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ON THE EXISTENCE OF THE SOLUTION OF RICCATI EQUATIONS ARISING IN LINEAR QUADRATIC MEAN FIELD DYNAMIC GAMES*

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Dedicated to Dr. Vasile Drăgan on the occasion of his 70th anniversary

Abstract

In this paper we obtain existence conditions for the solution of a class of generalized Riccati equations arising in finite horizon linear quadratic (LQ) mean-field games.
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1 Introduction

Mean field (MF) game theory provides a powerful tool to study non-cooperative games with a large population of players. It is a class of non-cooperative stochastic differential games, where there is a large number of players, who interact with each other through a mean field coupling term included in the cost function and/or each agent's dynamics. This theory attracted a phenomenal interest from the scientific community these last few years since the

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pioneering works [9, 13]. One of the main reason of such an interest is that It provides a powerful methodology for tackling complexity in large-population non-cooperative decision problems. For an overview of the general theory and applications, readers are referred to [1, 3, 7, 8].

A particularly attractive subclass of MF games is LQ mean field games. This is due to its tractable analytical structure [2, 10, 11, 17]. In the present work we are particularly interested by LQ zero-sum differential games driven by stochastic differential equations (SDEs) with McKean–Vlasov type for their remarkable capacity to characterize dynamical systems of large populations subject to a mean-field interaction. McKean–Vlasov SDEs have a long history since the works of [12] and [15] and find numerous applications in physics, biology, economics and finance, networks, and so on. The study of optimal control for such equations followed several routes from Pontryagin type maximum principle to dynamic programming approach and LQ theory and the related Riccati type equations. On the other hand, as pointed out by [16], there exists little research on game problems of McKean–Vlasov equations except few papers concerning Stackelberg type games, see [14] for example. Our present paper aims at enriching the research in this field. We take as a starting point the work developed in [16] which related to the problem of LQ zero-sum game for this class of systems (in the finite-horizon case). The authors established a closed-loop formulation for saddle points in the mixed-strategy-law form for this type of games. They also showed that the construction of the saddle point relies on the existence of unique solutions, to adequately defined generalized Riccati equations, that verify specific sign conditions. Hence, solutions of such Riccati equations play a crucial role. For these Riccati equations, they proposed a solvability result in a very particular case and they also pointed out that the solvability of such equations in the general case is a rather challenging problem and remains an open problem. Note also that in the result reported in [16], the sign conditions could only be verified a posteriori. In this paper we ambition to fill in the gapes by addressing the open problem of the existence of a unique solution of the involved generalized Riccati equations verifying specific sign conditions in the general setting. to develop our results, we follow a similar approach as in [5, 6].

This paper is organized as follows: In Section 2 we give the problem formulation. In Section 3 we formulate our main results. In Section 4, some concluding remarks are given.

Notations. A^T stands for the transpose of the matrix A. The notation $X \geq Y$ ($X > Y$, respectively), where X and Y are symmetric matrices, means that $X-Y$ is positive semi-definite (positive definite, respectively). In block matrices, $\star$ indicates symmetric terms: $\begin{pmatrix} A & B \\ B^T & C \end{pmatrix} = \begin{pmatrix} A & \star \\ B^T & C \end{pmatrix} = \begin{pmatrix} A & B \\ \star & C \end{pmatrix}$. The expression $MN\star$ is equivalent to MNM^T while $M\star$ is equivalent to MM^T.

2 Problem formulation

In this section, we will first recall some results related to LQ zero-sum mean field games. This material is mostly based on [16]. We will then present the problem we address in the rest of the paper.

We denote by $\mathbb{R}^n$ the n-dimensional Euclidean space equipped with the Euclidean inner product $\langle \cdot, \cdot \rangle$ and the induced Euclidean norm $|\cdot|$. $\mathbb{R}^{n\times m}$ is the collection of all $(n \times m)$ real matrices. $\mathbb{S}^n \subset \mathbb{R}^{n\times n}$ is the set of all $(n \times n)$ symmetric matrices. Let $T > 0$ be a fixed time horizon. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space, on which a r-dimensional standard Brownian motion $w(\cdot) = (w_1(\cdot), w_2(\cdot), \cdots, w_r(\cdot))^T$ is defined, and $\mathbb{F} = \{\mathcal{F}_s, 0 \leq s \leq T\}$ is the natural filtration generated by $w(\cdot)$ which is augmented by all the $\mathbb{P}$-null sets. We assume $w(0) = 0$ and $\mathcal{F} = \mathcal{F}_T$.
For any Euclidean space $\mathbb{R}^n$ (or $\mathbb{R}^{n\times m}$, $\mathbb{S}^n$) and any $t \in [0,\ T]$, we introduce some Banach/Hilbert spaces as follows:

- $L^\infty(t,\ T;\ \mathbb{R}^n) = \{A(\cdot) : [t,\ T] \to \mathbb{R}^n | A(\cdot)$ is a Lebesgue measurable deterministic process essentially bounded$\}$;
- $L^2_{\mathcal{F}_t}(\Omega;\ \mathbb{R}^n) = \{\xi : \Omega \to \mathbb{R}^n | \xi$ is an $\mathcal{F}_t$-measurable random variable satisfying $\mathbb{E}[|\xi|^2] < \infty\}$;
- $L^2_{\mathbb{F}}(t,\ T;\ \mathbb{R}^n) = \{u : \Omega \times [t,\ T] \to \mathbb{R}^n | u(\cdot)$is an $\mathbb{F}$-progressively measurable process such that $\mathbb{E} \int_t^T |u(s)|^2 ds < \infty\}$;
- $L^2_{\mathbb{F}}(\Omega;\ C([t,\ T]; \mathbb{R}^n)) = \{x : \Omega \times [t, T] \to \mathbb{R}^n | x(\cdot)$ is an $\mathbb{F}$-progressively measurable process which admits continuous paths and such that $\mathbb{E}\left[\sup_{s\in[t,\ T]} |x(s)|^2\right] < \infty\}$.

We denote $\mathcal{D} = \{(t, x_t) | t \in [0,\ T], x_t \in L^2_{\mathcal{F}_t}(\Omega;\ \mathbb{R}^n)\}$ and each element $(t,\ x_t)$ is called an initial pair. For any $(t,\ x_t) \in \mathcal{D}$, we consider the following controlled linear mean-field type stochastic differential equation (MF-SDE, for short):

$$\begin{cases} dx(s) = \left\{A_0(s)x(s) + \bar{A}_0(s)\mathbb{E}_t[x(s)] + B_0(s)u(s) + \bar{B}_0(s)\mathbb{E}_t[u(s)]\right\} ds \\ \quad + \sum_{j=1}^r \left\{A_j(s)x(s) + \bar{A}_j(s)\mathbb{E}_t[x(s)] + B_j(s)u(s) + \bar{B}_j(s)\mathbb{E}_t[u(s)]\right\} dw(s) \\ x(t) = x_t \end{cases} \tag{1}$$

$s \in [t,\ T]$, where $\mathbb{E}_t[\cdot] \equiv \mathbb{E}[\cdot|\mathcal{F}_t]$ is the conditional expectation with respect to $\mathcal{F}_t$, $A_0(\cdot)$, $\bar{A}_0(\cdot)$, $A_j(\cdot)$, $\bar{A}_j(\cdot) \in L^\infty(0,\ T;\ \mathbb{R}^n)$, and $B_0(\cdot)$, $\bar{B}_0(\cdot)$, $B_j(\cdot)$, $\bar{B}_j(\cdot)$ $\in L^\infty(0,\ T;\ \mathbb{R}^{n\times m})$, $1 \le j \le r$. The process $u(\cdot) \in L^2_{\mathbb{F}}(t,\ T;\ \mathbb{R}^m)$ is partitioned as: $u(\cdot) = (u_1(\cdot)^T, u_2(\cdot)^T)^T$, and $u_i(\cdot) \in \mathcal{U}_i[t,\ T] \equiv L^2_{\mathbb{F}}(t,\ T;\ \mathbb{R}^{m_i})$ is called the *admissible control* for Player i on the time interval $[t,\ T]$ ($i =$ 1, 2). Here, $m = m_1 + m_2$. Correspondingly, we have the decompositions: $B_0(\cdot) = [B_{01}(\cdot), B_{02}(\cdot)]$, $\bar{B}_0(\cdot) = [\bar{B}_{01}(\cdot), \bar{B}_{02}(\cdot)]$, $B_j(\cdot) = [B_{j1}(\cdot), B_{j2}(\cdot)]$, $\bar{B}_j(\cdot) = [\bar{B}_{j1}(\cdot), \bar{B}_{j2}(\cdot)]$, $1 \le j \le r$.

We also define the following quadratic objective functional:

$$\begin{aligned} \mathcal{J}(T, x_t; u(\cdot)) = \frac{1}{2}\mathbb{E}_t \big\{ &\langle Gx(T), x(T)\rangle + \langle \bar{G}\mathbb{E}_t[x(T)],\ \mathbb{E}_t[x(T)]\rangle \\ + \int_t^T &\big[\langle Q(s)x(s),\ x(s)\rangle + \langle \bar{Q}(s)\mathbb{E}_t[x(s)],\ \mathbb{E}_t[x(s)]\rangle \\ &+ \langle R(s)u(s),\ u(s)\rangle + \langle \bar{R}(s)\mathbb{E}_t[u(s)],\ \mathbb{E}_t[u(s)]\rangle\big] ds \big\} \end{aligned} \tag{2}$$

where $G, \bar{G} \in \mathbb{S}^n$, $Q(\cdot), \bar{Q}(\cdot) \in L^\infty(0,\ T;\ \mathbb{S}^n)$, and $R(\cdot), \bar{R}(\cdot) \in L^\infty(0,\ T;\ \mathbb{S}^m)$ with the block representations: $R(\cdot) = \mathrm{diag}\{R_1(\cdot),\ R_2(\cdot)\}$ and $\bar{R}(\cdot) = \mathrm{diag}\{\bar{R}_1(\cdot),\ \bar{R}_2(\cdot)\}$. Obviously, for any $(t, x_t) \in \mathcal{D}$ and any $u(\cdot) \in L^2_{\mathbb{F}}(t,\ T;\ \mathbb{R}^m)$, $\mathcal{J}(t, x_t; u(\cdot))$ is well-defined. In the considered game problem, on the one hand, Player 1 controls $u_1(\cdot)$ and wants to maximize (2). On the other hand, Player 2 controls $u_2(\cdot)$ and tries to minimize (2). Therefore, (2) can be regarded as a payoff for Player 1 and a cost for Player 2.

Definition 1. *i) Let $t \in [0,\ T]$. A non-anticipative strategy for Player 1 on the time interval $[t,\ T]$ is a mapping $\alpha_1 : \mathcal{U}_2[t,\ T] \to \mathcal{U}_1[t,\ T]$ such that for any $\mathbb{F}$-stopping time $t \le \tau \le T$ and any $u_2(\cdot), \tilde{u}_2(\cdot) \in \mathcal{U}_2[t,\ T]$, with $u_2(\cdot) \equiv \tilde{u}_2(\cdot)$ on $[t,\ \tau]$, it holds that $\alpha_1[u_2(\cdot)] \equiv \alpha_1[\tilde{u}_2(\cdot)]$ on $[t,\ \tau]$. A non-anticipative strategy $\alpha_2 : \mathcal{U}_1[t,\ T] \to \mathcal{U}_2[t,\ T]$ for Player 2 on $[t,\ T]$ is defined in the same way.*

ii) Let$\varphi_2 : [0,\ T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_1} \to \mathbb{R}^{m_2}$ be a Borel measurable mapping. If for any $(t, x_t) \in \mathcal{D}$ and any $u_1(\cdot) \in \mathcal{U}_1[t,\ T]$, the MF-SDE *obtained from (1) by replacing* $u_2(s)$ *with* $\varphi_2(s, x^{u_1,\varphi_2}(s), \mathbb{E}_t[x^{u_1,\varphi_2}(s)], u_1(s), \mathbb{E}_t[u_1(s)])$ *admits a unique solution* $x^{u_1,\varphi_2}(\cdot) \equiv x^{u_1,\varphi_2}(\cdot; t, x_t)$*, and*

$$\varphi_2(\cdot, x^{u_1,\varphi_2}(\cdot), \mathbb{E}_t[x^{u_1,\varphi_2}(\cdot)], u_1(\cdot), \mathbb{E}_t[u_1(\cdot)]) \in \mathcal{U}_2[t,\ T] \tag{3}$$

then the mapping φ_2 *is called an explicit strategy law for Player 2. In this case, it is clear that, for any* $(t, x_t) \in \mathcal{D}$*, the mapping* $\alpha_2^{\varphi_2}[\cdot] \equiv \alpha_2^{\varphi_2}[\cdot; t, x_t] : \mathcal{U}_1[t, T] \to \mathcal{U}_2[t, T]$ *defined by* $u_1(\cdot) \mapsto \varphi_2(\cdot, x^{u_1,\varphi_2}(\cdot), \mathbb{E}_t[x^{u_1,\varphi_2}(\cdot)], u_1(\cdot), \mathbb{E}_t[u_1(\cdot)])$ *is a non-anticipative strategy for Player 2 on* $[t,\ T]$*. We call* $\alpha_2^{\varphi_2}[\cdot]$ *the induced strategy of* φ_2 *at* (t, x_t)*. An explicit strategy law for Player 1 and its induced strategies can be defined analogously.*

iii) Let $\psi_1 : [0,\ T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{m_1}$ *be another Borel measurable mapping. If for any* $(t, x_t) \in \mathcal{D}$ *and any* $u_2(\cdot) \in \mathcal{U}_2[t,\ T]$*, the MF-SDE obtained from (1) by replacing* $u_1(s)$ *by* $\psi_1(s, x^{\psi_1,u_2}(s), \mathbb{E}_t[x^{\psi_1,u_2}(s)])$ *admits a unique solution* $x^{\psi_1,u_2}(\cdot) \equiv x^{\psi_1,u_2}(\cdot; t, x_t)$ *and*

$$\psi_1(\cdot, x^{\psi_1,u_2}(\cdot), \mathbb{E}_t[x^{\psi_1,u_2}(\cdot)]) \in \mathcal{U}_1[t,\ T] \tag{4}$$

then the mapping ψ_1 *is called an implicit strategy law for Player1. In this case, it is clear that, for any* $(t, x_t) \in \mathcal{D}$*, the mapping* $\alpha_1^{\psi_1}[\cdot] \equiv \alpha_1^{\psi_1}[\cdot;\ t,\ x_t]$ *defined by:* $u_2(\cdot) \mapsto \psi_1(\cdot, x^{\psi_1,u_2}(\cdot), \mathbb{E}_t[x^{\psi_1,u_2}()])$ *is a non-anticipative strategy for Player 1 on* $[t,\ T]$*. We call* $\alpha_1^{\psi_1}[\cdot]$ *the induced strategy of* ψ_1 *at* $(t,\ x_t)$*. An implicit strategy law for Player 2 and its induced strategies can be defined analogously.*

iv) From the definitions above, one can characterizes in a similar way a pair of (implicit–explicit) strategy laws (ψ_1, φ_2)*. Moreover,* $u_1^{\psi_1,\varphi_2}(\cdot)$ *defined by:*

$$u_1^{\psi_1,\varphi_2}(\cdot) = \psi_1(\cdot, x^{\psi_1,\varphi_2}(\cdot), \mathbb{E}_t[x^{\psi_1,\varphi_2}(\cdot)]) \in \mathcal{U}_1[t,\ T] \tag{5}$$

and $u_2^{\psi_1,\varphi_2}(\cdot)$ *defined:*

$$\begin{aligned} u_2^{\psi_1,\varphi_2}(\cdot) = \varphi_2 \Big(&\cdot, x^{\psi_1,\varphi_2}(\cdot), \mathbb{E}_t[x^{\psi_1,\varphi_2}(\cdot)], \psi_1(\cdot, x^{\psi_1,\varphi_2}(\cdot), \\ &\mathbb{E}_t[x^{\psi_1,\varphi_2}(\cdot)]), \mathbb{E}_t[\psi_1(\cdot, x^{\psi_1,\varphi_2}(\cdot), \mathbb{E}_t[x^{\psi_1,\varphi_2}(\cdot)])]\Big) \in \mathcal{U}_2[t,\ T] \end{aligned} \tag{6}$$

are called the induced controls of (ψ_1, φ_2) *at* (t, x_t)*.*

Definition 2. *Let (ψ_1, φ_2) be a pair of strategy laws. If for any $(t, x_t) \in \mathcal{D}$:*

$$J(t, x_t; u_1^{\psi_1,\varphi_2}(\cdot), u_2^{\psi_1,\varphi_2}(\cdot)) = \underset{u_2(\cdot)\in\mathcal{U}_2[t\ T]}{essinf} J(t, x_t; \alpha_1^{\psi_1}[u_2(\cdot)], u_2(\cdot))$$
$$= \underset{u_1(\cdot)\in\mathcal{U}_1[t\ T]}{esssup} J(t, x_t; u_1(\cdot), \alpha_2^{\varphi_2}[u_1(\cdot)]) \quad (7)$$

then (ψ_1, φ_2) is called a saddle point of the game. Often (ψ_1, φ_2) is named a zero sum Nash equilibrium strategy.

The control problem (MF-LQ). Find a saddle point for the zero-sum LQ mean-field dynamic game described by (1) and (2).

In [16], the authors showed that the solution of the game **(MF-LQ)** relies on the existence of unique solutions, to adequately defined generalized Riccati equations, that verify specific sign conditions. Such equations will be referred to in the rest of the paper as **(MF-GREs)**. They proposed a solvability result in a very particular case and they also pointed out that the solvability of such equations in the general case is a rather challenging problem and remains an open problem. Note also that in the result reported in [16], the sign conditions could only be verified a posteriori.

In this paper we ambition to fill in the gapes by addressing the problem of the existence of a unique solution of **(MF-GRE)** verifying specific sign conditions in the general setting.

3 Main results

On $\mathbb{S}^n$ we consider the matrix differential equation:

$$\dot{X}(t) + \mathcal{R}(t, X(t)) = 0 \quad (8)$$

$t \in [0,\ T]$. $\mathcal{R} : \mathrm{Dom}(\mathcal{R}) \to \mathbb{S}^n$ is described by:

$$\mathcal{R}(t, X) = A_0^T(t)X + XA_0(t) + \sum_{j=1}^{r} A_j^T(t)XA_j(t) + Q(t)$$
$$- \left(XB_0(t) + \sum_{j=1}^{r} A_j^T(t)XB_j(t) + L(t) \right) \left(R(t) + \sum_{j=1}^{r} B_j^T(t)XB_j(t) \right)^{-1} \star \quad (9)$$

and:

$$\mathrm{Dom}(\mathcal{R}) = \left\{ (t, X) \in [0,\ T] \times \mathbb{S}^n | \det \left(R(t) + \sum_{j=1}^{r} B_j^T(t) X B_j(t) \right) \neq 0 \right\} \tag{10}$$

where in (9): $t \to A_j(t) : [0,\ T] \to \mathbb{R}^{n\times n}$, $t \to B_j(t) : [0,\ T] \to \mathbb{R}^{n\times m}$, $0 \le j \le r$, $t \to L(t) : [0,\ T] \to \mathbb{R}^{n\times m}$, $t \to Q(t) : [0,\ T] \to \mathbb{S}^n$, $t \to R(t) : [0,\ T] \to \mathbb{S}^m$ are continuous matrix valued functions.
We set

$$\begin{array}{l} B_j(t) = \left(\begin{array}{cc} B_{j1}(t) & B_{j2}(t) \end{array} \right), B_{jk}(t) \in \mathbb{R}^{n\times m_k}, 0 \le j \le r, \\ L(t) = \left(L_1(t) \quad L_2(t) \right), L_k(t) \in \mathbb{R}^{n\times m_k}, k = 1, 2, \\ R(t) = \left(\begin{array}{cc} R_{11}(t) & R_{12}(t) \\ R_{12}^T(t) & R_{22}(t) \end{array} \right), R_{lj}(t) \in \mathbb{R}^{m_l\times m_j}, l, j = 1, 2. \end{array} \tag{11}$$

Let $X_T(t)$ be the solution of (8) satisfying the given terminal condition $X_T(T) = G$, where $G \in \mathbb{S}^n$. In this work, we are interested by solutions $X_T(\cdot) : \mathcal{I} \subset [0,\ T] \to \mathbb{S}^n$ of (8) satisfying the following sign conditions:

$$\mathbb{R}_{22}(t, X_T(t)) := R_{22}(t) + \sum_{j=1}^{r} B_{j2}^T(t) X_T(t) B_{j2}(t) > 0 \tag{12}$$

$$\begin{aligned} \mathbb{R}_{22}^{\sharp}(t, X_T(t)) := & R_{11}(t) + \sum_{j=1}^{r} B_{j1}^T(t) X_T(t) B_{j1}(t) \\ & - \left(R_{12}(t) + \sum_{j=1}^{r} B_{j1}^T(t) X_T(t) B_{j2}(t) \right) \left(\mathbb{R}_{22}(t, X_T(t)) \right)^{-1} \\ & \times \left(R_{12}(t) + \sum_{j=1}^{r} B_{j1}^T(t) X_T(t) B_{j2}(t) \right)^T < 0 \end{aligned} \tag{13}$$

for all $t \in \mathcal{I}$.

Remark 1. *We point out here that the GRE considered in this work is more general that the one considered in [16] by allowing cross terms in the quadratic objective functional.*

The main objective of this work is to give conditions which guarantee the fact that the solution $X_T(t)$ is well defined and verifies the sign conditions

(12), (13) for any $t \in [0;\ T]$. Before doing so we need to introduce several auxiliary results.
Consider the following SDE:

$$\begin{aligned} dx(t) &= (A_0(t)x(t) + B_{01}(t)u_1(t) + B_{02}(t)u_2(t))dt \\ &+ \sum_{j=1}^{r}(A_j(t)x(t) + B_{j1}(t)u_1(t) + B_{j2}(t)u_2(t))dw(t) \end{aligned} \tag{14}$$

$x(0) = x_0$, and the cost functional:

$$\mathcal{J}(T, x_0, u_1, u_2) = \mathbb{E}\left[\langle Gx_u(T), x_u(T)\rangle + \int_0^T \langle \mathbb{Q}(t)\begin{pmatrix} x_u(t) \\ u_1(t) \\ u_2(t) \end{pmatrix}, \begin{pmatrix} x_u(t) \\ u_1(t) \\ u_2(t) \end{pmatrix}\rangle dt\right] \tag{15}$$

where $\mathbb{Q}(t) = \begin{pmatrix} Q(t) & L_1(t) & L_2(t) \\ \star & R_{11}(t) & R_{12}(t) \\ \star & \star & R_{22}(t) \end{pmatrix}$, $0 \le t \le T$. Let $K : [0;\ T] \to \mathbb{R}^{m_2 \times n}, W : [0;\ T] \to \mathbb{R}^{m_2 \times m_1}$ be continuous matrix valued functions. Setting formally $u_2(t) = K(t)x(t) + W(t)u_1(t)$ in (14) and (15), we obtain:

$$\begin{aligned} dx(t) &= (A_{0K}(t)x(t) + B_{0W}(t)u_1(t))dt + \\ &+ \sum_{j=1}^{r}(A_{jK}(t)x(t) + B_{jW}(t)u_1(t))dw_j(t), \quad x(0) = x_0 \in \mathbb{R}^n \end{aligned} \tag{16}$$

$$\begin{aligned} \mathcal{J}_{KW}(T, x_0, u_1) &= \mathbb{E}\Big[\langle Gx_{u_1}(T), x_{u_1}(T)\rangle \\ &+ \int_0^T \langle \begin{pmatrix} Q_K(t) & L_{KW}(t) \\ L_{KW}^T(t) & R_W(t) \end{pmatrix}\begin{pmatrix} x_{u_1}(t) \\ u_1(t) \end{pmatrix}, \begin{pmatrix} x_{u_1}(t) \\ u_1(t) \end{pmatrix}\rangle dt\Big] \end{aligned} \tag{17}$$

where $x_{u_1}(t)$ is the solution of the initial value problem (16) corresponding to the input $u_1(t)$ and:

$$\begin{cases} A_{jK}(t) &= A_j(t) + B_{j2}(t)K(t), \\ B_{jW}(t) &= B_{j1}(t) + B_{j2}(t)W(t), \quad 0 \le j \le r \\ Q_K(t) &= Q(t) + L_2(t)K(t) + K^T(t)L_2^T(t) + K^T(t)R_{22}(t)K(t) \\ L_{KW}(t) &= L_1(t) + K^T(t)R_{12}^T(t) + (L_2(t) + K^T(t)R_{22}(t))W(t) \\ R_W(t) &= \begin{pmatrix} I_{m_1} \\ W(t) \end{pmatrix}^T \begin{pmatrix} R_{11}(t) & R_{12}(t) \\ R_{12}^T(t) & R_{22}(t) \end{pmatrix}\begin{pmatrix} I_{m_1} \\ W(t) \end{pmatrix} \end{cases} \tag{18}$$

To the pair formed by the system (16) and the quadratic functional (17), we associate the following GRE:

$$
\begin{aligned}
-\dot{Y}(t) = {} & A_{0K}^T(t)Y(t) + Y(t)A_{0K}(t) + \sum_{j=1}^{r} A_{jK}^T(t)Y(t)A_{jK}(t) \\
& -\Big(Y(t)B_{0W}(t) + \sum_{j=1}^{r} A_{jK}^T(t)Y(t)B_{jW}(t) \\
& + L_{KW}(t)\Big)\Big(R_W(t) + \sum_{j=1}^{r} B_{jW}^T(t)Y(t)B_{jW}(t)\Big)^{-1}\Big(B_{0W}^T(t)Y(t) + \\
& + \sum_{j=1}^{r} B_{jW}^T(t)Y(t)A_{jK}(t) + L_{KW}^T(t)\Big) + Q_K(t).
\end{aligned}
\tag{19}
$$

Let us define the set Σ_{KWT} as the set of continuous matrix valued functions $K : [0;\ T] \to \mathbb{R}^{m_2 \times n}$ and $W : [0;\ T] \to \mathbb{R}^{m_2 \times m_1}$ with the property that the solution $Y_{KWT}(\cdot)$ of the corresponding GRE (19) satisfying the terminal condition $Y_{KWT}(T) = G$ is well defined on the whole interval $[0;\ T]$ and satisfies the sign condition $R_W(t) + \sum_{j=1}^{r} B_{KW}^T(t)Y_{KWT}(t)B_{KW}(t) < 0$, $\forall t \in [0;\ T]$.

Set:

$$
\mathbb{D}(t)[\tilde{Z}(t)] = \begin{pmatrix} \dot{\tilde{Z}}(t) + \mathcal{L}(t)[\tilde{Z}(t)] + Q(t) & D_{12}(t, \tilde{Z}(t)) \\ \star & D_{22}(t, \tilde{Z}(t)) \end{pmatrix} \tag{20}
$$

where $D_{12}(t, \tilde{Z}(t)) = \tilde{Z}(t)B_{02}(t) + \sum_{j=1}^{r} A_j^T(t)\tilde{Z}(t)B_{j2}(t) + L_2(t)$, $D_{22}(t, \tilde{Z}(t)) = R_{22}(t) + \sum_{j=1}^{r} B_j 2^T(t)\tilde{Z}(t)B_{j2}(t)$ and $\mathcal{L}[\cdot]$ is the Lyapunov type operator defined by:

$$
\mathcal{L}[\tilde{Z}(t)] = A_0^T(t)\tilde{Z}(t) + \tilde{Z}(t)A_0(t) + \sum_{j=1}^{r} A_j^T(t)\tilde{Z}(t)A_j(t) \tag{21}
$$

and $\tilde{Z} : [0;\ T] \to \mathbb{S}^n$ is an arbitrary differentiable matrix valued function.

We associate to the matrix valued function $\mathbb{D}(t)[\cdot]$ the following RDE on $\mathbb{S}^n$:

$$
\dot{X}(t) + \mathcal{R}_{\mathbb{D}}(t, X(t)) = 0 \tag{22}
$$

where:

$$\begin{aligned}\mathcal{R}_{\mathbb{D}}(t,X(t)) = &\ \mathcal{L}(t)[X(t)] + Q(t)\\ &- \left(X(t)B_{02}(t) + \sum_{j=1}^{r} A_j^T(t)X(t)B_{j2}(t) + L_2(t)\right)\\ &\times \left(R_{22}(t) + \sum_{j=1}^{r} B_{j2}^T(t)X(t)B_{j2}(t)\right)^{-1} \star \end{aligned} \tag{23}$$

We are now in position to state the main result of this paper.

Theorem 1. *Assume that:*

$$\begin{pmatrix} Q(t) & L_2(t) \\ \star & R_{22}(t) \end{pmatrix} \geq 0 \tag{24}$$

for $0 \leq t \leq T$. We denote $\mathcal{I}(T)$ the maximal interval $\mathcal{I}(T) \subset [0;\ T]$ where the solution $X_T(t)$ of RDE (8) satisfying $X_T(T) = G$ is well defined and satisfies the sign conditions (12)-(13). Then the following are equivalent:

i) $\mathcal{I}(T) = [0;\ T]$*;*

ii) the set Σ_{KWT} is not empty and there exists a differentiable function $Z : [0;\ T] \to \mathbb{S}^n$ satisfying:

$$\begin{cases} -\dot{Z}(t) \leq \mathcal{R}(t, Z(t)) \\ R_{22}(t) + \sum_{j=1}^{r} B_{j2}^T(t)Z(t)B_{j2}(t) > 0 \end{cases} \tag{25}$$

$t \in [0;\ T]$ *and* $G \geq Z(T)$*;*

iii) the set Σ_{KWT} is not empty and there exists a differentiable function $\tilde{Z} : [0;\ T] \to \mathbb{S}^n$ satisfying:

$$\begin{cases} \mathbb{D}(t)[\tilde{Z}(t)] \geq 0 \\ R_{22}(t) + \sum_{j=1}^{r} B_{j2}^T(t)\tilde{Z}(t)B_{j2}(t) > 0 \end{cases} \tag{26}$$

$t \in [0;\ T]$ *and* $G \geq \tilde{Z}(T)$.

Proof. i)⇒ii) If $X_T(t)$ is well defined for any $t \in [0;\ T]$, this means that $F^{X_T}(t)$ is well defined on the interval $[0;\ T]$. We set:

$$\begin{cases} K^{X_T}(t) \triangleq (\mathbb{V}_{22}^{X_T}(t))^{-1} \begin{pmatrix} \mathbb{V}_{21}^{X_T}(t) & \mathbb{V}_{22}^{X_T}(t) \end{pmatrix} F^{X_T}(t) \\ W^{X_T}(t) \triangleq -(\mathbb{V}_{22}^{X_T}(t))^{-1}\mathbb{V}_{21}^{X_T}(t) \end{cases} \tag{27}$$

then we infer that $(K^{X_T}(t),\ W^{X_T}(t)) \in \sum_{KWT}$. The second part of the implication is trivial.

ii)⇒i). First, if $(K(\cdot),\ W(\cdot)) \in \Sigma_{KWT}$ and if $Y_{KWT}(\cdot)$ is the corresponding solution of RDE (19) satisfying $Y_{KWT}(T) = G$, then, one obtains

$$X_T(t) \leq Y_{KWT}(t), \quad \forall t \in \mathcal{I}(T) \tag{28}$$

The proof of this result follows similar lines as in the proof of Lemma 2 and Lemma 1 from [6].
Let us now show that $X_T(t) \geq Z(t), \quad \forall t \in \mathcal{I}(T)$. It follows from (25) that there exists a matrix function $Q_Z(t) \geq 0$, $t \in [0;\ T]$; such that:

$$-\dot{Z}(t) = R(t, Z(t)) - Q_Z(t), \quad t \in [0;\ T] \tag{29}$$

Define: $\tilde{Q}(t) = Q(t) - Q_Z(t),\ t \in [0;\ T]$. It is clear that:

$$\begin{pmatrix} Q(t) & L(t) \\ \star & R(t) \end{pmatrix} \geq \begin{pmatrix} \tilde{Q}(t) & L(t) \\ \star & R(t) \end{pmatrix}, \quad t \in [0;\ T] \tag{30}$$

Now, by using the comparison theorem (Theorem 1) from [6], one obtaines that:

$$X_T(t) \geq Z(t), \quad \forall t \in \mathcal{I}(T) \tag{31}$$

Using (31) and (25) one see that $X_T(t)$ verifies the sign condition (12) on $\mathcal{I}(T)$. This result, in conjunction with (28) yields:

$$\begin{aligned} & R_W(t) + \sum_{j=1}^{r} B_{jW}^T(t) X_T(t) B_{jW}(t) \\ & - \left(R_{12}(t) + W^T(t) R_{22}(t) + \sum_{j=1}^{r} B_{jW}^T(t) X_T(t) B_{j2}(t) \right) \\ & \times \left(R_{22}(t) + \sum_{j=1}^{r} B_{j2}^T(t) X_T(t) B_{j2}(t) \right)^{-1} \star < 0,\ t \in \mathcal{I}(T) \end{aligned} \tag{32}$$

The left hand side of (32) is the Schur complement of the (2,2)-block of the matrix:

$$\Phi(t) = \mathcal{W}^T(t) \mathbb{R}(t, X_T(t)) \mathcal{W}(t) \tag{33}$$

where $\mathcal{W}(t) = \begin{pmatrix} I_{m_1} & 0 \\ W(t) & I_{m_2} \end{pmatrix}$
and $\mathbb{R}(t, X_T(t)) = R(t) + \sum_{j=1}^{r} B_j^T(t) X_T(t) B_j(t)$. Using (33) one can

show by direct calculation that the left hand side of (32) is the Schur complement of the (2,2)-block of the matrix $\mathbb{R}(t, X_T(t))$. This yields that $X_T(t)$ verifies the sign condition (13) on $\mathcal{I}(T)$. In addition, from (28) and (31) one obtains that $X_T(t)$ is bounded and hence it can be extended to $[0;\ T]$. This completes the proof of ii)⇒i).

iii)⇒i) Using Schur complement property, one obtains from (26) that $\hat{Q}(t) \geq 0$, $t \in [0;\ T]$, where $\hat{Q}(t) = \mathcal{R}_{\mathbb{D}}(t, \tilde{Z}(t)) + \dot{\tilde{Z}}(t)$. Hence, $\tilde{Z}(t)$ verifies the equation:

$$\dot{\tilde{Z}}(t) + \tilde{\mathcal{R}}_{\mathbb{D}}(t, \tilde{Z}(t)) = 0,\ t \in [0;\ T] \tag{34}$$

where $\tilde{\mathcal{R}}_{\mathbb{D}}(t, \tilde{Z}(t)) = \mathcal{R}_{\mathbb{D}}(t, \tilde{Z}(t)) - \hat{Q}(t)$, $t \in [0;\ T]$. By applying Lemma 5.1.1 from [4] to (34), one obtains that $\tilde{Z}(t)$ verifies the equation:

$$\begin{aligned}
&\dot{\tilde{Z}}(t) + (A_0(t) + B_{02}(t)\Gamma^{X_T}(t))^T \tilde{Z}(t) + \tilde{Z}(t)(A_0(t) + B_{02}(t)\Gamma^{X_T}(t)) \\
&- \left(\Gamma^{X_T}(t) - F^{\tilde{Z}}(t)\right)^T \left(R_{22}(t) + \sum_{j=1}^{r} B_{j2}^T(t)\tilde{Z}(t)B_{j2}(t)\right) \star \\
&+ \begin{pmatrix} I \\ \Gamma^{X_T}(t) \end{pmatrix}^T \begin{pmatrix} Q(t) - \hat{Q}(t) & L_2(t) \\ \star & R_{22}(t) \end{pmatrix} \begin{pmatrix} I \\ \Gamma^{X_T}(t) \end{pmatrix} \\
&+ \begin{pmatrix} I \\ \Gamma^{X_T}(t) \end{pmatrix}^T \\
&\times \begin{pmatrix} \sum_{j=1}^{r} A_j^T(t)\tilde{Z}(t)A_j(t) & \sum_{j=1}^{r} A_j^T(t)\tilde{Z}(t)B_{j2}(t) \\ \star & \sum_{j=1}^{r} B_{j2}^T(t)\tilde{Z}(t)B_{j2}(t) \end{pmatrix} \star = 0
\end{aligned} \tag{35}$$

$t \in \mathcal{I}(T)$, where

$$\begin{aligned}
F^{\tilde{Z}}(t) = &- \left(R_{22}(t) + \sum_{j=1}^{r} B_{j2}^T(t)\tilde{Z}(t)B_{j2}(t)\right)^{-1} \\
&\times \left(\tilde{Z}(t)B_{02}(t) + \sum_{j=1}^{r} A_j^T(t)\tilde{Z}(t)B_{j2}(t) + L_2(t)\right)^T
\end{aligned} \tag{36}$$

$$\Gamma^{X_T}(t) = (\mathbb{V}_{22}^{X_T}(t))^{-1} \left(\mathbb{V}_{21}^{X_T}(t)F_1^{X_T}(t) + \mathbb{V}_{22}^{X_T}(t)F_2^{X_T}(t)\right) \tag{37}$$

$$\begin{cases} \mathbb{V}_{21}^{X_T}(t) = (\mathbb{R}_{22}^{X_T}(t))^{\frac{-1}{2}}(\mathbb{R}_{12}^{X_T}(t))^T \\ \mathbb{V}_{22}^{X_T}(t) = (\mathbb{R}_{22}^{X_T}(t))^{\frac{1}{2}} \\ \mathbb{R}_{22}^{X_T}(t) = R_{22}(t) + \sum_{j=1}^r B_{j2}^T(t)X_T(t)B_{j2}(t) \\ \mathbb{R}_{12}^{X_T}(t) = R_{12}(t) + \sum_{j=1}^r B_{j1}^T(t)X_T(t)B_{j2}(t) \end{cases} \tag{38}$$

and

$$\begin{cases} F_1^{X_T}(t) = \begin{pmatrix} I & 0 \end{pmatrix} F^{X_T}(t) \\ F_2^{X_T}(t) = \begin{pmatrix} 0 & I \end{pmatrix} F^{X_T}(t) \\ F^{X_T}(t) = -\Big(R(t) + \sum_{j=1}^r B_j^T(t)X_T(t)B_j(t)\Big)^{-1} \\ \times \Big(X_T(t)B_0(t) + \sum_{j=1}^r A_j^T(t)X_T(t)B_j(t) + L(t)\Big)^T \end{cases} \tag{39}$$

On the other hand and using similar manipulations as in the proof of Proposition 5 from [6], one obtains the following version of (8) satisfied by $X_T(\cdot)$:

$$\begin{aligned} &\dot{X}_T(t) + (A_0(t) + B_{02}(t)\Gamma^{X_T}(t))^T X_T(t) + X_T(t)(A_0(t) + B_{02}(t)\Gamma^{X_T}(t)) \\ &+ \begin{pmatrix} I \\ \Gamma^{X_T}(t) \end{pmatrix}^T \begin{pmatrix} Q(t) & L_2(t) \\ \star & R_{22}(t) \end{pmatrix} \begin{pmatrix} I \\ \Gamma^{X_T}(t) \end{pmatrix} \\ &+ \begin{pmatrix} I \\ \Gamma^{X_T}(t) \end{pmatrix}^T \\ &\times \begin{pmatrix} \sum_{j=1}^r A_j^T(t)X_T(t)A_j(t) & \sum_{j=1}^r A_j^T(t)X_T(t)B_{j2}(t) \\ \star & \sum_{j=1}^r B_{j2}^T(t)X_T(t)B_{j2}(t) \end{pmatrix} \star \\ &+ (F_1^{X_T})^T(t)(\mathbb{V}_{11}^{X_T}(t))^2 F_1^{X_T}(t) = 0 \end{aligned} \tag{40}$$

$t \in \mathcal{I}(T)$, where

$$\begin{cases} \mathbb{V}_{11}^{X_T}(t) = [\mathbb{R}_{12}^{X_T}(t)(\mathbb{R}_{22}^{X_T}(t))^{-1}(\mathbb{R}_{12}^{X_T}(t))^T - \mathbb{R}_{11}^{X_T}(t)]^{\frac{1}{2}} \\ \mathbb{R}_{11}^{X_T}(t) = R_{11}(t) + \sum_{j=1}^r B_{j1}^T(t)X_T(t)B_{j1}(t) \end{cases} \tag{41}$$

Define $P_T(t) = X_T(t) - \tilde{Z}(t)$, $t \in \mathcal{I}(T)$, hence from (35) and (40), it follows that $P_T(\cdot)$ verifies:

$$
\begin{aligned}
&\dot{P}_T(t) + (A_0(t) + B_{02}(t)\Gamma^{X_T}(t))^T P_T(t) + P_T(t)(A_0(t) + B_{02}(t)\Gamma^{X_T}(t)) \\
&+ \begin{pmatrix} I \\ \Gamma^{X_T}(t) \end{pmatrix}^T \\
&\times \begin{pmatrix} \sum_{j=1}^r A_j^T(t)P_T(t)A_j(t) & \sum_{j=1}^r A_j^T(t)P_T(t)B_{j2}(t) \\ \star & \sum_{j=1}^r B_{j2}^T(t)P_T(t)B_{j2}(t) \end{pmatrix} \star \\
&+ \left(\Gamma^{X_T}(t) - F^{\tilde{Z}}(t)\right)^T \left(R_{22}(t) + \sum_{j=1}^r B_{j2}^T(t)\tilde{Z}(t)B_{j2}(t) \right) \star \\
&+ \begin{pmatrix} I \\ \Gamma^{X_T}(t) \end{pmatrix}^T \begin{pmatrix} \hat{Q}(t) & 0 \\ \star & 0 \end{pmatrix} \star + (F_1^{X_T})^T(t)(\mathbb{V}_{11}^{X_T}(t))^2 F_1^{X_T}(t) = 0
\end{aligned}
\tag{42}
$$

$t \in \mathcal{I}(T)$. One then deduces from the equation above that $P_T(t) \geq 0$ and hence $X_T(t) \geq \tilde{Z}(t)$, $t \in \mathcal{I}(T)$. The rest of the proof is similar to the proof of the implication ii)⇒i).

i)⇒iii) The non emptiness of $\sum_{KWT}$ has been proved in i)⇒ii). Finally, using the parametrization given in Remark 6 from [6], one can show that $X_T(t)$ satisfies (26).

□

Remark 2.

- *As we mentioned before, in [16] the authors considered a very specific case. Indeed the proposed results are valid for a very restricted class of GREs. On the other hand, there proposed solution doesn't take into account the sign condition that has to be verified by the GRE solution and could just be checked a posteriori. Our proposed existence condition incorporate a priori the sign conditions and applies to a large class of GREs.*

- *One of the most remarkable feature of our proposed existence conditions is their formulation as linear matrix inequalities (LMIs) feasibility problem (see iii) from Theorem 1). Recalling the convex nature of LMI problems, the proposed conditions offers a high advantage from the numerical point of view in addition to their theoretical value.*

4 Conclusion

In this paper, we proposed existence condition for the solution of a class of GRE arising in finite-horizon LQ zero-sum dynamic mean field games. One of the remarkable features of the proposed conditions are their generality as well as their numerical tractability. Our future efforts will be devoted to propose a closed-loop formulation for saddle points for this class of mean field game in the infinite-horizon case.

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