear programming (LP). However, the analysis of the complexity of the static output-feedback stabilization (SOFS) problem for general deterministic stationary systems remains unfinished. It is demonstrated in [3] that this problem is decidable. Also in [4] it is shown that if a priori bounds on the gain of the control is imposed, the stabilization problem becomes NP-hard. Of course this result does not imply that the SOFS problem itself is NP-hard. However, inspired by our reported work [11], a general result in [12] has traced the SOFS problem in a general way: find two positive definite matrices X, Y satisfying two Linear Matrix Inequality (LMI) $F_1(X) < 0, F_2(Y) < 0$ and an equality constraint $XY = I$. They demonstrated that this problem is NP-hard, that is, there does not exist any algorithm that solves it in polynomial-time. Another closely related problem consists of finding a control that is stabilizing for several systems simultaneously. Such problem is demonstrated also to be

NP-hard [33] to solve. The classic tutorial [31] and the recent survey update [28] provide more details on this SOFS subject.

In [30] the SOFS problem (for stationary deterministic systems $\Pi = 0$) is reduced to solve a standard bilinear matrix equation. Other classic alternative can be found in [15, 26, 35] where the SOFS problem is treated by poles placement techniques. Our approach in this paper for Markovian jump systems is promising and extend our previous works on stationary deterministic systems [11]. This approach consists of formulating and treating the SOFS problem in terms of optimizing a scalar product of positive definite matrices under LMI constraints. This can viewed as a generalization of the classic complementarity problem in the Euclidean space [34, 17] to the cone of symmetric positive matrices. The algorithm that we have developed has a great succes for many examples known from the literature as well as others that are generated randomly in many thousands of numerical tests (see [11]). Moreover, this complementarity based approach has proven to be highly efficient and successful with comparison to the D-K iteration algorithm [6, 25, 27]. Also, it outperformes other well-known LMI-based methods such as the alternating projections method [13], the projection method of[25], the min-max algorithm [32] and the XY-centering algorithm [14].

For jump systems, to our knowledge, [18] has provided the first attempt to numerically resolve the static output-feedback stabilization problem based on an average quadratic criterion. Specifically, such approach is based on the solution of coupled equations that depend on the gains of the output-feedback stabilizing controls. The developed algorithm has been done in a similar way as for the DK-iteration. Other related method can be found in [20, 21].

In this paper, for the case of known constante transition rates, the SOFS problem is formulated in terms of optimizing a sum of scalar products of positive definite matrices under coupled LMI constraints. It is shown that there is a mean-square stabilizing static output-feedback control,if and only if, the global minimum is achieved and equals $n \times N$. For the delicate case of uncertain time-varying transition rates, a quasi similar but only sufficient condition is provided. The properties of the underlying cone complementarity algorithm are studied and illustrated by an example.

2 Problem Formulation and Preliminaries

Here, we are interested in stabilizing in the mean-square sense system (1) by a static output-feedback control of the form

$$u(t) = K(r(t))y(t), \text{ where } K(r(t)) = K_i \text{ where } r(t) = i, i = 1, ..., N. \quad (2)$$

If Π is constant, such control law exists, if and only if, the following inequalities hold in the matrix variables K_i, P_i (see for instance [1, 10]

$$\begin{array}{c} (A_i + B_i K_i C_i)^T P_i + P_i(A_i + B_i K_i C_i) + \sum_{j=1}^{N} \pi_{ij} P_j < 0, \\ P_1 > 0, ..., P_N > 0, \; i = 1, \dots, N. \end{array} \quad (3)$$

We note that (3) represents non convex Bilinear Matrix Inequalities (BMIs) that involves the variables K_i, P_i. There is no change of variables which can bring this BMI into convex conditions like it has been done for the case $C_i = I$ in [1, 10].

In the sequel, the basic idea that we will develop consists in eliminating the gain matrices K_i and obtaining equivalent conditions to (3) under a cone complementarity condition which will then be reformulated as an optimization problem. In order to show this, we will apply the following well-known elimination lemma (also known as projection lemma in the literature).

Lemma 1 *Let $(G, V, U) \in \mathbf{R}^{n\times n} \times \mathbf{R}^{n\times p} \times \mathbf{R}^{q\times n}$, then there exists a matrix $X \in \mathbf{R}^{p\times q}$ such that*

$$G + UXV + V^T X^T U^T < 0$$

if and only if

$$U_\perp^T G U_\perp < 0, \; \textit{and } V_\perp G V_\perp^T < 0$$

where $U_\perp$ (resp. $V_\perp$) represents an orthogonal matrix Z (resp. W) of maximal rank such that $Z^T U = 0$ (resp. $W V^T = 0$).

Now, coming back to the case of an output dynamic-feedback control

$$\begin{array}{rcl} \frac{dx_c}{dt}(t) & = & A_c(r(t))x_c + B_c(r(t))y \\ u(t) & = & C_c(r(t))x_c + D_c(r(t))y \end{array} \quad (4)$$

where

$$\begin{bmatrix} A_c(r(t)) & B_c(r(t)) \\ C_c(r(t)) & D_c(r(t)) \end{bmatrix} = \begin{bmatrix} A_c^i & B_c^i \\ C_c^i & D_c^i \end{bmatrix}$$

when $r(t) = 1, \ldots, N$.

This case can be formulated equivalently as a static output-feedback control law of the form (2) for an augmented system. Indeed, if we set $\tilde{x}^T = (x^T x_c^T)^T$, the augmented closed-loop system can be expressed as

$$\frac{d\tilde{x}}{dt}(t) = (\tilde{A}(r(t))_{+}\tilde{B}(r(t))\tilde{K}(r(t))\tilde{C}(r(t))$$

for which the associated matrices are given by

$$\tilde{A}(r(t)) := \begin{bmatrix} A_c(r(t)) & 0 \\ 0 & 0 \end{bmatrix}, \tilde{B}(r(t)) := \begin{bmatrix} 0 & B_c(r(t)) \\ I & 0 \end{bmatrix},$$

$$\tilde{C}(r(t)) := \begin{bmatrix} 0 & I \\ C(r(t)) & 0 \end{bmatrix}.$$

and the mode dependent matrix gain is given by

$$\tilde{K}(r(t)) := \begin{bmatrix} A_c(r(t)) & B_c(r(t)) \\ C_c(r(t)) & D_c(r(t)) \end{bmatrix} = \begin{bmatrix} A_c^i & B_c^i \\ C_c^i & D_c^i \end{bmatrix} \text{ when } r(t) = 1, \ldots, N.$$

3 Synthesis for known transition rates matrix Π

In this section, we assume that the transition rate matrix Π is fully known and constant. We provide different conditions that are necessary and sufficient for the existence of a stabilizing static output-feeedback control in the mean-square sense.

Theorem 1 *There exists a control law of the form (2) which is mean-square stabilizing for system (1), if and only if, there exist symmetric matrices $P_1, Q_1, \ldots, P_N, Q_N$ satisfying $P_i = Q_i^{-1} > 0$ for $i = 1, \ldots, N$ and such that*

$$\begin{aligned} C_{i\perp}^T(A_i^T P_i + P_i A_i + \sum_{j=1}^{N} \pi_{ij} P_j) C_{i\perp} < 0, \\ B_{i\perp}(A_i Q_i + Q_i A_i^T + \pi_{ii} Q_i + \sum_{j \neq i}^{N} \pi_{ij} Q_i Q_j^{-1} Q_i) B_{i\perp}^T < 0. \end{aligned} \tag{5}$$

Proof. a necessary and sufficient condition for the closed-loop system to be stable in the mean square is given by the BMI condition (3), then by using the elimination Lemma 1, we equivalently obtain

$$C_{i\perp}^T(A_i^T P_i + P_i A_i + \sum_{j=1}^{N} \pi_{ij} P_j) C_{i\perp} < 0,$$

$$B_{i\perp}(A_i P_i^{-1} + P_i^{-1} A_i^T + \pi_{ii} P_i^{-1} + \sum_{j\neq i}^{N} \pi_{ij} P_i^{-1} P_j^{-1} P_i^{-1}) B_{i\perp}^T < 0.$$

by making the change of variables $Q_i = P_i^{-1}$ we obtain the conditions (5) ■

Remark 1 *If the conditions (5) are not satisfied then the SOFS problem has no solution. Note that in the case of LTI stastionary systems (*$\Pi = 0$*), these conditions are exactly the conditions of stabilizability and detectability in the deterministic sense.*

The inequalities (5) can be expressed in terms of LMIs in the variables P_i, Q_i. For this purpose one can use the well-known Schur lemma.

Lemma 2 *Given matrices* $X = X^T$*,* Y *and* $Z = Z^T$ *with appropriate dimensions. Then, if* $Z > 0$*, we have*

$$\begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} > 0 (resp. \geq 0), \text{ is equivalent to } X - YZ^{-1}Y^T > 0 (resp. \geq 0) \tag{6}$$

Note that by simple manipulation via this Schur's lemma the second inequalitis in the conditions (5) can be equivalently expressed in terms of LMIs. However, the equality constraint $P_i = Q_i^{-1}$ is still not possible to be represented by an LMI. In fact, this is the hard non-convex constraint that makes very hard the numerical solvability of the stabilization problem by static output-feedback. In another point of view, this constraint represents the case of the maximal singularity of the following matrix $M(P_i, Q_i)$ which has a minimal rank equals n.

$$M(P_i, Q_i) := \begin{bmatrix} P_i & I \\ I & Q_i \end{bmatrix} \in \mathbf{R}^{n \times n}. \tag{7}$$

Indeed, by using Schur complement one can show that $P_i = Q_i^{-1}$ is equivalent to $\mathbf{rank}(M(P_i, Q_i)) = n$. Such fact will not be used in our treatment, but instead we make use of the following equivalence

$$M(P_i, Q_i)) \geq 0, \mathbf{Trace}(P_i Q_i) = n \text{ if and only if } P_i = Q_i^{-1} \tag{8}$$

Next, we are going to show that the static ouput-feedback problem is equivalent to optimizing a scalar product under LMI constraints.

Theorem 2 *There exists a control law of the form (2) that is mean square stabilizing for system (1), if and only if*

$$\left\{ \begin{array}{l} \min \sum_{i=1}^{N} \mathbf{Trace}(P_i Q_i) = n \times N \\ \textbf{subject to } (5) \textit{ and } \left[\begin{array}{cc} P_i & I \\ I & Q_i \end{array} \right] \geq 0,\ i = 1, \ldots, N. \end{array} \right. \quad (9)$$

Proof. Suppose there is a stabilizing control of the form (2). According to Theorem 1, there then exists $P_1, Q_1, > 0 \ldots, , P_N, Q_N > 0$ satisfying $P_i = Q_i^{-1}$. Then, we have trivially

$$\sum_{i=1}^{N} \mathbf{Trace}(P_i Q_i) = \sum_{i=1}^{N} \mathbf{Trace}(I) = n \times N.$$

Conversely, suppose that the global minimum is equal to $n \times N$ and achieved by $P_1, Q_1, > 0 \ldots, , P_N, Q_N > 0$.

For the rest of the proof we use the following identity

$$\sum_{i=1}^{N} \mathbf{Trace}(P_i Q_i) = \sum_{i=1}^{N} \mathbf{Trace}(Q_i^{1/2} P_i Q_i^{1/2}) = n \times N. \quad (10)$$

Note that by using Schur Lemma the LMIs

$$\left[\begin{array}{cc} P_i & I \\ I & Q_i \end{array} \right] \geq 0,\ i = 1, \ldots, N.$$

after simple manipulation, are equivalent to

$$Q_i^{1/2} P_i Q_i^{1/2} \geq I,\ i = 1, \ldots, N.$$

which implies that

$$\mathbf{Trace}(Q_i^{1/2} P_i Q_i^{1/2}) \geq n,\ i = 1, \ldots, N.$$

From the identity (10) one can deduce that for each i we have

$$Tr((Q_i^{1/2} P_i Q_i^{1/2}) = n.$$

Thus, by appealing to the property (8) this is equivent to the equalities $P_i = Q_i^{-1}$ ∎

4 Synthesis for uncertain transition rate matrix Π

In this section, we assume that the transition rates matrix is possibly time-varying and uncertain. It is only known that $\Pi(t)$ takes values in a polytope domain $D = \mathbf{Co}(\Pi^1, ..., \Pi^L)$, where each Π^k represents a known constant transition rates matrix.

In this case, we only have sufficient conditions for the existence of a mean-square stabilzing control of the form (2). These sufficient conditions can be established based on the robust stability result in [1, 10] , see also the Appendix of [10] for general robust stability result. Then, for $\Pi \in \mathbf{Co}(\Pi^1, \ldots, \Pi^L)$ such robust stability conditions are given by the inequalities

$$\begin{gathered}(A_i + B_iK_iC_i)^T P_i + P_i(A_i + B_iK_iC_i) + \sum_{j=1}^{N} \pi_{ij}^k P_j < 0, \\ P_1 > 0, ..., P_N > 0,\ i = 1, \ldots, N,\ k = 1, \ldots, L.\end{gathered} \tag{11}$$

We notice that each gain K_i appears in L different inequalities. Thereby, we cannot apply the elimination lemma, which is generally true only for a single inequality. Instead of the conditions (11), we introduce othe conditions with aditional matrix variables that allow us to eliminate the gains K_i. Thus, this consists in introducing positive definite matrices $S_i, i = 1, \ldots N$, such that

$$\begin{gathered}(A_i + B_iK_iC_i)^T P_i + P_i(A_i + B_iK_iC_i) + S_i < 0 \\ \sum_{j=1}^{N} \pi_{ij}^k P_j \leq S_i, \\ P_1 > 0, ..., P_N > 0,\ i = 1, \ldots, N,\ k = 1, \ldots, L.\end{gathered} \tag{12}$$

Then, we apply the elimination Lemma to $(A_i + B_iK_iC_i)^T P_i + P_i(A_i + B_iK_iC_i) + S_i < 0$ and finally obtain the following conditions

$$\begin{gathered}C_{i\perp}^T (A_i^T P_i + P_iA_i + S_i)C_{i\perp} < 0 \\ B_{i\perp}(A_iQ_i + Q_iA_i^T + Q_iS_iQ_i)B_{i\perp}^T < 0, \\ \sum_{j=1}^{N} \pi_{ij}^k P_j \leq S_i, \\ P_1 > 0, ..., P_N > 0,\ i = 1, \ldots, N,\ k = 1, \ldots, L.\end{gathered} \tag{13}$$

In order to reformulate the second inequality in (13) as an LMI, we introduce $T_i := S_i^{-1} > 0$ and obtain via the Shur Lemma

$$\begin{array}{c} C_{i\perp}^T(A_i^T P_i + P_i A_i + S_i)C_{i\perp} < 0 \\ \left[\begin{array}{cc} B_{i\perp}^T(A_i Q_i + Q_i A_i^T)B_{i\perp}^T & B_{i\perp}Q_i \\ Q_i B_{i\perp}^T & -T_i \end{array}\right] < 0 \\ \sum_{j=1}^{N} \pi_{ij}^k P_j \leq S_i, \\ P_1 > 0, ..., P_N > 0,\ i = 1, \ldots, N,\ k = 1, \ldots, L. \end{array} \tag{14}$$

Now, owing to the previous development for the case when Π is known, we can establish the following result in similar way with the same line of argument.

Theorem 3 *There exists a control law of the form (2) that is mean-square stabilizing of system (1) with uncertain* $\Pi \in \mathbf{Co}(\Pi^1, ..., \Pi^L)$ *, if*

$$\left\{\begin{array}{l} \min \sum_{i=1}^{N} \mathbf{Trace}(P_i Q_i + S_i T_i) = 2 \times n \times N \\ \textbf{subject to } (14) \textit{ and } \left[\begin{array}{cc} P_i & I \\ I & Q_i \end{array}\right] \geq 0,\ \left[\begin{array}{cc} S_i & I \\ I & T_i \end{array}\right] \geq 0, i = 1, \ldots, N. \end{array}\right. \tag{15}$$

5 Cone complementarity algorithm

In the sequel, we first focus on the case when Π is known and constant.

The function $\sum_{i=1}^{N} \mathbf{Trace}(P_i Q_i)$ is not convex. This lack of convexity can make the search for a global minimum difficult. At a point $(P_i^0, Q_i^0), i = 1, ..., N$, the derivative of this function is given by $\sum_{i=1}^{N} \mathbf{Trace}(P_i Q_i^0 + P_i^0 Q_i)$. The algorithm that we are going to propose is based on the minimization of this derivative at different feasible points. More precisely, these points are generated by a family of LMIs problems. This algorithm generates a decreasing sequence $t_k = \sum_{i=1}^{N} \mathbf{Trace}(P_i^{k+1} Q_i^k + P_i^k Q_i^{k+1}) \geq 2 \times n \times N$. If there exists k such that $t_k = 2 \times n \times N$ then the global minimum is reached.

The scheme of our algorithm is as follows

Algorithm 1 *Fix a tolerence accuracy, for instance $\epsilon = 10^{-6}$ (or smaller if necessary).*

1. *Initialize $P_i^0 = I, Q^0 = I,\ i = 1, \ldots, N$*
2. *Find $P_i, Q_i,\ i = 1, \ldots, N$ solution to the SDP problem*

$$\begin{cases} \min \sum_{i=1}^{N} \mathbf{Trace}(P_i Q_i^{k-1} + P_i^{k-1} Q_i) \\ \textbf{subject to } (5) \textit{ and } \begin{bmatrix} P_i & I \\ I & Q_i \end{bmatrix} \geq 0,\ i = 1, \ldots, N. \end{cases} \tag{16}$$

3. *if $\sum_{i=1}^{N} \|P_i - Q_i^{-1}\| < \epsilon$ (or as another criteria $\sum_{i=1}^{N} \mathbf{Trace}(P_i Q_i) - nN < \epsilon$), stop, and compute the gains K_i by solving the LMIs*

$$(A_i + B_i K_i C_i)^T P_i + P_i(A_i + B_i K_i C_i) + \sum_{j=1}^{N} \pi_{ij} P_j < 0,$$

   else if, set $P_i \to P_i^{k+1},\ Q_i \to Q_i^{k+1}$ and return to step 2.

Remark 2 *To initialize the algorithm, we can also start from any feasible point. If we choose the initialization $P_i^0 = Q^0 = I,\ i = 1, \ldots, N$, then step 2 of the algorithm consists in minimizing $\sum_{i=1}^{N} \mathbf{Trace}(P_i + Q_i)$ which is the sum of the eigenvalues of all the matrices P_i, Q_i.*

In order to study the convergence properties of our algorithm, we need the following result, see for instance [24, 29].

Lemma 3 *For all positive definite matrices $X > 0, Y > 0$, we have*

$$\min_{\begin{bmatrix} V & I \\ I & W \end{bmatrix} \geq 0} \mathbf{Trace}(VX + WY) = 2\mathbf{Trace}(X^{1/2} Y X^{1/2})^{1/2} \tag{17}$$

The minimum is reached by

$$V_* = X^{-1/2}(X^{1/2} Y X^{1/2})^{1/2} X^{-1/2},$$
$$W_* = V_*^{-1} = Y^{-1/2}(Y^{1/2} X Y^{1/2})^{1/2} Y^{-1/2}.$$

Now, we are in a place to state the following result

Theorem 4 *Algorithm 1 generates a decreasing sequence*

$$t_k := \sum_{i=1}^{N} \mathbf{Trace}(P_i^{k+1}Q_i^k + P_i^k Q_i^{k+1}),$$

Moreover, for all $k > 0$, we have that $t_k \geq 2 \times n \times N$. If there exists k^* such that $t_{k^*} = 2 \times n \times N$, then $P_i^{k^*} Q_i^{k^*} = I$, $i = 1, \ldots, N$. In this case, the problem of static output-feedback stabilization is solved.

Proof. Since for $k \geq 1$ P_i^k, Q_i^k, $i = 1, \ldots, N$ are feasible solution and P_i^{k+1}, Q_i^{k+1}, $i = 1, \ldots, N$ are optimal solution to

$$\begin{cases} \min \displaystyle\sum_{i=1}^{N} \mathbf{Trace}(P_i Q_i^k + P_i^k Q_i) \\ \textbf{subject to } LMIs \text{ and } \begin{bmatrix} P_i & I \\ I & Q_i \end{bmatrix} \geq 0, \; i = 1, \ldots, N. \end{cases}$$

We can deduce that

$$t_{k+1} \leq \sum_{i=1}^{N} \mathbf{Trace}(P_i^{k+1}Q_i^k + P_i^k Q_i^{k+1}) = t_k.$$

The rest of the proof follows from Lemma 3 and from the fact that if $X \geq Y^{-1}$ then $\mathbf{Trace}(XY) \geq n$ and that the egality is satisfied if and only if $XY = I$

■

Next, for the case of uncertain Π, we provide another cone complementarity algorithm that is based on Theorem 3. Its properties are similar to the previous Algorithm 1 and can be shown in the same line of aegument.

Algorithm 2 *Fix a tolerence accuracy, for instance $\epsilon = 10^{-6}$ (or smaller if necessary).*

1. *Initialize $P_i^0 = Q_i^0 = S_i^0 = T_i^0 = I$, $i = 1, \ldots, N$*

2. *Find P_i, Q_i, S_i, T_i $i = 1, \ldots, N$ solution to the SDP problem*

$$\begin{cases} \min \displaystyle\sum_{i=1}^{N} \mathbf{Trace}(P_i Q_i^{k-1} + P_i^{k-1} Q_i + S_i T_i^{k-1} + S_i^{k-1} T_i) \\ \textbf{subject to } (14) \; and \begin{bmatrix} P_i & I \\ I & Q_i \end{bmatrix} \geq 0, \begin{bmatrix} S_i & I \\ I & T_i \end{bmatrix} \geq 0, \\ i = 1, \ldots, N. \end{cases} \tag{18}$$

3. if $\sum_{i=1}^{N} \|P_i - Q_i^{-1}\| + \|S_i - T_i^{-1}\| < \epsilon$ *(or as another criteria* $\sum_{i=1}^{N} \mathbf{Trace}(P_iQ_i + S_iT_i) - 2nN < \epsilon$*), stop, and compute the gains* K_i *by solving the LMIs*

$$(A_i + B_iK_iC_i)^T P_i + P_i(A_i + B_iK_iC_i) + S_i < 0,$$

else if, set $P_i \to P_i^{k+1}$, $Q_i \to Q_i^{k+1}$, $S_i \to S_i^{k+1}$, $T_i \to T_i^{k+1}$ *and return to step 2.*

6 Numerical illustration

We illustrate the proposed Algorithm 2 by solving an example that represents a jump system with uncertain transition rates. We seek a control law of the form (2) for system (1) whose modes and matrix of transition rates $\Pi \in \mathbf{Co}(\Pi^1, \Pi^2)$ are characterized as follows.

$$A_1 = \begin{bmatrix} 0,4002 & 0,5193 & 0,4281 \\ 0,5373 & 0,7715 & 0,9773 \\ 0,4461 & 0,6543 & 0,4778 \end{bmatrix}, \ A_2 = \begin{bmatrix} 0,3291 & 0,5306 & 0,9304 \\ 0,4597 & 0,7149 & 0,9692 \\ 0,7432 & 0,3725 & 0,2553 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0,7036 & 0,1614 & 0,8162 \\ 0,2387 & 0,517 & 0,3862 \\ 0,5233 & 0,1522 & 0,4395 \end{bmatrix}, \ B_1 = \begin{bmatrix} 0,3272 & 0,8719 \\ 0,7135 & 0,7582 \\ 0,5768 & 0,8366 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0,6998 & 0,6056 \\ 0,3891 & 0,3621 \\ 0,9352 & 0,5395 \end{bmatrix}, \ B_3 = \begin{bmatrix} 0,4551 & 0,8310 \\ 0,4017 & 0,4216 \\ 0,2404 & 0,4724 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0,2898 & 0,2164 & 0,5274 \\ 0,1050 & 0,8020 & 0,6688 \end{bmatrix}, \ C_2 = \begin{bmatrix} 0,8636 & 0,7116 & 0,3828 \\ 0,2438 & 0,2637 & 0,0810 \end{bmatrix},$$

$$C_3 = \begin{bmatrix} 0,4598 & 0,7954 & 0,0032 \\ 0,2329 & 0,4913 & 0,2661 \end{bmatrix}.$$

The transition rate matrix Π is uncertain and supposed to varies between Π_1 and Π_2, that is, $\Pi \in \mathbf{Co}(\Pi_1, \Pi_2)$, where

$$\Pi_1 = \begin{bmatrix} -1,9095 & 0,9635 & 0,9460 \\ 0,4252 & -0,5963 & 0,1712 \\ 0,1868 & 0,7959 & -0,9827 \end{bmatrix},$$

$$\Pi_2 = \begin{bmatrix} -1,2361 & 0,5097 & 0,7264 \\ 0,3475 & -0,6769 & 0,3294 \\ 0,2925 & 0,8343 & -1,1268 \end{bmatrix}.$$

We have applied Algorithm 2 to this example and found a global minimum in 2 iterations. The computed corresponding stabilizing control for all $\Pi \in \mathbf{Co}(\Pi_1, \Pi_2)$ is given by the following gains

$$K_1 = 10^5 \begin{bmatrix} 0,0791 & 0,0450 & -0,2418 \\ -0,1092 & -0,0622 & 0,5887 \end{bmatrix},$$

$$K_2 = 10^5 \begin{bmatrix} 1,1697 & 0,4513 & -1,0587 \\ -2,8486 & -0,2753 & 0,6459 \end{bmatrix}.$$

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7 Conclusion

We have shown that the problem of robust static output-feedback control of jump systems can be formulated as an optimization problem. This formulation represents an extension of the problem of complementarity for the Euclidian case to the case of the cone of positive symmetric matrices. The search for a global minimum is based on a linearization algorithm.

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