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REMARKS ON TIME-SCALE DECOMPOSITION USING SINGULAR PERTURBATIONS WITH APPLICATIONS

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Dedicated to Dr. Vasile Drăgan on the occasion of his 70th anniversary

Abstract

In this paper, we point out important observations on time-scale decomposition of linear singularly perturbed systems. It has been established in the control literature that the asymptotically stable fast modes of a singularly perturbed system decay rapidly in a boundary layer interval when the perturbation parameter is very small hence the slow subsystem can serve as a good approximation of the original model. We observe that while this is the case in the steady state, it is not true during the transient response for a strictly proper system with highly damped and highly oscillatory modes. Instead, the fast subsystem provides a very good approximation of the original model's response but with a DC gain offset. We propose a correction to rectify the DC gain offset and illustrate the findings using an islanded microgrid electric power system model.

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keywords: Singular perturbations, time-scale decomposition, islanded microgrid

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1 Introduction

Singular perturbation theory has been widely used for time-scale separation and order reduction for systems where different time-scales are present. The most common case is when two time-scales, namely *slow* and *fast* characterize the dynamics of the given system. The latter form is the most researched variant of singular perturbation methods; see for example [12], [5], [14], [11] and references therein.

The premise of singular perturbation methods for order reduction lies in the fact that asymptotically stable fast dynamics decay rapidly in a boundary interval when a large separation between clusters of eigenvalues exists due to the presence of a sufficiently small perturbation parameter. As the fast dynamics die out, an $n+m$ size original model degenerates to a lower order size n which corresponds to the slow subsystem [12]. The latter serves as a good representation of the original full-order system [12].

In this paper, we make two remarks on singular perturbation theory by utilizing a model of a real physical system. We use the latter to illustrate observations in the time-scale decomposition of strictly proper singularly perturbed system with highly damped and highly oscillatory modes. First, even though the singular perturbation parameter in the considered case study is sufficiently small, the slow subsystem does not approximate the original model as theory suggests. Instead, the fast subsystem provides a very good approximation of the original model's response but with a DC gain offset. Second, we point out that a fast DC gain correction has to be employed to improve the transfer function approximation of the overall system when the classical singular perturbation technique is used. The DC gain is defined as the value of the transfer function evaluated at $s = 0$.

We propose a solution based on the DC gain of the fast subsystem to correct the offset. It should be pointed out that when the system is exactly decoupled using the Chang transformation [1], the DC gain offset is $\mathcal{O}(\varepsilon)$. That is, its steady-state response is only $\mathcal{O}(\varepsilon)$ apart from the corresponding response of the original system. The rest of this manuscript is organized as follows.

In Section 2, a review of singular perturbation theory and how it is used for time-scale decomposition and model order reduction is presented. Section 3 contains a motivating example of a real physical system, important results on model order reduction using singular perturbations and some observations. Further considerations of singular perturbation theory with respect to the highly oscillatory and highly damped mode example are presented in Section 4. Finally, Section 5 concludes the paper.

2 Review of the Singular Perturbation Method

We start with a linear time-invariant (LTI) strictly proper singularly perturbed system in mathematical form represented as

$$\begin{aligned} \dot{x}_1(t) &= A_1x_1(t) + A_2x_2(t) + B_1u(t) \\ \varepsilon\dot{x}_2(t) &= A_3x_1(t) + A_4x_2(t) + B_2u(t) \\ y(t) &= C_1x_1(t) + C_2x_2(t) \end{aligned} \tag{1}$$

where $x_1(t) \in \mathbb{R}^n$ and $x_2(t) \in \mathbb{R}^m$ are the slow and fast state variables respectively, $u(t) \in \mathbb{R}^p$ is the system control input, $y(t) \in \mathbb{R}^q$ are the system measurements, and ε is the small singular perturbation parameter $0 < \varepsilon \ll 1$. All the matrices in (1) are constant and of appropriate dimensions. Two time-scale LTI singularly perturbed systems have eigenvalues located in two disjoint groups: for example, slow $\mathcal{O}(1)$ eigenvalues close to the imaginary axis and fast $\mathcal{O}(\frac{1}{\varepsilon})$ eigenvalues far from it. The following standard assumption is imposed [12].

Assumption 1: Matrix A_4 is nonsingular.

When $\varepsilon = 0$ (a common strategy for order reduction), the following reduced-order system corresponding to the slow dynamics is obtained

$$\begin{aligned} \dot{\bar{x}}_1(t) &= A_0\bar{x}_1(t) + B_0u(t) \\ \bar{y}(t) &= C_0\bar{x}_1(t) + D_0u(t) \end{aligned} \tag{2}$$

where

$$\begin{aligned} A_0 &:= A_1 - A_2A_4^{-1}A_3 \quad & B_0 &:= B_1 - A_2A_4^{-1}B_2 \\ C_0 &:= C_1 - C_2A_4^{-1}A_3 \quad & D_0 &:= -C_2A_4^{-1}B_2 \end{aligned} \tag{3}$$

The approximated fast subsystem is [12]

$$\begin{aligned} \dot{\bar{x}}_2(\tau) &= A_4\bar{x}_2(\tau) + B_2u(\tau) \\ \bar{y}_f(\tau) &= C_2\bar{x}_2(\tau) \end{aligned} \tag{4}$$

where $\tau = t/\varepsilon$. According to the theory of singular perturbations [3], [12], the approximation obtained in (2)-(4) satisfies Equation (5).

$$\begin{aligned} x_1(t) &= \bar{x}_1(t) + \mathcal{O}(\varepsilon), \quad \forall t \geq t_0 \\ x_2(t) &= \bar{x}_2(\tau) - A_4^{-1}(A_3\bar{x}_1(t) + B_2u(t)) + O(\varepsilon), \quad \forall t \geq t_0 \end{aligned} \tag{5}$$

Hence, the smaller parameter ε, the better the approximation. Another effective method to obtain *exact* dynamic decoupling is by utilizing the transformation developed in [1]. Referred to as the Chang transformation, it diagonalizes the system by exposing the slow and fast dynamics. The transformation is given as follows.

$$\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} I - \varepsilon HL & \varepsilon H \\ L & I \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = T \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{6a}$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} I & \varepsilon H \\ -L & I - \varepsilon LH \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = T^{-1} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} \tag{6b}$$

After (6) is applied to (1), the decoupled system is then

$$\begin{aligned} \dot{z}_1(t) &= A_s z_1(t) + B_s u(t) \\ \varepsilon \dot{z}_2(t) &= A_f z_2(t) + B_f u(t) \\ y(t) &= C_s z_1(t) + C_f z_2(t) \end{aligned} \tag{7}$$

where

$$\begin{aligned} &A_s := A_1 - A_2 L \quad && B_s := B_1 - HB_2 - \varepsilon HLB_1 \\ &A_f := A_4 + \varepsilon LA_2 && B_f := B_2 + \varepsilon LB_1 \\ &C_s := C_1 - C_2 L && C_f := C_2 - \varepsilon C_2 LH + \varepsilon C_1 H \end{aligned} \tag{8}$$

Note that all quantities in (7)-(8) are off by $\mathcal{O}(\varepsilon)$ from the corresponding quantities defined in (2)-(4), that is

$$\begin{aligned} A_s &= A_0 + \mathcal{O}(\varepsilon), \quad & A_f &= A_4 + \mathcal{O}(\varepsilon) \\ B_s &= B_0 + \mathcal{O}(\varepsilon), & B_f &= B_2 + \mathcal{O}(\varepsilon) \\ C_s &= C_0 + \mathcal{O}(\varepsilon), & C_f &= C_2 + \mathcal{O}(\varepsilon) \end{aligned} \tag{9}$$

A noticeable difference between the reduced-order slow model defined in (2)-(3) and the decoupled system (7) is that the measurement in (7) lacks an input $u(t)$ while $\bar{y}(t)$ from (2) includes a $D_0 u(t)$ term.

Matrices L and H are obtained by solving the following equations.

$$\begin{aligned} A_4 L - A_3 - \varepsilon L(A_1 - A_2 L) = 0 \\ HA_4 - A_2 + \varepsilon(HLA_2 - A_1 H + A_2 LH) = 0 \end{aligned} \tag{10}$$

The reader can refer to [5] and [9] for methods on finding the solution of L and H equations.

In the following sections we investigate a model belonging to an islanded microgrid [8] and show that simulation results do not completely follow the aforementioned singular perturbation theory.

3 Motivating Example

To illustrate the shortcomings of singular perturbation theory for systems containing highly damped and highly oscillatory modes (see [2] for a study of singularly perturbed systems with such behavior), a motivating example of an islanded microgrid model follows.

3.1 Model Description

The sixth-order model belongs to a microgrid right after it has been disconnected from the main electric power grid at the point of common coupling [8] . Under balanced conditions the initial model is transformed into an $\alpha\beta$-reference frame, which is further simplified by applying a dq transformation leading to a six-order model. The state vector of dimension six is $x(t) = \begin{bmatrix} V_d(t) & V_q(t) & I_{td}(t) & I_{tq}(t) & I_{Ld}(t) & I_{Lq}(t) \end{bmatrix}$, the input vector is $u(t) = \begin{bmatrix} V_{td}(t) & V_{tq}(t) \end{bmatrix}' \in \mathbb{R}^2$, and the output vector is $y(t) = \begin{bmatrix} V_d(t) & V_q(t) \end{bmatrix}' \in \mathbb{R}^2$, where $V_d(t)$ and $V_q(t)$ represent the voltages at the point of common coupling after the dq transformation, $I_{td}(t)$ and $I_{tq}(t)$ represent the currents originating from the distributed generation (DG) unit, $I_{Ld}(t)$ and $I_{Lq}(t)$ represent load currents, and $V_{td}(t)$ and $V_{tq}(t)$ represent the voltages at the DG unit. A, B, and C representing the state, input, and output matrices respectively are given below.

$$A = \begin{bmatrix} -209.32 & 376.99 & 15908 & 0 & -15908 & 0 \\ -376.99 & -209.32 & 0 & 15908 & 0 & -15908 \\ -3333.3 & 0 & -5 & 376.99 & 0 & 0 \\ 0 & -3333.3 & -376.99 & -5 & 0 & 0 \\ 8.9366 & 0 & 0 & 0 & -3.1416 & 376.99 \\ 0 & 8.9366 & 0 & 0 & -376.99 & -3.1416 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 3333.3 & 0 \\ 0 & 3333.3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Matrix A is Hurwitz and its eigenvalues are shown in (11).

$$\lambda_{IM} = \{-3.15 \pm 377.0j, -107.16 \pm 7668.06j, -107.16 \pm 6914.07j\} \quad (11)$$

It is important to note that the original model is not in the standard singularly perturbed form. To achieve an explicit form, the following permutation matrix is introduced.

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The new state-space matrices are now $\bar{A} = PAP$, $\bar{B} = PB$, and $\bar{C} = CP$. Using the standard theory of singular perturbations covered in the previous section and the system's eigenvalues presented in Equation (11), it is evident that the eigenvalues are clustered in two disjoint groups: complex conjugate pairs located close to the imaginary axis that are considered *slow* and eigenvalues located far from the imaginary axis which are responsible for the system's *fast* dynamics. The small singular perturbation parameter is evaluated as follows.

$$\varepsilon = \frac{\mathrm{Re}\{\lambda_{max}^{slow}\}}{\mathrm{Re}\{\lambda_{min}^{fast}\}} \approx 0.03 \tag{12}$$

As discussed earlier, by eliminating the fast modes we can obtain an approximation of the system dynamics by using the slow modes only. We will show next using simulations that such an approximation will produce poor results.

3.2 Order-reduction via Singular Perturbations

A step input is considered for simulation purposes. The results for the original model due to the first input and first output are shown in Figure 1.

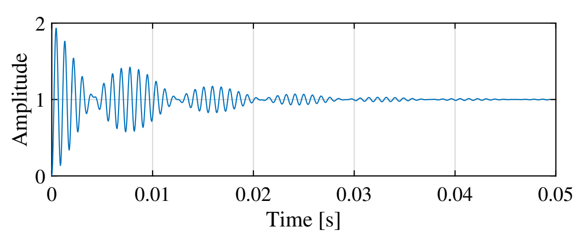

Figure 1: Step response of original system (1)

Matrices of the slow subsystem (2) are given as follows.

$$A_0 = \begin{bmatrix} -3.1566 & -378 \\ 378 & -3.1566 \end{bmatrix}, \quad B_0 = \begin{bmatrix} -0.01369 & 8.9604 \\ 8.9604 & 0.01369 \end{bmatrix}$$

$$C_0 = \begin{bmatrix} 0.1134 & -0.0017 \\ -0.0017 & -0.1134 \end{bmatrix}, \quad D_0 = \begin{bmatrix} 1.0027 & 0.0015 \\ -0.0015 & 1.0027 \end{bmatrix}$$

The corresponding step response of the slow approximate system (2)-(3) is shown in Figure 2. The response settles at one after some initial low magnitude jitter and it is evident that this response is not a satisfactory approximation of the original system. Namely, the responses in Figure 1 and Figure 2 are far apart. Note that a longer time has been selected for this simulation to show the steady-state.

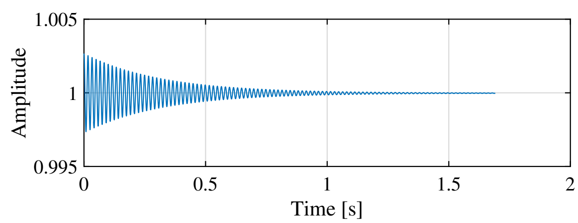

Figure 2: Step response of the *approximate slow* (2)-(3)

Next, system decoupling using the Chang transformation [1] is considered. Figure 3 shows the simulation result for the *slow* subsystem defined in (7). Again, this response does not approximate the original system. The output is similar to the approximate slow subsystem output presented in Figure 2 but with a different gain.

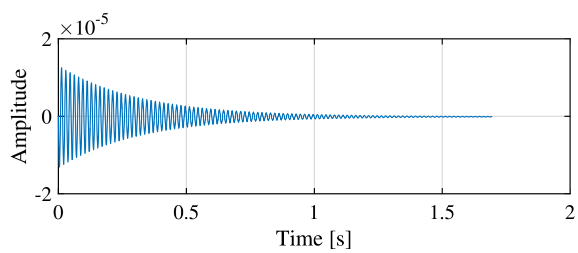

Figure 3: Step response of the exact *slow* subsystem obtained via the Chang transformation

It is interesting to note that the *output of the fast subsystem obtained via the Chang transformation provides a very good approximation of the original system*. Likewise, the fast subsystem (4) produces similar results but with a small steady-state error. The corresponding responses of the fast subsystem via the Chang transformation and the fast subsystem via the classical singular perturbation method are shown in Figure 4 and Figure 5 respectively. Clearly, these observations do not agree with the classical singular perturbation system theory.
Table 1 summarizes the errors of the reduced-order models with the original response. On the other hand, the fast modes obtained via the classical singular perturbation method ($\mathrm{Fast}_{\mathrm{SP}}$) and the Chang transformation ($\mathrm{Fast}_{\mathrm{Chang}}$) provide an accurate approximation of the full-order system's response. $\mathrm{Fast}_{\mathrm{SP}}$ is less accurate than $\mathrm{Fast}_{\mathrm{Chang}}$ due to the extra DC gain offset present in the response of the fast subsystem obtained via the classical singular perturbation method (4).

Table 1: Errors of the reduced-order models for the step responses

Reduced model	$\|Original - Reduced\|_2$
$\mathrm{Slow}_{\mathrm{SP}}$	5.6034×10^1
$\mathrm{Slow}_{\mathrm{Chang}}$	3.5184×10^2
$\mathrm{Fast}_{\mathrm{SP}}$	9.7452×10^{-2}
$\mathrm{Fast}_{\mathrm{Chang}}$	3.0078×10^{-4}

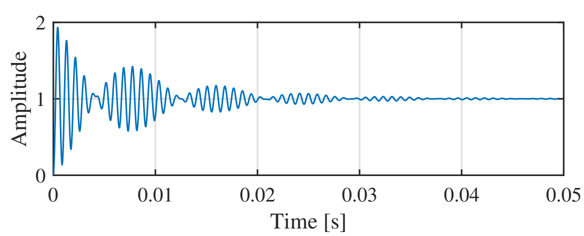

Figure 4: Step response of the exact *fast* subsystem obtained via the Chang transformation

3.3 Analytical Time-scale Decomposition Analysis

Earlier we showed via simulations that the slow subsystem is not a good candidate to approximate the original system even though ε was very small.

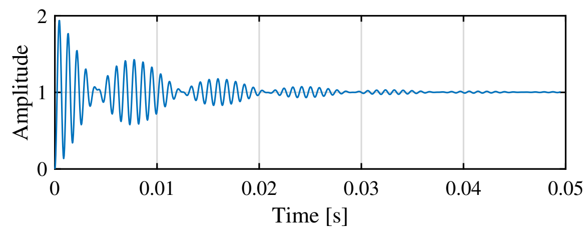

Figure 5: Step response of the *fast* subsystem (4)

Instead, simulation results showed that the fast subsystem approximates well the system dynamics. What follows, is an analytical attempt into understanding the anomaly that arose in the simulations.
We start by considering the eigenvalues of the islanded microgrid model. Using $\varepsilon \approx 0.03$, the model's eigenvalues (11) can be rewritten as

$$\lambda_{IM} = \{-3.15 \pm \frac{11.31}{\varepsilon} j, \frac{3.2148}{\varepsilon} \pm \frac{6.9013}{\varepsilon^2} j, \frac{3.2148}{\varepsilon} \pm \frac{6.2227}{\varepsilon^2} j\} \tag{13}$$

To facilitate the problem, we propose to investigate the system transformed into the modal canonical form. A similarity transformation T can be found such that the system matrix is transformed into the modal form [7]. The same conclusion can be drawn by transforming the system matrix A into the Schur form via the QR algorithm. The QR algorithm is considered the most efficient method for finding the system eigenvalues [6].
Note that the modal form is known to be numerically ill-conditioned when the eigenvalues are close to each other. Hence, in our case it is only used for theoretical considerations. The Schur form on the other hand is numerically well-conditioned even when the eigenvalues are repeated or close to each-other.
The modal form of the system considered in this paper is given by

$$\tilde{A}(\varepsilon) = \left[\begin{array}{cc|cc|cc} \alpha_1 & \frac{\beta_1}{\varepsilon} & 0 & 0 & 0 & 0 \\ -\frac{\beta_1}{\varepsilon} & \alpha_1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \frac{\alpha_2}{\varepsilon} & \frac{\beta_2}{\varepsilon^2} & 0 & 0 \\ 0 & 0 & -\frac{\beta_2}{\varepsilon^2} & \frac{\alpha_2}{\varepsilon} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \frac{\alpha_3}{\varepsilon} & \frac{\beta_3}{\varepsilon^2} \\ 0 & 0 & 0 & 0 & -\frac{\beta_3}{\varepsilon^2} & \frac{\alpha_3}{\varepsilon} \end{array}\right] \tag{14}$$

In state-space form, the system is represented as in (15). Note that for the sake of the argument we are not considering an input here.

$$\frac{dz(t)}{dt} \equiv \dot{z}(t) = \tilde{A}(\varepsilon)z(t) \tag{15}$$

Since the term $\frac{1}{\varepsilon}$ is present in all the elements of matrix $\tilde{A}$, (15) is multiplied by ε on both sides to obtain

$$\varepsilon\frac{dz(t)}{dt} = \varepsilon\tilde{A}(\varepsilon)z(t) \tag{16}$$

It is clear from (14) and (16) that all six state variables are fast due to the presence of ε. In addition, the last four state variables are much faster than the first two since the last four contain elements that are multiplied by $\frac{1}{\varepsilon}$. To obtain a standard singularly perturbed form, we employ a change of variables, namely $\frac{dt}{\varepsilon} = d\tau$. The state-space form (16) can be now be written as

$$\frac{dz(\tau)}{d\tau} = \left[\begin{array}{cc:cc:cc} \varepsilon\alpha_1 & \beta_1 & 0 & 0 & 0 & 0 \\ -\beta_1 & \varepsilon\alpha_1 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & \alpha_2 & \dfrac{\beta_2}{\varepsilon} & 0 & 0 \\ 0 & 0 & -\dfrac{\beta_2}{\varepsilon} & \alpha_2 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & \alpha_3 & \dfrac{\beta_3}{\varepsilon} \\ 0 & 0 & 0 & 0 & -\dfrac{\beta_3}{\varepsilon} & \alpha_3 \end{array}\right] \tag{17}$$

The standard singularly perturbed form is then given as follows.

$$\begin{aligned} \dot{z}_1(\tau) &= \begin{bmatrix} \varepsilon\alpha_1 & \beta_1 \\ -\beta_1 & \varepsilon\alpha_1 \end{bmatrix} z_1(\tau) \\ \varepsilon\dot{z}_2(\tau) &= \begin{bmatrix} \varepsilon\alpha_2 & \beta_2 \\ -\beta_2 & \varepsilon\alpha_2 \end{bmatrix} z_2(\tau) \\ \varepsilon\dot{z}_3(\tau) &= \begin{bmatrix} \varepsilon\alpha_3 & \beta_3 \\ -\beta_3 & \varepsilon\alpha_3 \end{bmatrix} z_3(\tau) \end{aligned} \tag{18}$$

We observe that in the original system there are no slow dynamics. The *fast* dynamics are represented by state variable $z_1(\tau)$ and the *very fast* dynamics are represented by $z_2(\tau)$ and $z_3(\tau)$. Analysis of this real physical system model shows that its response cannot be approximated by the slower dynamics as it is typical in singularly perturbed systems since here only fast and very fast modes are present. It is interesting to point out that using only the fast modes (belonging to $z_1(\tau)$) does not produce a good approximation. However, the fourth-order approximation based on the very fast modes represented by $z_2(\tau)$ and $z_3(\tau)$ produces an excellent approximation in the original coordinates meaning that the system's dynamics are contained within the *very fast* modes.

4 Discussions on the Singular Perturbation Approach for Order Reduction

As we saw in the earlier sections, we were unable to obtain a *reduced-order slow subsystem based* approximation of the original model using the classical singular perturbation technique (2) or the exact time-scale decomposition (6). While there is no error in the steady state, there is a large error during the transient when the classical approach is used for order reduction. This error is evident during the first 0.04 seconds which corresponds to the transient response.

It is worth noting that the original system lacks a feedthrough matrix, that is $D = 0$. On the other hand, the classical singular perturbation approach has an input ($D_0 \neq 0$) present at the output as is evident in (2). This fact provides no guarantee that the original output $y(t)$ can be approximated by $\bar{y}(t)$ and $y_f(\tau)$. As a matter of fact, in [3], it was incorrectly stated that $y(t) = \bar{y}(t) + y_f(\tau) + \mathcal{O}(\varepsilon)$. Depending what the input signal is, the extra term introduced in the reduced-order system could influence the approximation of the overall system's response.

It is easier to see the ramifications if we consider the problem in the frequency domain. For a strictly proper system, the transfer function is as follows.

$$H(s) = C(sI - A)^{-1}B \tag{19}$$

The reduced-order transfer function and the fast counterpart become

$$H_0(s) = C_0(sI - A_0)^{-1}B_0 + D_0 \tag{20a}$$

$$H_f^0(s) = C_2(sI - A_4)^{-1}B_2 \tag{20b}$$

The sum of the slow and fast transfer functions gives us the original system's transfer function. Namely, $H(s) = H_0(s) + H_f^0(s)$. The schematic for this configuration is presented in Figure 6 [3]. An issue that arises at this point is the DC gain of the overall response.

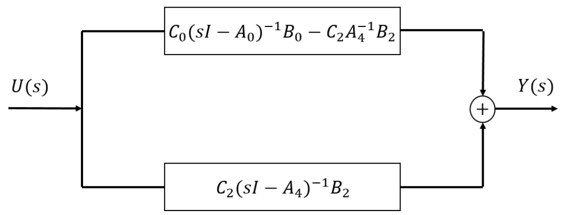

Figure 6: Current slow-fast decomposition architecture

As it was shown in [13], the DC gain of the overall system is given as the sum of the reduced-order systems' DC gain and that of the fast, that is

$$CA^{-1}B = C_0A_0^{-1}B_0 + C_2A_4^{-1}B_2 \tag{21}$$

It is easily seen from the block diagram in Figure 6 that the DC gain ($s = 0$) of the final transfer function $H(s)$ will have twice the gain of the fast subsystem. This can be observed in Figure 7 where simulation results on

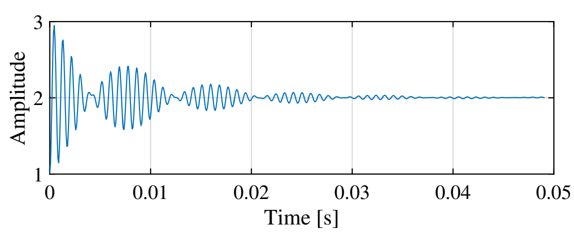

Figure 7: Step response of overall system using (2) and (4) approximations

the example considered earlier show that there is an offset in amplitude compared to the original response depicted in Figure 1. Namely, the response settles at two instead of one. To rectify this issue, we introduce a correction that offsets the gain coming from $H_f^0(s)$. An additional term is added to the transfer function of the fast subsystem as seen in the new scheme depicted in Figure 8. It was shown in [4] that such a corrected DC gain approach works well for reduced-order models obtained via balancing [4], [13]. The corrected response of the example considered earlier is shown in Figure 9. If the latter is compared to Figure 1, we notice that it is very accurate. Note that this would not be the case if the Chang transformation [1] is used. That is, we will have the following transfer functions for each of the decoupled

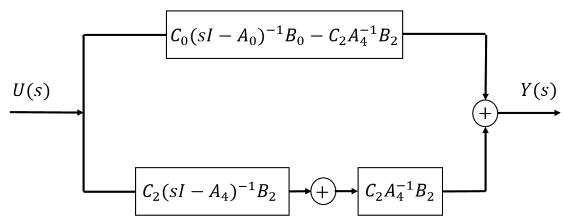

Figure 8: Corrected slow-fast decomposition architecture

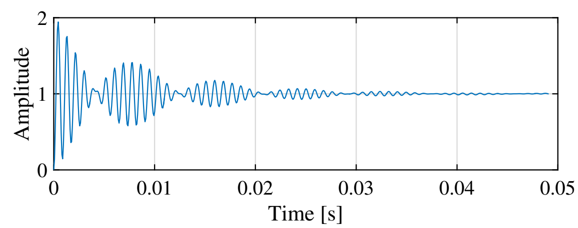

Figure 9: Step response of overall system with fast DC gain correction

subsystems.

$$H_s(s) = H_0(s) + \mathcal{O}(\varepsilon) \tag{22a}$$

$$H_f(s) = H_f^0(s) + \mathcal{O}(\varepsilon) \tag{22b}$$

Equation (22) implies that the slow and fast transfer functions obtain via the exact decoupling differ from their counterparts obtained via the classical singular perturbation technique by $\mathcal{O}(\varepsilon)$. In contrast to $H_0(s)$ obtained in (20a), $H_0(s)$ in (22) lacks the fast DC gain that comes from the D_0 term. To clearly observe this fact, we rewrite (22) as follows.

$$H_s(s) = C_0(sI - A_0)^{-1}B_0 + \mathcal{O}(\varepsilon) \tag{23a}$$

$$H_f(s) = C_2(sI - A_4)^{-1}B_2 + \mathcal{O}(\varepsilon) \tag{23b}$$

The block diagram of this setup is presented in Figure 10. We notice that a correction for the fast DC gain is not needed in this case. The only drawback is that the gain will be offset by $\mathcal{O}(\varepsilon)$.

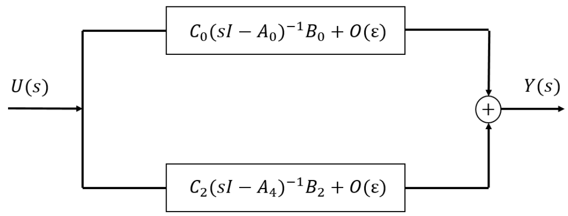

Figure 10: Exact slow-fast decomposition architecture

5 Conclusion

In this paper we presented a case when the classical singular perturbation techniques fail to successfully perform order reduction. While it is customary to decouple the system into slow and fast subsystems and use the slow subsystem as an approximation of the original system, we showed that for an islanded microgrid model with highly damped and highly oscillatory eigenvalues that was not the case. Instead, the system contains fast and very fast modes. The system's dominant dynamics are retained within the very fast modes and that is the reason why the corresponding subsystem provides a very good approximation of the original system. In addition, we discussed some implications of the classical singular perturbation approach for approximating the system's response with respect to the DC gain and proposed a method that rectifies the issue.

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