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LYAPUNOV FUNCTIONS FOR TRICHOTOMY WITH GROWTH RATES OF EVOLUTION OPERATORS IN BANACH SPACES*

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Dedicated to Dr. Vasile Drăgan on the occasion of his 70th anniversary

Abstract

The main objective of this paper is to give a characterization in terms of Lyapunov functions for trichotomy with growth rates of evolution operators in Banach spaces.

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1 Introduction

The concept of trichotomy, firstly arose in the work [19] of R.J. Sacker and G.R. Sell in 1976. They described trichotomy for linear differential systems

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by linear skew-product flows. Later, S. Elaydi and O. Hajek in [9] and [10] gave the notion of exponential trichotomy for differential equations and for nonlinear differential systems, respectively.

As a natural generalization of exponential dichotomy (firstly introduced by O. Perron in [18]) exponential trichotomy (see [2], [4], [16], [20] and the references therein) is one of the most complex asymptotic properties arising from the central manifold theory (see [5] and [20]). The general case of trichotomy with growth rates was studied by L. Barreira and C. Valls in [4], M. Megan, T. Ceauşu and V. Crai (Terlea) in [7] and [17], Y. Jiang, and F.-F. Liao [12]).

In the study of trichotomy the main idea is to obtain a decomposition of the space at every moment into three closed subspaces: the stable subspace, the unstable subspace and the central manifold.

In the previous studies of uniform and nonuniform trichotomies the growth rates are always assumed to be the same type functions for the stable, unstable and central manifold parts.

This paper considers a general concept of trichotomy (and as particular case the general concept of uniform trichotomy) that allows different growth rates for the stable, unstable and central manifold subspaces.

The main result of the paper is a characterization for nonuniform H-trichotomy respectively uniform H-trichotomy of evolution operators in Banach spaces. The characterizations are given in terms of Lyapunov functions.

In the study of the stability of trajectories (that begins in 1892 with the seminal work of Lyapunov [13], Lyapunov functions are very important. According to Coppel in [6], the connection between Lyapunov functions and uniform exponential dichotomies was first considered by Maĭzel' in [15]. For more recent works about characterizations of nonuniform contractions, dichotomies and trichotomies using Lyapunov functions, we refer to [1], [2], [3], [4], [21]. In 2011 L. Barreira and C. Valls [3] obtained an optimal characterization of the exponential (contraction/stability) behavior in terms of strict Lyapunov sequences. In particular, they construct explicitly strict Lyapunov sequences for each exponential contraction. In 2012 F.-F. Liao, Y. Jiang and Z. Xie in [11] extend this result to the the concept of nonuniform (μ, ν)-contraction/stability. Later, Y. Jiang and F.F. Liao in [12] gave a characterization of trichotomy with the same growth rate in the stable, unstable and central manifold parts, using strict Lyapunov functions by giving explicitly Lyapunov functions for each given trichotomy. Their work extends the result obtained by L. Barreira, J. Chu and C. Valls in [1] for nonuniform dichotomy. In [21] J. Zhang, M. Fan, H. Zhu made an exhaustive study of the concept of (h, k, μ, ν) dichotomy that includes: robustness, sufficient and necessary condition using Lyapunov functions, Hartman-Grobman theorem. Unfortunately, there is no general method to construct explicitly Lyapunov functions for a given dynamics.

We obtain characterization of the trichotomy with different growth rates, both uniform and nonuniform, in terms of Lyapunov functions. Since our concept incorporates the exponential respectively polynomial trichotomy notions, in particular, we obtain characterizations with Lyapunov functions for these concepts too.

2 Growth rates for evolution operators

Let X be a real or complex Banach space. The norms on X and on $\mathcal{B}(X)$, the Banach algebra of all linear and bounded operators on X, will be denoted by $\|\cdot\|$. We also denote by

$$\Delta = \{(t,s) \in \mathbb{R}_+^2, t \geq s \geq 0\}$$

.

Definition 1. *A map $U : \Delta \to \mathcal{B}(X)$ is called an evolution operator on X if:*

(e_1) *$U(t,t) = I$ (the identity operator on X), for every $t \geq 0$*

and

(e_2) *$U(t,t_0) = U(t,s)U(s,t_0)$, for all $(t,s),(s,t_0) \in \Delta$.*

Definition 2. *A map $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is called*

(i) a family of projectors on X if

$$P^2(t) = P(t), \text{ for every } t \geq 0;$$

(ii) invariant for the evolution operator $U : \Delta \to \mathcal{B}(X)$ if:

$$U(t,s)P(s)x = P(t)U(t,s)x,$$

for all $(t,s,x) \in \Delta \times X$;

(iii) strongly invariant for the evolution operator $U : \Delta \to \mathcal{B}(X)$ if it is invariant for U and for all $(t,s) \in \Delta$ the restriction of $U(t,s)$ on Range $P(s)$ is an isomorphism from Range $P(s)$ to Range $P(t)$.

Remark 1. *It is obvious that if P is strongly invariant for U then it is also invariant for U. The converse is not valid (see [14]).*

Remark 2. *If the family of projectors $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is strongly invariant for the evolution operator $U : \Delta \to \mathcal{B}(X)$ then ([14]) there exists a map $V : \Delta \to \mathcal{B}(X)$ with the properties:*

(v_1) *$V(t,s)$ is an isomorphism from Range $P(t)$ to Range $P(s)$,*

(v_2) $U(t,s)V(t,s)P(t)x = P(t)x$,

(v_3) $V(t,s)U(t,s)P(s)x = P(s)x$,

(v_4) $V(t,t_0)P(t) = V(s,t_0)V(t,s)P(t)$,

(v_5) $V(t,s)P(t) = P(s)V(t,s)P(t)$,

(v_6) $V(t,t)P(t) = P(t)V(t,t)P(t) = P(t)$,

for all $(t,s),(s,t_0) \in \Delta$ and $x \in X$.

Definition 3. *Let $P_1, P_2, P_3 : \mathbb{R} \to \mathcal{B}(X)$ be three families of projectors on X. We say that the family $\mathcal{P} = \{P_1, P_2, P_3\}$ is*

(i) orthogonal if

- (o_1) $P_1(t) + P_2(t) + P_3(t) = I$ *for every* $t \geq 0$ *and*
- (o_2) $P_i(t)P_j(t) = 0$ *for all* $t \geq 0$ *and all* $i, j \in \{1,2,3\}$ *with* $i \neq j$;

(ii) compatible with the evolution operator $U : \Delta \to \mathcal{B}(X)$ if

- (c_1) *P_1 is invariant for U and*
- (c_2) *P_2, P_3 are strongly invariant for U.*

If $\mathcal{P} = \{P_1, P_2, P_3\}$ is compatible with U then in what follows we shall denote by $V_j(t,s)$ the isomorphism (given by Remark 2) from Range $P_j(t)$ to Range $P_j(s)$ where $j \in \{2,3\}$.

Definition 4. *We say that a nondecreasing map $h : \mathbb{R}_+ \to [1,\infty)$ is a growth rate if*

$$\lim_{t\to\infty} h(t) = \infty.$$

As particular cases of growth rates we remark:

(r_1) *exponential rates*, i.e. $h(t) = e^{\alpha t}$ with $\alpha > 0$;

(r_2) *polynomial rates*, i.e. $h(t) = (t+1)^{\alpha}$ with $\alpha > 0$.

Let $\mathcal{P} = \{P_1, P_2, P_3\}$ be an orthogonal family of projectors which is compatible with the evolution operator $U : \Delta \to \mathcal{B}(X)$ and let $H = \{h_1, h_2, h_3, h_4\}$ be a set of growth rates.

Definition 5. *We say that the pair $(U, \mathcal{P})$ has a H-growth (and we denote H-g) if there exists a nondecreasing function $M : \mathbb{R}_+ \to [1, \infty)$ such that:*

$(h_1 g_1)$ $\quad h_1(s)\|U(t,s)P_1(s)x\| \leq M(s)h_1(t)\|x\|$

$(h_2 g_1)$ $\quad h_2(s)\|V_2(t,s)P_2(t)x\| \leq M(t)h_2(t)\|x\|$

$(h_3 g_1)$ $\quad h_3(s)\|U(t,s)P_3(s)x\| \leq M(s)h_3(t)\|x\|$

$(h_4 g_1)$ $\quad h_4(s)\|V_3(t,s)P_3(t)x\| \leq M(t)h_4(t)\|x\|$

for all $(t, s, x) \in \Delta \times X$.

Remark 3. *As particular cases of H-g we have:*

(i) the uniform-H-growth(and we denote by u-H-g) when the function M is constant;

(ii) the exponential growth (e.g.) and respectively uniform exponential growth (u.e.g) when h_1, h_2, h_3, h_4 are exponential rates;

(iii) the polynomial growth (p.g.) and respectively uniform polynomial growth (u.p.g.) when h_1, h_2, h_3, h_4 are polynomial rates;

A characterization of the H-g is given by

Proposition 1. *The pair $(U, \mathcal{P})$ has H-g if and only if there exists a nondecreasing function $M : \mathbb{R}_+ \to [1, \infty)$ such that:*

$(h_1 g_2)$ $\quad h_1(s)\|U(t,s)P_1(s)x\| \leq M(s)h_1(t)\|P_1(s)x\|$

$(h_2 g_2)$ $\quad h_2(s)\|P_2(s)x\| \leq M(t)h_2(t)\|U(t,s)P_2(s)x\|$

$(h_3 g_2)$ $\quad h_3(s)\|U(t,s)P_3(s)x\| \leq M(s)h_3(t)\|P_3(s)x\|$

$(h_4 g_2)$ $\quad h_4(s)\|P_3(s)x\| \leq M(t)h_4(t)\|U(t,s)P_3(s)x\|$

for all $(t, s, x) \in \Delta \times X$.

Proof. Necessity: The implications $(h_1 g_1) \Rightarrow (h_1 g_2)$ and $(h_3 g_1) \Rightarrow (h_3 g_2)$ result by replacing x by $P_1(s)x$ in $(h_1 g_1)$ respectively in $(h_3 g_1)$.

For $(h_2 g_1) \Rightarrow (h_2 g_2)$ and $(h_4 g_1) \Rightarrow (h_4 g_2)$ we observe that by Remark 2 and Definition 5 it follows that

$$h_j(s)\|P_j(s)x\| = h_j(s)\|V_j(t,s)U(t,s)P_j(s)x\| \leq M(t)h_j(t)\|U(t,s)P_j(s)x\|$$

for all $(t, s, x) \in \Delta \times X$ and $j \in \{2, 4\}$.

Sufficiency: We denote by

$$M_1(t) = \sup_{s\in[0,t]} M(s)(\|P_1(s)\| + \|P_2(s)\| + \|P_3(s)\|).$$

For implications $(h_i g_2) \Rightarrow (h_i g_1)$, where $i \in \{1, 3\}$ we observe that

$$h_i(s)\|U(t,s)P_i(s)x\| \leq M(s)h_i(t)\|P_i(s)x\| \leq M_1(s)h_i(t)\|x\|$$

for all $(t, s, x) \in \Delta \times X$ and $i \in \{1, 3\}$.

Similarly for $(h_j g_2) \Rightarrow (h_j g_1)$, where $j \in \{2, 4\}$ we have

$$\begin{aligned} h_j(s)\|V_j(t,s)P_j(t)x\| &= h_j(s)\|P_j(s)V_j(t,s)P_j(t)x\| \\ &\leq M(t)h_j(t)\|U(t,s)P_j(s)V_j(t,s)P_j(t)x\| \\ &\leq M_1(t)h_j(t)\|x\| \end{aligned}$$

for all $(t, s, x) \in \Delta \times X$. □

Corollary 1. *If the pair* $(U, \mathcal{P})$ *has an uniform-H-growth then there exists* $M \geq 1$ *such that*

(uHg_1) $\quad h_i(s)\|U(t,s)P_i(s)x\| \leq Mh_i(t)\|P_i(s)x\|$

(uHg_2) $\quad h_j(s)\|P_j(t)x\| \leq Mh_j(t)\|U(t,s)P_j(s)x\|$

for all $(t, s, x) \in \Delta \times X, i \in \{1, 3\}$ *where* $j \in \{2, 4\}$.

Proof. It results from the proof of the previous proposition. □

3 Trichotomy with growth rates

With the notations from the previous section we introduce the main concept studied in this paper by

Definition 6. *We say that the pair $(U, \mathcal{P})$ is H-trichotomic (and we denote H-t) if there exists a nondecreasing function $N : \mathbb{R}_+ \to [1, \infty)$ such that:*

(h_1t_1) $h_1(t)\|U(t,s)P_1(s)x\| \leq N(s)h_1(s)\|x\|$

(h_2t_1) $h_2(t)\|V_2(t,s)P_2(t)x\| \leq N(t)h_2(s)\|x\|$

(h_3t_1) $h_3(s)\|U(t,s)P_3(s)x\| \leq N(s)h_3(t)\|x\|$

(h_4t_1) $h_4(s)\|V_3(t,s)P_3(t)x\| \leq N(t)h_4(t)\|x\|$

for all $(t, s, x) \in \Delta \times X$.

In particular if the function N is constant then we obtain the *uniform-H-trichotomy* concept, denoted by u-H-t.

Remark 4. *As important particular cases of H-trichotomy we have:*

(i) (nonuniform) exponential trichotomy (e.t.) and respectively uniform exponential trichotomy (u.e.t.) when h_1, h_2, h_3, h_4 are exponential rates;

(ii) (nonuniform) polynomial trichotomy (p.t) and respectively uniform polynomial trichotomy (u.p.t.) when h_1, h_2, h_3, h_4 are polynomial rates;

(iii) (nonuniform) (h_1, h_2)-dichotomy ($(h_1, h_2) - d$) and respectively uniform (h_1, h_2)-dichotomy ($u - (h_1, h_2) - d$) for $P_3 = 0$;

(iv) (nonuniform) exponential dichotomy (e.d.) and respectively uniform exponential dichotomy (u.e.d.) when $P_3 = 0$ and h_1, h_2 are exponential rates;

(v) (nonuniform) polynomial dichotomy (p.d.) and respectively uniform polynomial dichotomy (u.p.d) for $P_3 = 0$ and h_1, h_2 are polynomial rates;

A characterization of H-trichotomy is given by

Proposition 2. *The pair $(U, \mathcal{P})$ is H-trichotomic if and only if there exists a nondecreasing function $N : \mathbb{R}_+ \to [1, \infty)$ such that:*

(h_1t_2) $h_1(t)\|U(t,s)P_1(s)x\| \leq N(s)h_1(s)\|P_1(s)x\|$

(h_2t_2) $h_2(t)\|P_2(s)x\| \leq N(t)h_2(s)\|U(t,s)P_2(s)x\|$

(h_3t_2) $h_3(s)\|U(t,s)P_3(s)x\| \leq N(s)h_3(t)\|P_3(s)x\|$

(h_4t_2) $h_4(s)\|P_3(s)x\| \leq N(t)h_4(t)\|U(t,s)P_3(s)x\|$

for all $(t,s,x) \in \Delta \times X$.

Proof. It is similar with the proof of the Proposition 1. □

Corollary 2. *If the pair* $(U,\mathcal{P})$ *has an uniform-H-trichotomy then there exists* $N \geq 1$ *such that*

(uh_1t_2) $h_1(t)\|U(t,s)P_1(s)x\| \leq Nh_1(s)\|P_1(s)x\|$

(uh_2t_2) $h_2(t)\|P_2(s)x\| \leq Nh_2(s)\|U(t,s)P_2(s)x\|$

(uh_1t_2) $h_3(s)\|U(t,s)P_3(s)x\| \leq Nh_3(t)\|P_3(s)x\|$

(uh_2t_2) $h_4(s)\|P_3(s)x\| \leq Nh_4(t)\|U(t,s)P_3(s)x\|$

for all $(t,s,x) \in \Delta \times X$.

Proof. It is similar with the proof of Corollary 1. □

4 Examples and counterexamples

In this section we consider the set $H = \{h_1, h_2, h_3, h_4\}$ and three orthogonal projectors $P_1, P_2, P_3 \in \mathcal{B}(X)$ (i.e. $P_1 + P_2 + P_3 = I, P_i^2 = P_i$ for every $i \in \{1,2,3\}$ and $P_iP_j = 0$ for $i \neq j$). The connections between the concepts of trichotomy and growths defined in the previous sections are given in the following diagram

$$\begin{array}{ccc} u.-H-t & \Rightarrow & H-t \\ \Downarrow & & \Downarrow \\ u.-H-g & \Rightarrow & H-g. \end{array}$$

The aim of this section is to show that the converse implications are not valid.

Example 1 (*A pair* $(U,\mathcal{P})$ *which has an uniform-H-growth and is not H-trichotomic*)**.** *Let* $U : \Delta \to \mathcal{B}(X)$ *be the evolution operator defined by*

$$U(t,s) = \frac{h_1(t)}{h_1(s)}P_1 + \frac{h_2(s)}{h_2(t)}P_2 + \frac{h_3(t)}{h_3(s)}\frac{h_4(s)}{h_4(t)}P_3 \tag{1}$$

for all $(t,s) \in \Delta$.

Then $\mathcal{P} = \{P_1, P_2, P_3\}$ *is compatible with* U *and the pair* $(U,\mathcal{P})$ *has the properties:*

$$(uh_1g_1)\ \ h_1(s)\|U(t,s)P_1(s)x\| = h_1(t)\|P_1(s)x\| \le h_1(t)\|P_1\|\|x\| \le Mh_1(t)\|x\|$$
$$(uh_2g_1)\ \ h_2(s)\|V_2(t,s)P_2(t)x\| = h_2(t)\|P_2(s)x\| \le h_2(t)\|P_2\|\|x\| \le Mh_2(t)\|x\|$$
$$(uh_3g_1)\ \ h_3(s)U(t,s)P_3(s)x\| = \frac{h_3(t)h_4(s)}{h_4(t)}\|P_3(s)x\| \le Mh_3(t)\|x\|$$
$$(uh_4g_1)\ \ h_4(s)V_3(t,s)P_3(t)x\| = \frac{h_4(t)h_3(s)}{h_3(t)}\|P_3(s)x\| \le Mh_4(t)\|x\|$$

for all $(t,s,x) \in \Delta \times X$, *where*

$$M = \|P_1\| + \|P_2\| + \|P_3\|.$$

In consequence the pair $(U,\mathcal{P})$ *has a u-H-g and hence H-g .*

The pair is not H-t, because if we assume on the contrary then by (h_1t_1) *there exists a function* $N : \mathbb{R}_+ \to [1,\infty)$ *such that*

$$\frac{h_2^2(t)}{h_2(s)}\|P_2x\| = h_2(t)\|V_2(t,s)P_2x\| \le N(s)h_2(s)\|P_2x\|$$

which implies

$$h_2^2(t) \le N(s)h_2^2(s),\ \textit{for all}\ (t,s) \in \Delta.$$

This inequality is not valid (it is sufficient to consider $s = 0$ *and* $t \to \infty$*). Hence,* $(U,\mathcal{P})$ *is not H-t and in consequence is not u-H-t.*

In conclusion this example shows that in general the implications

$$u - H - g \Rightarrow u - H - t$$

and

$$H - g \Rightarrow H - t$$

are not valid.

Example 2 (*A pair* $(U,\mathcal{P})$ *which is H-trichotomic and has not an uniform H-growth*)**.** *Let* $U : \Delta \to \mathcal{B}(X)$ *be the evolution operator defined by*

$$U(t,s) = \frac{h_1^2(s)}{h_1^2(t)}P_1 + \frac{h_2^2(s)}{h_2^2(t)}P_2 + \frac{h_2^3(s)h_3(t)}{h_2^3(t)h_3(s)}\frac{h_4(s)}{h_4(t)}P_3 \qquad (2)$$

for all $(t,s) \in \Delta$.

We observe that if denote

$$N(t) = (\|P_1\| + \|P_2\| + \|P_3\|)h_2^3(t)$$

then

$$(h_1t_1)\ \ h_1(t)\|U(t,s)P_1(s)x\| = \frac{h_1^2(s)}{h_1(t)}\|P_1(s)x\| \leq h_1(s)\|P_1\|\|x\| \\ \leq N(s)h_1(s)\|x\|,$$

$$(h_2t_1)\ \ h_2(t)\|V_2(t,s)P_2(s)x\| = \frac{h_2^3(t)}{h_2^2(s)}\|P_2(s)x\| \leq N(t)h_2(s)\|x\|,$$

$$(h_3t_1)\ \ h_3(s)\|U(t,s)P_3(s)x\| = h_3(t)\frac{h_2^3(s)h_4(s)}{h_2^3(t)h_4(t)}\|P_3(s)x\| \leq N(s)h_3(t)\|x\|,$$

$$(h_4t_1)\ \ h_4(s)\|V_3(t,s)P_3(s)x\| = h_4(t)\frac{h_3(s)h_2^3(t)}{h_3(t)h_2^3(s)}\|P_3(s)x\| \leq N(t)h_4(t)\|x\|,$$

for all $(t,s,x) \in \Delta \times X$.

Finally it follows that the pair $(U,\mathcal{P})$ *is H-trichotomic.*

If we assume that $(U,\mathcal{P})$ *has an uniform H-growth then by* (uh_2g_1) *it results that there exits* $M \geq 1$ *such that*

$$\frac{h_2^2(t)}{h_2(s)}\|P_2x\| = h_2(s)\|V_2(t,s)P_2x\| \leq Mh_2(t)\|x\|, \text{ for all } (t,s) \in \Delta.$$

Thus we obtain

$$h_2^2(t)\|P_2\| \leq Mh_2^2(s), \text{ for all } (t,s) \in \Delta.$$

which is impossible for $s = 0$ *and* $t \to \infty$.

We conclude that $(U,\mathcal{P})$ *has not H-growth.*

Furthermore, it follows that the implications

$$H - t \Rightarrow u - H - t$$
$$H - g \Rightarrow u - H - g$$
$$H - t \Rightarrow u - H - g$$

are not valid.

5 The main results

In this section we give characterizations of H-trichotomy and u-H-trichotomy in terms of a certain type of Lyapunov functions. Firstly we introduce

Definition 7. *A function* $L : \mathbb{R}_+ \times X \to \mathbb{R}_+$ *is called a H-Lyapunov function for the pair* $(U, \mathcal{P})$ *if there exists a nondecreasing map* $T : \mathbb{R}_+ \to [1, \infty)$ *such that*

(L_0) $\|x\| \leq L(t, x) \leq T(t)\|x\|$;

(h_1L) $h_1(t)L(t, U(t, s)P_1(s)x) \leq T(s)h_1(s)L(s, x)$;

(h_2L) $h_2(t)L(s, V_2(t, s)P_2(t)x) \leq T(t)h_2(s)L(t, x)$;

(h_3L) $h_3(s)L(t, U(t, s)P_3(s)x) \leq T(s)h_3(t)L(s, x)$;

(h_4L) $h_4(s)L(s, V_3(t, s)P_3(t)x) \leq T(t)h_4(s)L(t, x)$,

for all $(t, x) \in \mathbb{R}_+ \times X$.

Remark 5. *As particular cases we have:*

- *If* T *is a constant then we have the concept of uniform-H-Lyapunov function.*
- *If the growth rates are of exponential type then the function is called exponential Lyapunov.*
- *If the growth rates are of polynomial type then the function is called polynomial Lyapunov.*

We begin with

Lemma 1. *If the pair* $(U, \mathcal{P})$ *has a H-growth then the function* $L : \mathbb{R}_+ \times X \to \mathbb{R}_+$ *defined by*

$$\begin{aligned} L(t,x) = &\sup_{\tau \geq t} \frac{h_1(t)}{h_1(\tau)} \|U(\tau,t)P_1(t)x\| + \sup_{r \leq t} \frac{h_2(r)}{h_2(t)} \|V_2(t,r)P_2(t)x\| \\ &+ \sup_{\tau \geq t} \frac{h_3(t)}{h_3(\tau)} \|U(\tau,t)P_3(t)x\| + \sup_{r \leq t} \frac{h_4(r)}{h_4(t)} \|V_3(t,r)P_3(t)x\| \end{aligned} \tag{3}$$

satisfies the condition (L_0).

Proof. If $(U, \mathcal{P})$ has a H-growth then by Definition 5 it follows that there is a nondecreasing function $M : \mathbb{R}_+ \to [1, \infty)$ such that

$$L(t, x) \leq T(t)\|x\|,$$

where $T(t) = 4M(t)$, for all $(t, x) \in \mathbb{R}_+ \times X$. On the other hand from the definition of L it results that

$$L(t, x) \geq \|P_1(t)x\| + \|P_2(t)x\| + \|P_3(t)x\| \geq \|P_1(t)x + P_2(t)x + P_3(t)x\| = \|x\|,$$

for all $(t, x) \in \mathbb{R}_+ \times X$. □

The main result is the characterization of the concept using a Lyapunov function and is given in the following theorem.

Theorem 1. *The pair $(U, \mathcal{P})$ is H-trichotomic if and only if there exists a H-Lyapunov function $L : \mathbb{R}_+ \times X \to \mathbb{R}_+$ for $(U, \mathcal{P})$.*

*Proof. Necessity:*If the pair $(U, \mathcal{P})$ is H-trichotomic then it has a H-growth and by the previous lemma we have the function $L : \mathbb{R}_+ \times X \to \mathbb{R}_+$ given by (3) that satisfies (L_0).

In what follows we shall denote $T(t) = 2N(t)$, where the function N is given by Definition 6.

(h_1L) We observe that by the definition of L and the condition (h_1t_1) it results

$$h_1(t)L(t, U(t, s)P_1(s)x) = h_1(t) \sup_{\tau \geq t} \frac{h_1(t)}{h_1(\tau)} \|U(\tau, t)P_1(t)U(t, s)P_1(s)x\|$$
$$= \sup_{\tau \geq t} \frac{h_1^2(t)}{h_1(\tau)} \|U(\tau, s)P_1(s)x\| \leq N(s) \frac{h_1(t)h_1(s)}{h_1(\tau)} \|x\| \leq N(s)h_1(s)\|x\|$$
$$\leq T(s)h_1(s)L(t, x).$$

(h_2L) Similarly, the inequality (h_2t_1) implies that

$$h_2(t)L(s, V_2(t, s)P_2(t)x) = h_2(t) \sup_{r \leq s} \frac{h_2(r)}{h_2(s)} \|V_2(s, r)P_2(s)V_2(t, s)P_2(t)x\|$$
$$= h_2(t) \sup_{r \leq s} \frac{h_2(r)}{h_2(s)} \|V_2(t, r)P_2(t)x\| \leq N(t)h_2(s)\|x\|$$
$$\leq T(t)h_2(s)L(t, x),$$

for all $(t, s, x) \in \Delta \times X$.

(h_3L) Firstly, we observe that from (h_3t_1) we obtain

$$h_3(s)L(t,U(t,s)P_3(s)x) = h_3(s)(\sup_{\tau\geq t}\frac{h_3(t)}{h_3(\tau)}\|U(\tau,t)P_3(t)U(t,s)P_3(s)x\|$$
$$+\sup_{r\leq t}\frac{h_4(r)}{h_4(t)}\|V_3(t,r)P_3(t)U(t,s)P_3(s)x\|)$$
$$= h_3(s)(\sup_{\tau\geq t}\frac{h_3(t)}{h_3(\tau)}\|U(\tau,s)P_3(s)x\|$$
$$+\sup_{r\leq t}\frac{h_4(r)}{h_4(t)}\|V_3(t,r)P_3(t)U(t,s)P_3(s)x\|)$$
$$\leq N(s)h_3(t)L(s,x) + h_3(s)\sup_{r\leq t}\frac{h_4(r)}{h_4(t)}\|V_3(t,r)P_3(t)U(t,s)P_3(s)x\|, \tag{4}$$

for all $(t,s,x)\in\Delta\times X$.

Case I. If $s\leq r\leq t\leq\tau$ then by (h_3t_1) we have that

$$h_3(s)\sup_{s\leq r\leq t}\frac{h_4(r)}{h_4(t)}\|V_3(t,r)U(t,s)P_3(s)x\|$$
$$= h_3(s)\sup_{s\leq r\leq t}\frac{h_4(r)}{h_4(t)}\|V_3(t,r)U(t,r)U(r,s)P_3(s)x\|$$
$$\leq N(s)h_3(s)\sup_{s\leq r\leq t}h_3(s)\frac{h_4(r)}{h_4(t)}\frac{h_3(r)}{h_3(s)}\|x\|$$
$$\leq N(s)h_3(t)L(s,x), \tag{5}$$

for all $(t,s,x)\in\Delta\times X$.

Case II. Similarly, if $r\leq s\leq t\leq\tau$ then by (h_3t_1) we obtain

$$h_3(s)\sup_{r\leq t}\frac{h_4(r)}{h_4(t)}\|V_3(t,r)U(t,s)P_3(s)x\|$$
$$= h_3(s)\sup_{r\leq t}\frac{h_4(r)}{h_4(t)}\|V_3(s,r)V_3(t,s)P_3(t)U(t,s)P_3(s)x\|$$
$$= h_3(s)\sup_{r\leq t}\frac{h_4(r)}{h_4(t)}\|V_3(s,r)P_3(s)x\|$$
$$\leq N(s)h_3(s)\sup_{r\leq t}\frac{h_4(r)}{h_4(t)}\frac{h_4(s)}{h_4(r)}\|x\|$$
$$\leq N(s)h_3(t)L(s,x), \tag{6}$$

for all $(t,s,x)\in\delta\times X$.

The inequalities (4), (5) and (6) show that (h_4L) is satisfied for $T(t) = 2N(t)$.

(h₄L) Using Remark 2 and the inequality (h_4t_1) it follows that

$$\begin{aligned}
&h_4(s)L(s, V_3(t,s)P_3(t)x) = h_4(s)L(s, P_3(s)V_3(t,s)P_3(t)x)\\
&= h_4(s)(\sup_{\tau\geq s} \frac{h_3(s)}{h_3(\tau)}\|U(\tau,s)P_3(s)V_3(t,s)P_3(t)x\|\\
&+ \sup_{r\leq s} \frac{h_4(r)}{h_4(s)}\|V_3(s,r)P_3(s)V_3(t,s)P_3(t)x\|)\\
&= h_4(s)\sup_{\tau\geq s} \frac{h_3(s)}{h_3(\tau)}\|U(\tau,s)V_3(t,s)P_3(t)x\|\\
&+ h_4(s)\sup_{r\leq s} \frac{h_4(r)}{h_4(s)}\|V_3(t,r)P_3(t)x\|\\
&\leq h_4(s)\sup_{\tau\geq s} \frac{h_3(s)}{h_3(\tau)}\|U(\tau,s)V_3(t,s)P_3(t)x\|\\
&+ N(t)h_4(s)\sup_{r\leq s} \frac{h_4(r)}{h_4(s)}\frac{h_4(t)}{h_4(r)}\|x\|\\
&\leq h_4(s)\sup_{\tau\geq s} \frac{h_3(s)}{h_3(\tau)}\|U(\tau,s)V_3(t,s)P_3(t)x\| + N(t)h_4(t)L(t,x), \qquad (7)
\end{aligned}$$

for all $(t,s,x) \in \Delta \times X$.

Case I. If $\tau \geq t \geq s$ then by (e_2), Remark 2 and (h_3t_4) we obtain that

$$\begin{aligned}
&h_4(s)\sup_{\tau\geq s} \frac{h_3(s)}{h_3(\tau)}\|U(\tau,s)V_3(t,s)P_3(t)x\|\\
&= h_4(s)\sup_{\tau\geq s} \frac{h_3(s)}{h_3(\tau)}\|U(\tau,t)P_3(t)x\|\\
&\leq N(t)h_4(s)\sup_{r\geq s} \frac{h_3(s)}{h_3(\tau)}\frac{h_3(\tau)}{h_3(t)}\|x\| \leq N(t)h_4(t)\|x\|\\
&\leq N(t)h_4(t)L(t,x), \qquad (8)
\end{aligned}$$

for all $(t,s,x) \in \Delta \times X$.

Case II. Similarly, if $t \geq \tau \geq s$ then by Remark 2 and (h_4t_1) it result that

$$\begin{aligned}
&h_4(s)\sup_{t\geq\tau\geq s}\frac{h_3(s)}{h_3(\tau)}\|U(\tau,s)V_3(t,s)P_3(t)x\| \\
&= h_4(s)\sup_{t\geq\tau\geq s}\frac{h_3(s)}{h_3(\tau)}\|U(\tau,s)V_3(\tau,s)P_3(\tau)V_3(t,\tau)P_3(t)x\| \\
&= h_4(s)\sup_{t\geq\tau\geq s}\frac{h_3(s)}{h_3(\tau)}\|V_3(t,\tau)P_3(t)x\| \\
&\leq N(t)h_4(s)\sup_{t\geq\tau\geq s}\frac{h_3(s)}{h_3(\tau)}\frac{h_4(t)}{h_4(\tau)}\|x\| \\
&\leq N(t)h_4(t)L(t,x), \qquad (9)
\end{aligned}$$

for all $(t,s,x)\in\Delta\times X$.

The inequalities (7), (8) and (9) show that (h_4L) is satisfied.

Sufficiency: Using Definition 7 we obtain that

$$\begin{aligned}
h_1(t)\|U(t,s)P_1(s)x\| &\leq h_1(t)L(t,U(t,s)P_1(s)x) \\
&\leq T(s)h_1(s)L(s,x)\leq T^2(s)h_1(s)\|x\|; \\
h_2(t)\|V_2(t,s)P_2(t)x\| &\leq h_2(t)L(s,V_2(t,s)P_2(t)x) \\
&\leq T(t)h_2(s)L(t,x)\leq T^2(t)h_2(s)\|x\|; \\
h_3(s)\|U(t,s)P_3(s)x\| &\leq h_3(s)L(t,U(t,s)P_3(s)x) \\
&\leq T(s)h_3(t)L(s,t)\leq T^2(s)h_3(t)\|x\|; \\
h_4(s)\|V_3(t,s)P_4(t)x\| &\leq h_4(s)L(t,V_3(t,s)P_4(t)x) \\
&\leq T(t)h_4(t)L(t,x)\leq T^2(t)h_4(t)\|x\|,
\end{aligned}$$

for all $(t,s,x)\in\Delta\times X$.

Finally, by Definition 6 it results that $(U,\mathcal{P})$ is H-trichotomic. □

As a particular case, we obtain a characterization of uniform trichotomy with different growth rates with Lyapunov functions.

Corollary 3. *The pair $(U,\mathcal{P})$ is uniformly-H-trichotomic if and only if there exists an uniform H-Lyapunov function for $(U,\mathcal{P})$.*

Proof. It results from the proof of Theorem 1 for the particular case when the functions M, N and T are constant. □

Since the concept of trichotomy with different growth rates incorporates the exponential and polynomial notions, further we will give characterizations of these using Lyapunov functions.

Corollary 4. *The pair* $(U, \mathcal{P})$ *is exponential trichotomic if and only if there exists an exponential Lyapunov function for* $(U, \mathcal{P})$.

Corollary 5. *The pair* $(U, \mathcal{P})$ *is polynomial trichotomic if and only if there exists a polynomial Lyapunov function for* $(U, \mathcal{P})$.

Remark 6. *In the particular case when* $P_3 = 0$ *we obtained a characterization in [8] of the concept of* (h_1, h_2)*-dichotomy using Lyapunov type norms.*

References

[1] L. Barreira, J. Chu, C. Valls, *Lyapunov Functions for General Nonuniform Dichotomies*, Milan J. Math. Vol. **81** (2013), 153-169.

[2] L. Barreira, D. Dragičević, C. Valls, *Characterization of nonuniform exponential trichotomies for flows*, J. Math.Anal.Appl. **434** (2016), 376–400.

[3] L. Barreira, C. Valls, *Lyapunov functions versus exponential contractions*, Mathematische Zeitschrift, June 2011, Volume **268**, Issue 12, 187-196.

[4] L. Barreira, C. Valls, *Lyapunov functions for trichotomies with growth rates*, J.Math.Anal.Appl **2 48** (2010), 151–183.

[5] J. Carr, *Applications of Central Manifolds Theory*, Appl. Math. Sci., Springer, vol **35**(1981).

[6] W. A. Coppel, *Dichotomy in Stability Theory*, Lecture Notes in Mathematics, Vol. 629, Springer-Verlag, New York/Berlin, 1978.

[7] V. Crai, *On the robustness of some general concepts of dichotomy and trichotomy for evolution operators in Banach spaces*, Acta Sci. Math. (Szeged), **83** (2017), 487-509.

[8] V. Crai, *Lyapunov type norms for strong (h,k)- dichotomy*, Buletinul Stiitific al Universitii Politehnica Timioara, Romnia Seria Matematică-Fizică, Volume **62** (76), Issue 1, (2017), 5–12.

[9] S. Elaydi, O. Hajek, *Exponential trichotomy of differential systems*, J. Math. Anal. Appl. **129**(1988), 362–374.

[10] S. Elaydi, O. Hajek, *Exponential dichotomy and exponential trichotomy of nonlinear differential equation*, Differential and Integral Equations, vol **3** (1990), 1201–1224.

[11] F.-F. Liao, Y. Jiang and Z. Xie, *A Generalized Nonuniform Contraction and Lyapunov Function*, Appl. Anal., Vol. 2012 (2012), Article ID 613038, 14 pp.

[12] Y. Jiang, F.-F. Liao, *Lyapunov functions for general nonuniform trichotomy with different growth rates*, Electronic Journal of Differential Equations, Vol. 2014 (2014), No. **244**, 1–17.

[13] A. Lyapunov, *The General Problem of the Stability of Motion*, Taylor and Francis, 1992.

[14] N. Lupa, M. Megan, *Exponential dichotomies of evolution operators in Banach Spaces*, Monatshefte fur Mathematik **174 (2)**(2014), 265–284.

[15] A. Maǐzel', *On stability of solutions of systems of differential equations*, Ural. Politehn. Inst. Trudy **51** (1954), 20–50.

[16] M. Megan, C. Stoica, *On uniform exponential trichotomy of evolution operators in Banach spaces*, Integr.Equ.Operator Theory **60** (2008), 499-506.

[17] M. Megan, T. Ceausu, V. Crai, *On* (h, k, μ, ν)*-trichotomy in Banach spaces*, Stud. Univ. Babes-Bolyai Math. **64** (2019), No. 3, 387–398.

[18] O. Perron, *Die Stabilitatsfrage bei Differentialgleichungen*, Math. Z., **32** (1930), 703–728.

[19] R. J. Sacker, G. R. Sell, *Existence of dichotomies and invariant splittings for linear differential systems*, Journal of Differential Equations **22** (1976), 497–522.

[20] B. Sasu, A.L. Sasu, *Nonlinear criteria for existence of the exponential trichotomy in infinite dimensional spaces*, Nonlinear Analysis Theory, Methods and Applications, vol **74**(2011), 5097–5110.

[21] J. Zhang, M. Fan, H. Zhu, *Nonuniform* (h, k, μ, ν)*-dichotomy with applications to nonautonomous dynamical systems*, . Math. Anal. Appl. **452** (2017,) 505–551.