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MEAN SQUARE ASYMPTOTIC STABILITY OF DISCRETE-TIME LINEAR FRACTIONAL ORDER SYSTEMS*

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Dedicated to Dr. Vasile Drăgan on the occasion of his 70^{th} anniversary

Abstract

This paper considers stability problems for discrete-time linear fractional -order systems (LFOSs) with Markovian jumps and/ or multiplicative noise. For the case of LFOSs with finite delays and Markovian jumps, we provide sufficient conditions for the mean-square asymptotic (MSA) stability or instability of the system by using Lyapunov type equations. In the absence of the Markovian perturbations, we use Z-transform and operator spectral properties to derive instability criteria for fractional-order systems with multiplicative random perturbations and either finite or infinite delays. Four numerical results accompanied by computer simulations illustrate the effectiveness of the theoretical results.
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1 Introduction

Although fractional calculus (FC) has a long history, only recently it has attracted high scientific attention due to its new and unexpected applications in engineering and in other fields of science and technology such as electrochemistry, biophysics, quantum mechanics, radiation physics, control theory and so forth (see [11],[8], [22], [18],[10] and the references therein for several examples).

Among the hottest topics in the field we find stability problems for differential/difference systems of fractional order. Even in the deterministic case, the most general stability results concerning fractional order systems provide only sufficient conditions for the asymptotic stability/ or instability of solutions [1],[3], [2], [4], [6]. Some necessary and sufficient conditions for the asymptotic stability of the deterministic LFOSs are obtained in [13] under very restrictive conditions.

Many of these results concern autonomous systems and are based on either LMI (linear matrix inequality) techniques [16] or Laplace transform theory ([12], [15]).

Since many real-world phenomena are influenced by random factors, the study of LFOSs with random perturbations becomes an important and useful issue. Moreover, the class of LFOSs with Markovian jumps seems to be the most suitable mathematical model for many physical processes that suffer abrupt and unpredictable changes in their behavior.

Motivated by these considerations, in this paper we consider a class of time-invariant finite-dimensional LFOSs affected simultaneous by multiplicative noise and Markovian jumps and we study the MSA properties of their solutions. As far as we know, this subject is new in the case of stochastic systems with Markovian jumps. By using a technique based on Lyapunov type equations (which is closed related to the LMI theory) and an idea from [17], we derive (separately) sufficient and necessary conditions for the mean-square asymptotic (MSA) stability of LFOSs with finite delays. These conditions could be extended to the general case of LFOS's with infinite delays, but their verification becomes very complicated in infinite dimensions and we hope to refine them in a future work.

Similar results concerning MSA stability were obtained before in [17] for one-dimensional LFOSs with finite delays and multiplicative random perturbations. Recently, the so called "region stability" properties of these LFOSs are studied in [9] in a multidimensional framework. The results from [9] are mainly based on operator spectrum and LMIs techniques and provide deterministic characterizations of various types of region stability of solutions. Some of these "region stability" properties imply MSA stability and, consequently, certain stability criteria from [9] ensure the MSA stability of these stochastic LFOSs.

In our case, the MSA stability problem is solved by using an expanded-state model of the LFOS and a technique based on the mean-square representation of solutions (see for e.g [19]). The MSA stability and instability of the system is then discussed in terms of solvability of certain associated Lyapunov-type equations.

In the absence of the Markov jumps, we study the MSA instability of the LFOS with multiplicative noise by appealing to operator spectral properties and to Z transform theory. Although we treat separately the cases of LFOSs with finite and infinite delays, the approach is the same. We take the expectation in the stochastic LFOS and we get a deterministic system, which asymptotic properties are studied by using the results from [12] and Z transform theory. The obtained instability criteria are closely related to the spectral properties of certain coefficient operators. The theoretical results are, finally, illustrated by four numerical examples and by Matlab simulations.

2 Preliminaries

Operators

The LFOSs discussed in the sequel are defined on the real Hilbert space $\mathbf{R}^d, d \in \mathbf{N} - \{0\}$. As usual, we shall write $\langle ., . \rangle$ for the inner product and $\|.\|$ for norms of elements and operators, unless indicated otherwise. We denote by $L\left(\mathbf{R}^d\right)$ the real linear space of all linear operators from $\mathbf{R}^d$ to $\mathbf{R}^d$. For any $T \in L\left(\mathbf{R}^d\right)$ we denote by T^* the adjoint operator of T.

The subspace of $L\left(\mathbf{R}^d\right)$ formed by all self adjoint operators will be denoted by $S\left(\mathbf{R}^d\right)$. An operator $A \in S(\mathbf{R}^d)$ is called *non-negative* and we write $A \geq 0$, if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathbf{R}^d$. If $A \geq 0$ is also invertible, we shall write $A > 0$. Throughout this paper I_d will be the identity operator on $\mathbf{R}^d$. Let $\mathcal{Z} = \{1, 2, .., D\}$, where $D \in \mathbf{N} - \{0\}$ is fixed, and let $l^{\mathcal{Z}}_{L\left(\mathbf{R}^d\right)} = \{g = \{g_i \in L\left(\mathbf{R}^d\right)\}_{i \in \mathcal{Z}}\}$. It is well known (see [5]) that $l^{\mathcal{Z}}_{L\left(\mathbf{R}^d\right)}$ is a real Hilbert space with the usual term-wise addition, the (real) scalar multiplication and the inner product $\langle g, h \rangle = \sum\limits_{i \in \mathcal{Z}} Tr\left(h^*\left(i\right) g\left(i\right)\right)$. For any $A \in l^{\mathcal{Z}}_{L\left(\mathbf{R}^d\right)}, B \in l^{\mathcal{Z}}_{L\left(\mathbf{R}^d\right)}$, the product AB is defined by $\left(AB\right)\left(i\right) = A\left(i\right) B\left(i\right), i \in \mathcal{Z}$ and $AB \in l^{\mathcal{Z}}_{L\left(\mathbf{R}^d\right)}$. Also $A^{[*]}$ denotes the element of $l^{\mathcal{Z}}_{L\left(\mathbf{R}^d\right)}$ defined by $A^{[*]}(i) = A(i)^*, i \in \mathcal{Z}$.

An element $X \in l^{\mathcal{Z}}_{L(\mathbf{R}^d)}$ is said to be *non-negative* (we write $X \succeq 0$) iff $X(i) \geq 0$ for all $i \in \mathcal{Z}$. An element $\{X_i\}_{i\in\mathbf{N}} \in l^{\mathcal{Z}}_{L(\mathbf{R}^d)}$ is called *positive* (we write $X \succ 0$) if and only if there is $\gamma > 0$ such that $X(i) - \gamma I_d \geq 0$ for all $i \in \mathcal{Z}$. A linear and bounded operator $\Gamma \in L\left(l^{\mathcal{Z}}_{L(\mathbf{R}^d)}\right)$ is said to be *positive* if $\Gamma(X) \succeq 0$ for all $X \succeq 0$.

Note that, in the rest of the paper, we do not distinguish between the matrix and the linear operator defined by it.

Z transform

Let us recall that the transform Z of a sequence $y_n \in \mathbf{R}^d, n \in \mathbf{N}$ is a function $Z(y_n)$ defined by

$$Z(y_n)(z) = \sum_{n=0}^{\infty} y_n z^{-n},$$

for all $z \in \mathbf{C}$ for which the series is convergent. The domain of definition of $Z(y_n)$ is called the region of convergence (ROC) of $Z(y_n)$. Abel theorem for power series ensures that the ROC of $Z(y_n)$ is either the wide set or the outside of a circle (see [14] for e.g.). Sometimes, if no confusion is possible, we will use the notation $Z(y)(z)$ or $Y(z)$ for $Z(y_n)(z)$.

Random variables

Let $(\Omega, \mathcal{F}, P)$ be a probability space. If ξ is an integrable random variable on $(\Omega, \mathcal{F}, P)$, we write $E[\xi]$ for its mean (expectation). If η is another integrable random variable and $\mathcal{F}_1 \subset \mathcal{F}$ is a $\sigma-$ algebra, we denote by $E[\xi|\eta = x]$ and $E[\xi|_{\mathcal{F}_1}]$ the conditional expectations on the event $\eta = x$ and on $\mathcal{F}_1$, respectively. We will denote by $L^2(\Omega, \mathbf{R}^d)$, the Hilbert space of all $\mathbf{R}^d$ valued random variables ξ with the property that

$$\|\xi\|_2 =^{def} \sqrt{E\left[\|\xi\|^2\right]} < \infty.$$

3 Linear discrete-time fractional order systems with infinite time delays

Let $\mathbf{R}^*_+ = \{x \in \mathbf{R}, x > 0\}$, $\alpha \in \mathbf{R}^*_+, \alpha < 2$ be fixed and let $\left(\begin{array}{c}\alpha\\ j\end{array}\right)$, $j \in \mathbf{N}$ denote the generalized binomial coefficient

$$\left(\begin{array}{c}\alpha\\ j\end{array}\right) := \begin{cases} 1, j = 0 \\ \frac{\alpha(\alpha-1)\cdot\ldots\cdot(\alpha+1-j)}{j!}, j > 0 \end{cases}$$

The fractional systems studied in the sequel are defined by the following fractional-order Grünwald–Letnikov operator:

$$\Delta^{[\alpha]}x_k = \frac{1}{h^\alpha}\sum_{j=0}^{k}(-1)^j\left(\begin{array}{c}\alpha\\ j\end{array}\right)x_{k-j},$$

where $h \in \mathbf{R}_+^*$ is the sampling period or time increment.

Throughout this paper we assume that the following hypothesis is satisfied.

(H1) *(i)* $A, B \in l^{\mathcal{Z}}_{L(\mathbf{R}^d)}$

(ii) $\{r_n\}_{n\in\mathbf{N}}$ is an homogeneous Markov chain on $(\Omega, \mathcal{F}, P)$ with the finite state space $\mathcal{Z}$ =$\{1, 2, .., D\}, D \in \mathbf{N} - \{0\}$, the transition probability matrix

$$Q = \{q_{i,j} := P\left(r_{n+1} = j|_{r_n=i}\right)\}_{(i,j)\in\mathcal{Z}\times\mathcal{Z}}, n \in \mathbf{N}$$

and $P\left(r_k = i\right) > 0, k \in \mathbf{N}, i \in \mathcal{Z}$.

(iii) $\{\xi_k\}_{k\in\mathbf{N}}$ is a sequence of real-valued, mutually independent random variables on $(\Omega, \mathcal{F}, P)$ such that $\{\xi_k\}_{k\in\mathbf{N}}$ is independent of $\{r_k\}_{k\in\mathbf{N}}$ and

$$E\left[\xi_k\right] = 0, E\left[\xi_k^2\right] = b < \infty$$

for all $k \in \mathbf{N}$.

We note here that condition $P\left(r_k = i\right) > 0$ ensures the nontrivial computation of the conditional mean $E\left[\xi|r_n = i\right], \xi \in L^2\left(\Omega, \mathbf{R}^d\right)$.

Now, we consider the discrete-time fractional system

$$\begin{aligned}\Delta^{[\alpha]}x_{k+1} &= A\left(r_k\right)x_k + \xi_k B\left(r_k\right)x_k, k \in \mathbf{N} &(1)\\ x_0 &= x \in \mathbf{R}^d. &(2)\end{aligned}$$

As in [1], we multiply (1) by h^α and, since

$$h^\alpha\Delta^{[\alpha]}x_{k+1} = \sum_{j=0}^{k+1}(-1)^j\left(\begin{array}{c}\alpha\\ j\end{array}\right)x_{k+1-j} = x_{k+1} - \sum_{j=0}^{k}(-1)^j\left(\begin{array}{c}\alpha\\ j+1\end{array}\right)x_{k-j},$$

the equation (1) becomes

$$x_{k+1} = h^\alpha A\left(r_k\right)x_k + \sum_{j=0}^{k}(-1)^j\left(\begin{array}{c}\alpha\\ j+1\end{array}\right)x_{k-j} + \xi_k h^\alpha B\left(r_k\right)x_k.$$

If we denote $\mathbf{A}(i) = h^{\alpha} A(i), \mathbf{B}(i) = h^{\alpha} B(i)$, $i \in \mathcal{Z}$, $c_j := (-1)^j \begin{pmatrix} \alpha \\ j+1 \end{pmatrix}$ and $A_j = c_j I_{\mathbf{R}^d}$, $j \in \mathbf{N}$, the system (1) can be equivalently rewritten as

$$x_{k+1} = \mathbf{A}(r_k) x_k + \sum_{j=0}^{k} A_j x_{k-j} + \xi_k \mathbf{B}(r_k) x_k, \tag{3}$$

$$x_0 = x \in \mathbf{R}^d. \tag{4}$$

Let $\mathcal{F}_k$ and $\mathcal{G}_k$ be the σ-algebras generated by $\{\xi_n, 0 \leq n \leq k\}$ and $\{r_n, 0 \leq n \leq k\}$, respectively, and let $\mathcal{H}_k = \mathcal{F}_k \vee \mathcal{G}_k$ for all $k \geq 0$ ([5]). As in the case of classical linear stochastic systems, it can be proved that (3)-(4) has a unique solution x_k which belongs to $L^2(\Omega, \mathbf{R}^d)$ (see [21]) and is $\mathcal{H}_k$-measurable.

The mean-square representation of the solution x_k leads to an infinite dimensional system with a complicated form, which properties are very difficult to determine. However, the rapidly decrease to 0 of the sequence $\{A_j\}_{j \in \mathbf{N}}$, representing the coefficients of the delay terms from (3)-(4), makes from a system with a finite number N of time delays a good enough model for the most practical purposes.

Therefore, we associate with (3)-(4) the stochastic system

$$x_{k+1} = \mathbf{A}(r_k) x_k + \sum_{j=0}^{N} c_j x_{k-j} + \xi_k \mathbf{B}(r_k) x_k, \tag{5}$$

$$x_0 = x \in \mathbf{R}^d, x_{-p} = 0, p \in \{1, .., N\} \tag{6}$$

and we study the mean square asymptotic stability properties of its solutions. The stability notions that we use in the sequel are the ones defined below.

Definition 1 *The solution x_k of the system (3)-(4) (or (5)-(6)) is said to be MSA stable if $\lim_{k \to \infty} \|x_k\|_2 = 0$ for all $x_0 \in \mathbf{R}^d$. If the solution of the stochastic fractional system (3)-(4)(or (5)-(6)) is not MSA stable, we will say that it is mean square unstable.*

4 Linear discrete-time fractional order systems with finite delays

In this section we study the MSA stability of the stochastic systems (5)-(6) by using a classical technique based on Lyapunov equations. We first associate with (5)-(6) an equivalent linear *expanded-state* stochastic system and we obtain a solution mean-square representation, which involves a Lyapunov type operator. The solvability of certain algebraic Lyapunov equations defined by this operator will be the key tool in the analysis of the MSA stability of the discussed LFOSs.

4.1 An expanded state model

In this section we use a technique based on Lyapunov equations to derive necessary and sufficient conditions for the asymptotic stability of the system (5)-(6).

First, we rewrite the LFOSs (5)-(6) as a regular linear stochastic system. We introduce the linear operators $\mathcal{A}(i), \mathcal{B}(i) : (\mathbf{R}^d)^{N+1} \to (\mathbf{R}^d)^{N+1}, i \in \{1, .., D\}$ defined by the matrices

$$\mathcal{A}(i) = \begin{pmatrix} \mathbf{A}(i) + c_0 I_{\mathbf{R}^d} & c_1 I_{\mathbf{R}^d} & \dots & c_N I_{\mathbf{R}^d} \\ I_{\mathbf{R}^d} & 0 & . & 0 \\ . & I_{\mathbf{R}^d} & . & . \\ . & . & I_{\mathbf{R}^d} & 0 \end{pmatrix},$$

$$\mathcal{B}(i) = \begin{pmatrix} \mathbf{B}(i) & 0 & . & . & 0 \\ 0 & 0 & . & . & . \\ . & . & . & . & . \\ 0 & . & . & . & 0 \\ {\scriptstyle 1} & & & & {\scriptstyle N+1} \end{pmatrix}.$$

and the expanded state vector $X_k^T = \left(\underset{1}{x_k^T}, x_{k-1}^T, \dots, \underset{N+1}{x_{k-N}^T} \right) \in (\mathbf{R}^d)^{N+1}$, where the superscript T denotes the transpose.

Then (5)-(6) can be equivalently rewritten as

$$X_{k+1} = \mathcal{A}(r_k) X_k + \xi_k \mathcal{B}(r_k) X_k, k \geq 0 \tag{7}$$

$$X_0^T = \left(\underset{1}{x_0^T}, 0, \dots, \underset{N+1}{0} \right). \tag{8}$$

Let us define the linear operators

$$\begin{aligned} \mathcal{P}(R)(i) &= \sum_{j\in\mathcal{Z}} q_{ij}R(j), i\in\mathcal{Z}, R\in l^{\mathcal{Z}}_{L\left(\left(\mathbf{R}^d\right)^{N+1}\right)}, \\ \Lambda(R)(i) &= \left(\mathcal{A}^{[*]}\mathcal{P}(R)\mathcal{A}\right)(i), R\in l^{\mathcal{Z}}_{L\left(\left(\mathbf{R}^d\right)^{N+1}\right)} \qquad (9)\\ \Gamma(R)(i) &= b\left(\mathcal{B}^{[*]}\mathcal{P}(R)\mathcal{B}\right)(i), R\in l^{\mathcal{Z}}_{L\left(\left(\mathbf{R}^d\right)^{N+1}\right)} \qquad (10)\end{aligned}$$

The following representation result is a direct consequence of Theorem 3.1 from [5] or of Theorem 4 from [20].

Lemma 1 *For any* $S\in l^{\mathcal{Z}}_{L\left(\left(\mathbf{R}^d\right)^{N+1}\right)}$ *we have*

$$E\left[\langle S(r_{k+1})X_{k+1}, X_{k+1}\rangle\,|_{r_0=i}\right] = E\left[\langle(\Lambda+\Gamma)(S)(r_k)X_k, X_k\rangle\,|_{r_0=i}\right]. \qquad (11)$$

4.2 Lyapunov equations and mean square asymptotic stability

4.2.1 Main results

Following [17], we associate with (11) the Lyapunov equation

$$R = \Lambda(R) + U, \qquad (12)$$

where U is a sequence of operators with the following property:

(P1) i) $U\in l^{\mathcal{Z}}_{L\left(\left(\mathbf{R}^d\right)^{N+1}\right)}$ is a sequece of self-adjoint operators with the property that there is $\mathbf{U}(i)\in l^{\mathcal{Z}}_{L\left(\mathbf{R}^d\right)}, i\in\mathcal{Z}$ such that

$$U(i) = \begin{pmatrix} \mathbf{U}(i) & 0 & . & . & 0\\ 0 & 0 & . & . & .\\ . & . & . & . & .\\ 0 & . & . & . & 0\end{pmatrix} \text{ for all } i\in\mathcal{Z} \text{ and}$$

ii) $\mathbf{U}\succ 0$.

We note that (P1) i) implies that $\mathbf{U}(i)$, $i\in\mathcal{Z}$ is also a sequence of self-adjoint operators. If R is a solution of the Lyapunov equation (12), then we will denote by $R_1(i), i\in\mathcal{Z}$ the block matrix defined by the first d rows and d columns of the matrix $R(i), i\in\mathcal{Z}$.

Now we can prove the main result of this section, which gives a sufficient condition for the mean square asymptotic stability of (5)-(6).

Theorem 1 *Assume that there is U with the property (P1) such that equation (12) has a non-negative solution $R \in l^{\mathcal{Z}}_{L\left(\left(\mathbf{R}^d\right)^{N+1}\right)}$. If*

$$\mathbf{U}(i) - b\mathbf{B}^{[*]}(i)\left(\sum_{j\in\mathcal{Z}} q_{ij} R_1(j)\right)\mathbf{B}(i) > 0, \tag{13}$$

for all $i \in \{1,..,D\}$, where R_1 is the sequence of block matrices of R defined above , then the unique solution of (5)-(6) is MSA stable.

Proof. From Lemma 1 and (12) it follows that

$$\begin{aligned} E\left[\langle R(r_{k+1})X_{k+1}, X_{k+1}\rangle |_{r_0=i}\right] &= E\left[\langle(\Lambda+\Gamma)(R)(r_k)X_k, X_k\rangle |_{r_0=i}\right] \\ = E\left[\langle R(r_k)X_k, X_k\rangle |_{r_0=i}\right] &+ E\left[\langle \Gamma(R)(r_k)X_k, X_k\rangle |_{r_0=i}\right] \\ &- E\left[\langle U(r_k)X_k, X_k\rangle |_{r_0=i}\right] \end{aligned} \tag{14}$$

for all $i \in \{1,..,D\}$. A direct computation and the special form of $\mathcal{B}$ lead us to the conclusion that

$$\langle \Gamma(R)(i)X_k, X_k\rangle = \left\langle b\mathbf{B}^{[*]}(i)\left(\sum_{j\in\mathcal{Z}} q_{ij}R_1(j)\right)\mathbf{B}(i)x_k, x_k\right\rangle. \tag{15}$$

Condition (13) implies that there is $\gamma > 0$ such that

$$\mathbf{U}(i) - b\mathbf{B}^{[*]}(i)\left(\sum_{j\in\mathcal{Z}} q_{ji}R_1(j)\right)\mathbf{B}(i) \geq \gamma I_d.$$

Hence

$$\begin{gathered} E\left[\langle U(r_k)X_k, X_k\rangle - \langle \Gamma(R)(r_k)X_k, X_k\rangle |_{r_0=i}\right] = \\ E\left[\left\langle\left(\mathbf{U}(r_k) - b\mathbf{B}^{[*]}(r_k)\left(\sum_{j\in\mathcal{Z}} q_{r_k j}R_1(j)\right)\mathbf{B}(r_k)\right)x_k, x_k\right\rangle |_{r_0=i}\right] \geq \\ \gamma E\left[\langle x_k, x_k\rangle |_{r_0=i}\right] \end{gathered}$$

and, from (14), we obtain

$$\gamma E\left[\|x_k\|^2 |_{r_0=i}\right] + E\left[\langle R(r_{k+1})X_{k+1}, X_{k+1}\rangle |_{r_0=i}\right] \leq E\left[\langle R(r_k)X_k, X_k\rangle |_{r_0=i}\right].$$

Summing up the above inequality for $k = 0$ to n, we get,

$$\gamma \sum_{k=0}^{n} E\left[\|x_k\|^2 |_{r_0=i}\right] + E\left[\langle R(r_{n+1})X_{n+1}, X_{n+1}\rangle |_{r_0=i}\right] \le E\left[\langle R(r_0)X_0, X_0\rangle |_{r_0=i}\right].$$

Since R is non-negative, we deduce that

$$\sum_{k=0}^{n} E\left[\|x_k\|^2 |_{r_0=i}\right] \le \gamma^{-1} E\left[\langle R(r_0)X_0, X_0\rangle |_{r_0=i}\right]$$

for all $n \in \mathbf{N}, i \in \{1, .., D\}$. Thus, the series of positive numbers $\sum_{k=0}^{\infty} E\left[\|x_k\|^2 |_{r_0=i}\right]$ is convergent for all $i \in \{1, .., D\}$ and we conclude that

$$\lim_{k\to\infty} E\left[\|x_k\|^2 |_{r_0=i}\right] = 0$$

for all $i \in \{1, .., D\}$. Since $E\left[\|x_k\|^2\right] = \sum_{i\in\mathcal{Z}} E\left[\|x_k\|^2 |_{r_0=i}\right] p_i, p_i = P(r_0 = i)$, $i \in \{1, .., D\}$, it follows that $\lim_{k\to\infty} \|x_k\|_2 = 0$, i.e. the system (5)-(6) is MSA stable.

Proposition 1 *The following statements are equivalent:*

a) There is U with the property (P1) such that equation (12) has a non-negative solution $R \in l^{\mathcal{Z}}_{L\left(\left(\mathbf{R}^d\right)^{N+1}\right)}$ and

$$\mathbf{U}(i) - b\mathbf{B}^{[*]}(i)\left(\sum_{j\in\mathcal{Z}} q_{ij}R_1(j)\right)\mathbf{B}(i) > 0, i \in \mathcal{Z}, \tag{16}$$

where R_1 is defined as in the above theorem.

b) There is U with the property (P1) such that equation

$$R = (\Lambda + \Gamma)(R) + U \tag{17}$$

has a non-negative solution $R \in l^{\mathcal{Z}}_{L\left(\left(\mathbf{R}^d\right)^{N+1}\right)}$.

Proof. Assume that a) holds. We note that condition (16) is equivalent with $\mathbf{U} - b\mathbf{B}^{[*]}\left(\sum_{j\in\mathcal{Z}} q_{(.)j}R_1(j)\right)\mathbf{B} \succ 0$. By denoting $U_0 = U - \Gamma(R)$, we get $R = (\Lambda + \Gamma)(R) + U_0$. Since $\mathbf{U}_0(i), i \in \mathcal{Z}$, the block matrix defined by the first d rows and d columns of the matrix $U_0, i \in \mathcal{Z}$, coincides with $\mathbf{U} - b\mathbf{B}^{[*]}\left(\sum_{j\in\mathcal{Z}} q_{(.)j} R_1(j)\right)\mathbf{B}$, we deduce that U_0 has the property (P1) and b) follows.

Assume that b) is satisfied. Then, there is U_0 with the property (P1) such that $R = (\Lambda + \Gamma)(R) + U_0$ has a non-negative solution $R \in l^{\mathcal{Z}}_{L\left(\left(\mathbf{R}^d\right)^{N+1}\right)}$. By taking $U = U_0 + \Gamma(R)$ in (12), the equation (12) will have the same non-negative solution R. Arguing as in the proof of the above theorem, we deduce that U has the property (P1) i). Since $\mathbf{U}(i) - b\mathbf{B}^{[*]}(i)\left(\sum_{j\in\mathcal{Z}} q_{ij} R_1(j)\right)\mathbf{B}(i) = \mathbf{U}_0(i) > 0$, it follows that U also satisfies (P1) ii) and the proof is complete.

The following result gives a condition which ensures that (5)-(6) is not MSA stable.

Theorem 2 *If $U \in l^{\mathcal{Z}}_{L\left(\left(\mathbf{R}^d\right)^{N+1}\right)}$ is a non-negative operator such that equation*

$$R = (\Lambda + \Gamma)(R) - U \tag{18}$$

has a solution $R \in l^{\mathcal{Z}}_{L\left(\left(\mathbf{R}^d\right)^{N+1}\right)}$ with the property that

$$R_1(i) + U_1(i) > 0 \tag{19}$$

for all $i \in \mathcal{Z}$, then the system (5)-(6) is not MSA stable. Here, (as in the above theorem) $R_1(i)$ and $U_1(i), i \in \mathcal{Z}$ are the block matrices defined by the first d rows and d columns of the matrices $R(i)$ and $U(i), i \in \mathcal{Z}$, respectively.

Proof. Let $i \in \mathcal{Z}$ and $U \in l^{\mathcal{Z}}_{L\left(\left(\mathbf{R}^d\right)^{N+1}\right)}$ be a non-negative operator which satisfies the hypotheses of the theorem. Arguing as in the proof of Theorem 1, we have

$$\begin{aligned} E\left[\langle R(r_{k+1})X_{k+1}, X_{k+1}\rangle \,|_{r_0=i}\right] &= E\left[\langle (\Lambda+\Gamma)(R)(r_k)X_k, X_k\rangle \,|_{r_0=i}\right] \\ &= E\left[\langle R(r_k)X_k, X_k\rangle \,|_{r_0=i}\right] + E\left[\langle U(r_k)X_k, X_k\rangle \,|_{r_0=i}\right] \end{aligned}$$

Summing the above inequality for $k = 0$ to n, we get,

$$\sum_{k=0}^{n} E\left[\langle U(r_k)X_k, X_k\rangle \,|_{r_0=i}\right] + \langle R(i)X_0, X_0\rangle = E\left[\langle R(r_{n+1})X_{n+1}, X_{n+1}\rangle \,|_{r_0=i}\right].$$

The properties of the operator U and the last equation imply that

$$E\left[\langle R\left(r_{n+1}\right)X_{n+1},X_{n+1}\rangle\,|_{r_0=i}\right]\geq\langle R\left(i\right)X_0,X_0\rangle+\langle U\left(i\right)X_0,X_0\rangle=\\\langle\left(R_1\left(i\right)+U_1\left(i\right)\right)x_0,x_0\rangle>0$$

for all $n\in\mathbf{N}$ and $x_0\neq 0$. From (19) it follows that there is $\gamma>0$ such that $\langle\left(R_1\left(i\right)+U_1\left(i\right)\right)x_0,x_0\rangle>\gamma\left\|x_0\right\|^2$ for all $x_0\in\mathbf{R}^d$. On the other hand, $E\left[\langle R\left(r_{n+1}\right)X_{n+1},X_{n+1}\rangle\,|_{r_0=i}\right]\leq E\left(\|R\|\left\|X_{n+1}\right\|^2\right)$ and we deduce that $\|R\|\,E\left[\left\|X_{n+1}\right\|^2|_{r_0=i}\right]>\gamma\left\|x_0\right\|^2$ for all $n\in\mathbf{N}$, $x_0\in\mathbf{R}^d$ and $i\in\mathcal{Z}$. Therefore, $\limsup_{k\to\infty}E\left[\left\|X_k\right\|^2|_{r_0=i}\right]$ cannot be zero. Consequently, $\limsup_{k\to\infty}E\left[\left\|x_k\right\|^2|_{r_0=i}\right]$ is not zero and the conclusion follows by virtue of (H1) (ii) ($P\left(r_k=i\right)>0,k\in\mathbf{N},i\in\mathcal{Z}$) and Definition 1.

4.2.2 Procedural issues

For any $i\in\mathcal{Z}$, let Q_i be the matrix obtained from the probability matrix Q by replacing with 0 all the elements from the lines $j\in\mathcal{Z}-\{i\}$. We define the matrix

$$L=\sum_{i\in\mathcal{Z}}Q_i\otimes^K\left(\mathcal{A}^{[*]}\left(i\right)\otimes^K\mathcal{A}^{[*]}\left(i\right)\right)+b\sum_{i\in\mathcal{Z}}Q_i\otimes^K\left(\mathcal{B}^{[*]}\left(i\right)\otimes^K\mathcal{B}^{[*]}\left(i\right)\right)\quad(20)$$

where $\otimes^K$denotes the Kronecker product of two matrices.

Now let $vec(X)$ be the vectorization of a matrix X, i.e the vector formed by stacking the columns of X into a single column vector. Then equation (17) can be equivalently rewritten as

$$vec\left(R\right)=L\cdot vec\left(R\right)+vec\left(U\right),\quad(21)$$

where $vec\left(T\right)=\begin{pmatrix}vec(T\left(1\right))\\ .\\ vec\left(T\left(D\right)\right)\end{pmatrix},T=R,U.$

Similarly, (18) is equivalent to

$$vec\left(R\right)=L\cdot vec\left(R\right)-vec\left(U\right).\quad(22)$$

It is not difficult to see that the above linear and non-homogeneous systems are easier to implement and to solve than equations (21) and (22). Consequently, they are useful for many numerical computations. These systems will be used it in the last section to check the MSA stability/instability in some numerical examples.

5 Mean-square asymptotic instability of LFOs with multiplicative noise

In this section we provide sufficient conditions for the asymptotic instability of both systems (3)-(4) and (5)-(6), in the special case when their coefficients are not affected by Markovian jumps. In contrast with Theorem 2, the new conditions are easier to check and ensure that $\limsup_{k\to\infty} \|x_k\|_2 = \infty$. However, they do not apply for systems with Markovian jumps, as Theorem 2 does.

5.1 Fractional systems with infinite delays

In this section we assume that $\mathbf{A}, B \in L\left(\mathbf{R}^d\right)$ and (H1) iii) holds, without any reference to the Markov chain.

We consider the following stochastic fractional order system

$$x_{k+1} = \mathbf{A}x_k + \sum_{i=0}^{k} A_i x_{k-i} + \xi_k B x_k \tag{23}$$

$$x_0 = x \in \mathbf{R}^d. \tag{24}$$

By taking the expectation in (23) and by using (H1) iii), we obtain the deterministic fractional system

$$E\left[x_{k+1}\right] = \mathbf{AE}\left[x_k\right] + \sum_{i=0}^{k} A_i E\left[x_{k-i}\right] \tag{25}$$

$$E\left[x_0\right] = x \in \mathbf{R}^d.$$

The asymptotic behavior of the above deterministic system was studied in [12] by using Z-transform techniques. The next result follows from Proposition 38 b) in [12] and provides a sufficient condition for the mean square instability of (23).

Proposition 2 *Let $\mathcal{R}$ denote the set of roots of the equation*

$$\det\left(z\left(1-\frac{1}{z}\right)^{\alpha} I_d - \mathbf{A}\right) = 0. \tag{26}$$

If there is $z \in \mathcal{R}$ with the property $|z| > 1$, then $\limsup_{k\to\infty} \|x_k\|_2 = \infty$, where x_k is the unique solution of (23)-(24).

Proof. From Proposition 38 b) in [12], it follows that $\limsup_{k\to\infty} E[x_k] = \infty$. Now we apply the well known inequality

$$\|\mathbf{E}[x_k]\|^2 \le \mathbf{E}\left[\|x_k\|^2\right] \tag{27}$$

and we get the conclusion

In the last section we provide an example (see Example 3) which illustrates the applicability of Proposition 2.

5.2 Fractional systems with finite delays

As in the case of LFOSs with infinite delays, we use Z -transform to establish sufficient conditions for the stochastic instability of LFOSs with finite delays.

We assume that $\mathbf{A}, B \in L\left(\mathbf{R}^d\right)$ and (H1) iii) holds and we consider system (5)-(6) in the absence of the Markovian jumps. We get the following system

$$\begin{aligned} x_{k+1} &= \mathbf{A}x_k + \sum_{i=0}^{N} c_i x_{k-i} + \xi_k B x_k && (28)\\ x_0 &= x \in \mathbf{R}^d, x_{-p} = 0, p \in \{1, .., N\} .. && (29) \end{aligned}$$

By taking the expectation, we obtain the deterministic system

$$E[x_{k+1}] = \mathbf{A}E[x_k] + \sum_{i=0}^{N} c_i E[x_{k-i}]. \tag{30}$$

If we denote by $W(z)$ the transform Z of the sequence $\{E[x_k]\}_{k\in\mathbf{N}}$ and we use the shifting property of the transform Z, we get

$$Z\left(\sum_{l=0}^{N} c_l E[x_{k-l}|_{r_0=i}]\right) = c(z)\, W(z),$$

where $c(z) = \sum_{n=0}^{N} c_n z^{-n}$. Applying Z transform to (30), we obtain the following algebraic equation

$$\begin{aligned} zW(z) - zE[x_0] &= \mathbf{A}W(z) + c(z)W(z) \Leftrightarrow \\ ((z - c(z)) I_d - \mathbf{A}) W(z) &= zx_0. && (31) \end{aligned}$$

Proposition 3 *Let $\mathcal{R}$ denote the set of roots of the equation*

$$\det\left(\left(z-c\left(z\right)\right)I_d-\mathbf{A}\right)=0. \tag{32}$$

If there is $z_0\in\mathcal{R}$ such that $|z_0|>1$, then $\limsup_{k\to\infty}\|x_k\|_2=\infty$.

Proof. The function $f\left(z\right)=z\left(\left(z-c\left(z\right)\right)I_d-\mathbf{A}\right)^{-1}x_0$ is rational and holomorphic on $\mathbf{C}-\mathcal{R}$. Then, on $\left(\mathbf{C}-\mathcal{R}\right)\cap ROC\left(W\left(z\right)\right)$ we have

$$W\left(z\right)=f\left(z\right).$$

From the hypothesis, the complex number $z_0\in\mathcal{R}$ satisfying $|z_0|>1$ is a pole for $f\left(z\right)$. Let R be the radius of convergence of $W\left(z\right)$. We know that $W\left(z\right)$ is holomorphic outside the circle $|z|=R$. If $R<|z_0|$, then $\lim_{z\to z_0}f\left(z\right)=\lim_{z\to z_0}W\left(z\right)=W\left(z_0\right)$, which contradicts the hypothesis that z_0 is a pole for $f\left(z\right)$. It follows that $R\geq|z_0|>1$. From the Cauchy-Hadamard's theorem, we have $R=\limsup_{k\to\infty}\sqrt[k]{\|E\left[x_k\right]\|}\geq|z_0|>1$, which implies that $\sup_{n\geq k}\|E\left[x_n\right]\|\geq|z_0|^k$ for k sufficiently large. Then $\limsup_{k\to\infty}E\left[x_k\right]=\infty$ and the conclusion follows from (27).

6 Numerically solutions

In this section we provide some numerical examples which illustrates the theory. We first consider the system (5)-(6) for different values of coefficients and a sequence $\{\xi_k\}_{k\in\mathbf{N}}$ of real-valued, identically and uniformly distributed random variables from $L^2\left(\Omega,\mathbf{R}\right)$, which is generated by Matlab. The Markov chain and the solution's graph are generated by a computer program written by the author for this purpose.

Example 1 *Let us consider system (5)-(6) for $d=2, h=1$, $\alpha=1/2$, $A\left(1\right)=\begin{pmatrix}-2/5 & 0\\ 0 & 1/5\end{pmatrix}$, $A\left(2\right)=\begin{pmatrix}1 & 2\\ -1 & 3\end{pmatrix}$, $B=\frac{1}{10}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$, $x_0=(10,20)$ and $N=1$. The Markov chain has the state space $\mathcal{Z}=\{1,2\}$, the transition matrix $Q=\begin{pmatrix}0.5 & 0.5\\ 0.7 & 0.3\end{pmatrix}$ and the initial distribution $p=(0.8,0.2)$.*

We will apply Theorem 2 to prove that (5)-(6) is MSA instable. We consider (18) for $U\left(i\right)=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$, $i\in\{1,2\}$. The equation (18) *can be equivalently rewritten as (22). We use a Matlab program to generate the operator L and function **linsolve** to find the solution R of (18). We obtain the block matrices $R_1(1) = \begin{pmatrix} -0.9281 & -0.0408 \\ -0.0408 & 0.0335 \end{pmatrix}$ and $R_1(2) = \begin{pmatrix} 4.3206 & -0.7727 \\ -0.7727 & 2.0263 \end{pmatrix}$. The sets of eigenvalues of the matrices*

$$\begin{aligned} R_1(1) + U_1(1) &= \begin{pmatrix} 0.0719 & -0.0408 \\ -0.0408 & 1.0335 \end{pmatrix}, \\ R_1(2) + U_1(2) &= \begin{pmatrix} 5.3206 & -0.7727 \\ -0.7727 & 3.0263 \end{pmatrix} \end{aligned}$$

are $S_1 = \{1.0352, 7.0172 \times 10^{-2}\}$ and $S_2 = \{5.5566, 2.7903\}$, respectively, and contain only positive numbers. It follows that condition (19) holds and, by Theorem 2, the solution x_k of (5)-(6) is MSA unstable. A Matlab simulation of 10 instances of the sequence $x_k, k = \overline{1,30}$ is shown in Fig.1 (a) and confirms the theoretical result.

Example 2 *Now, we consider system (5)-(6) for $d = 2, h = 1, \alpha = 1/10$, $A(1) = \frac{-1}{5} I_d, A(2) = -I_d, B(1) = I_d$, $B(2) = 0$, $b = 1, x_0 = (10, 20), N = 9$ and the same Markov chain as in Example 1. Now we solve equation (17) for a sequence of operators U having the property (P1) with $\mathbf{U}(i) = I_2$, $i = 1, 2$. A computer program, similar to that used for solving (18), leads us to the conclusion that (17) has a non-negative solution $R \in l^{\mathcal{Z}}_{L\left(\left(\mathbf{R}^d\right)^{N+1}\right)}$. For reason of space we do not give the elements of R for $N = 9$ (because the associate matrix will have 100 entries). Instead, we give the sequence R for the case when $N = 1$. We have*

$$\begin{aligned} R(1) &= \begin{pmatrix} 18.419 & 0 & -0.0956 & 0 \\ 0 & 18.419 & 0 & -0.0956 \\ -0.0956 & 0 & 0.0347 & 0 \\ 0 & -0.0956 & 0 & 0.0347 \end{pmatrix}, \\ R(2) &= \begin{pmatrix} 15.841 & 0 & -0.7275 & 0 \\ 0 & 15.841 & 0 & -0.7275 \\ -0.7275 & 0 & 0.0357 & 0 \\ 0 & -0.7275 & 0 & 0.0357 \end{pmatrix}. \end{aligned}$$

The eigenvalues of $R(1)$ and $R(2)$ are 18.4193, 18.4193, 0.0342, 0.03420 and 15.8746, 15.8746, 0.0023, 0.0023, respectively. We conclude, that R is nonnegative and, by Theorem 1, the stochastic system is MSA stable (in both

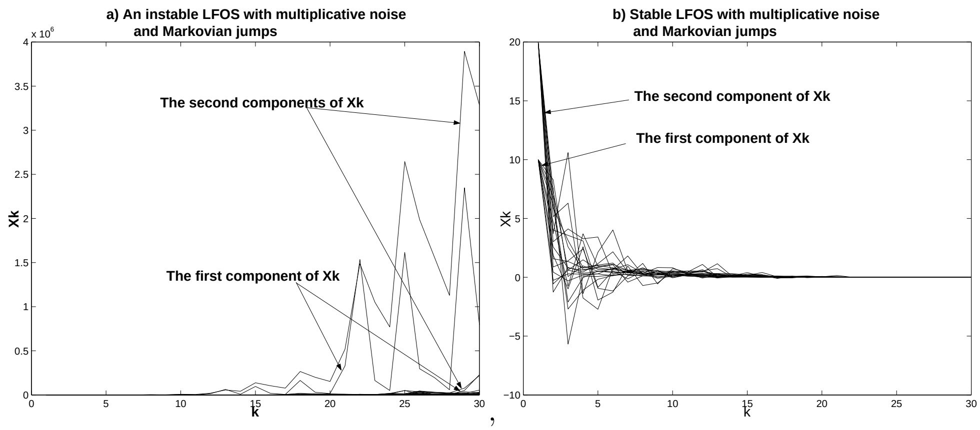

Figure 1: Finite delays case - the states $x_k, k = 1, ..30$ of the system (5)-(6).

cases $N = 1$ *and* $N = 9$). *For* $N = 9$, *the first* 30 *values of 10 instances of the solution* x_k *of the stochastic system (5)-(6) are shown in Fig.1 (b) .*

The next two examples consider fractional systems without Markovian jumps.

Example 3 *We analyze the asymptotic behavior of system (23) -(24) under the assumption that* $d = 2, \alpha = 1/2,\ A = \begin{pmatrix} 2 & -\frac{3}{2} \\ 1 & -1 \end{pmatrix}, B = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$, $h = 1, b = 1, x_0 = (10, 20)$. *In the case* $\alpha = 1/2$, *the matrix* $A^{\alpha}(z) = \mathbf{A} - z\left(1 - \frac{1}{z}\right)^{\alpha} I_d$ *becomes* $A^{\alpha}(z) = \mathbf{A} - \left(\sqrt{z^2 - z}\right) I_d$. *Since the eigenvalues of the matrix* A *are* $\frac{1}{2}\sqrt{3} + \frac{1}{2}, \frac{1}{2} - \frac{1}{2}\sqrt{3}$, *the equation* $\det(A^{\alpha}(z)) = 0$ *implies that* $z^2 - z = \left(\frac{1}{2}\sqrt{3} + \frac{1}{2}\right)^2$. *The solutions of the last equation are* $z_1 = -0.954\,66, z_2 = 1.954\,7$ *and one of them is placed outside the unit circle. From Proposition 3 it follows that solution* x_k *of the stochastic fractional system (23) -(24) is mean square unstable. By simulating the first 50 elements of 10 instances of* x_k, *we get the graph from Fig.2 (a).*

Example 4 *Now we consider the system (28)-(29) with* $\alpha = 1/2$, $N = 9$ *and the same numerical values for the coefficients and for the random sequences as in Example 3. In this case we have to determine the set* $\mathcal{R}$ *of solutions of equation* $\det((z - c(z)) I_d - \mathbf{A}) = 0$. *Obviously* $\mathcal{R}$ *is the reunion of the solutions sets of the following two algebraic equations*

$$z - c(z) = \lambda_k, k = 1, 2 \tag{33}$$

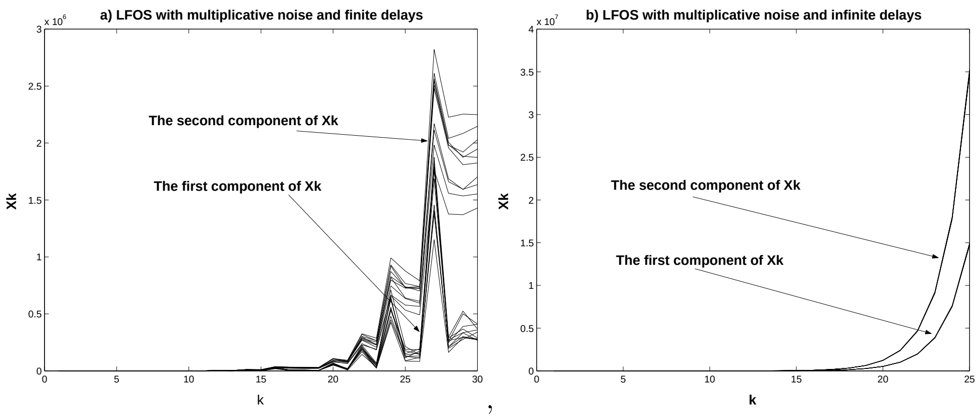

Figure 2: The states $x_k, k = 1, ..30$ of the systems (23) -(24) and (28)-(29), respectively.

where $\lambda_k, k = 1, 2$ *are the eigenvalues of the matrix* $\mathbf{A}$. *Since* $\lambda_1 = \frac{1}{2}\sqrt{3} + \frac{1}{2}, \lambda_2 = \frac{1}{2} - \frac{1}{2}\sqrt{3}$, *one of the equations (33) is*

$$z - (0.5 - 0.1250\frac{1}{z} + 0.1250\frac{1}{z^2} -$$
$$0.0625\frac{1}{z^3} - 0.0391\frac{1}{z^4} + 0.0273\frac{1}{z^5} - 0.0205\frac{1}{z^6} +$$
$$0.0161\frac{1}{z^7} - 0.0131\frac{1}{z^8} + 0.0109\frac{1}{z^9}) = \frac{1}{2}\sqrt{3} + \frac{1}{2}.$$

A solution of the above equation is $z = 1.822, 2$ *and lies outside the unit circle* $C = \{z, |z| = 1\}$. *Since the hypotheses of Proposition 3 are fulfilled, we conclude that the solution* x_n *of the stochastic fractional system (28)-(29) is MSA unstable. A computer simulation of* 10 *instances of this solution is shown in Fig.2 (b).*

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