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ON THE H_2 STATIC OUTPUT FEEDBACK CONTROL FOR HIDDEN MARKOV JUMP LINEAR SYSTEMS*

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Dedicated to Dr. Vasile Drăgan on the occasion of his 70th anniversary

Abstract

In this chapter we discuss the H_2 control for Markov jump linear systems in a context of partial observation of both the Markov chain and the state variable. The controller is static and depends on an observed variable that provides the only information of the Markov variable in a context of hidden Markov chains. We propose a new design condition in terms of linear matrix inequalities considering rank constraints in suitable system matrices that are easily fulfilled. Next we investigate the case in which the detector provides perfect estimations of the Markov chain and all the states are available to the controller. Finally we compare this result with the so-called two-step procedure for hidden Markov jump linear systems in an academic example of a

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1 Introduction

Systems subject to abrupt changes in their dynamics have been receiving a great deal of attention in the past decades. One of the main reasons concerns the presence of faults in critical applications, which motivates to develop a deeper understanding of how to detect and consequently act in such cases. Specially if the changes are not deterministic, as it is usually the case, the use of Markov jump linear systems (MJLS) to model these situations is appealing. By now there is a large body of works on MJLS such as [13, 5, 6, 22, 2] and the reference therein, to name a few.

The aforementioned works deal mainly with the case in which the Markov chain can be measured, the so-called mode-dependent or perfect observation case. However there are instances in which the Markov chain is not available to the control, as in applications of Active Fault-tolerant Control Systems (AFTCS) in [21, 1] in which the main jump process is a fault process. This setting imposes challenging problems in a vast array of applications going from control to filtering, see, for instance, [42, 15, 17, 18, 12, 16, 3], and the references therein. Among the approaches dealing with the so-called *partial observation setting* of the Markov chain used in the aforementioned works, we can point out the *cluster* and *mode-independent* cases. In the former setting, the modes of operation are grouped into distinguishable and disjoint sets so that the controller/filter would jump according to the set in which the Markov chain is currently operating; and the latter assumes that there is no information at all concerning the jump process so that there is only one controller for all possible modes of operation. More recently, an alternative formulation introduced in [4] that encompasses the aforementioned approaches has come into a great focus in the literature. The idea is to study the problem in the context of hidden Markov chains, see for instance, [34], so that even though the Markov chain is hidden, there is still some information provided by a type of *detector*, for instance, the output of a Fault-detection and Isolation device (FDI), that could be used in the control of the system. We can mention a few works such as [7, 43, 29, 41] concerning state-feedback control; [27, 32, 37] for filtering; [9, 20] for dynamic output feedback control;

and [38, 39, 10, 11, 45] for static output feedback control. It is worth to point out that, even though the use of hidden Markov chains in MJLS can be found in the literature with different names such as *detector approach*, *hidden MJLS*, *asynchronous control*, and *mismatched control*, among others, the modeling is essentially the same. On another vein, we can also mention the approach used in [26] for dealing with the partial observation setting of the Markov chain.

In this work we tackle a challenging control problem that has, so far, defied an optimal convex formulation, namely, *static output feedback control*. We mention an interesting survey about the subject in [35] which points out the fact that the problem is a Bilinear Matrix Inequality (BMI) type that is, in general, NP-Hard, even for linear time-invariant (LTI) systems. Nonetheless there are workarounds that have been proposed and provide at least sub-optimal approaches for the control design, such as the two-step procedure in [31, 23, 24, 25], the use of rank restrictions in suitable matrices as in [46, 44, 40, 45], or the approach in [19, 33] that studies the stabilizability problem taking a parallel with the linear quadratic regulator theory. In [11] it was proposed a two-step procedure for the H_2 static output feedback control for hidden MJLS in terms of Linear Matrix Inequalities (LMIs), which relies on an initial choice of a stabilizing state-feedback controller. In this work we propose an alternative way of solving the problem that does not depend on this initial state-feedback controler but, instead, it resorts to rank constraints similarly as done in [40, 45], which seems to be a natural fashion of obtaining a solution to the control problem, since it directly concerns the measurements of the states. We show that these new condition, when applied to the case of perfect measurement of the states and the Markov chain, yields to the $\mathcal{H}_2$ state-feedback design conditions of [29], which in turns leads to the optimal control in the case we have a perfect detector of the Markov chain. We also provide a comparison between this new condition and the two-step procedure presented in [11].

We now do a brief comparison between our work and [40] and [45]. Even though both papers rely on rank constraints and similar controller parametrizations, the paper [40] considered the $\mathcal{H}_\infty$ static output feedback control for continuous-time exponential hidden Markov jump systems. Thus its main focus is on robust control and in the continuous-time setting. As for [45], it deals with discrete-time linear quadratic control for hidden MJLS in the finite and infinite-horizon settings with hard constraints on the norms of the control input and the state. We instead focus on the $\mathcal{H}_2$ static output feedback control with an emphasis on the stochastic formulation of the infinite-horizon setting.

This work is organized as follows. In Section 2 we introduce the notation and in Section 3 we introduce the system, the basic definitions and the problem formulation. In Section 4 we present the main results, namely, a sub-optimal design condition in terms of LMIs for calculating a static output feeddback controller that depends only on the detector that guarantees a bound on the $\mathcal{H}_2$ norm of the closed-loop system, as well as a corollary that investigates the relation between our work and the results of [29] for the $\mathcal{H}_2$ state-feedback control. We also briefly revisit the result of [11] for our setting. Finally in Section 5 we perform a numerical comparison between our result and the one given in [11] in an academic application of systems subject to faults and state our final remarks in Section 6.

2 Notation

The real n-dimensional Euclidean space is represented by $\mathbb{R}^n$ and $\mathbb{B}(\mathbb{R}^m, \mathbb{R}^n)$ is the space of $n \times m$ real matrices. The operator $(\cdot)'$ represents the transpose of a matrix, I_n is the identity matrix of size $n \times n$, $0_{n\times m}$ is the zero matrix of size $n \times m$, $\mathbf{diag}(\cdot)$ is a block diagonal matrix, $\mathrm{Tr}(\cdot)$ is the trace of a square matrix, and for $G \in \mathbb{B}(\mathbb{R}^n) \triangleq \mathbb{B}(\mathbb{R}^n, \mathbb{R}^n)$ we set $\mathrm{Her}(G) \triangleq G + G'$. The symbol $\bullet$ represents a symmetric block for partitioned symmetric matrices. For N and M positive integers, we set $\mathbb{N} \triangleq \{1, \ldots, N\}$ and $\mathbb{M} \triangleq \{1, \ldots, M\}$. The set $\mathbb{H}^{n,m}$ is the linear space of all N-sequence of real matrices $V = (V_1, V_2, ..., V_N)$, $V_i \in \mathbb{B}(\mathbb{R}^n, \mathbb{R}^m)$, $i \in \mathbb{N}$ and, for simplicity, $\mathbb{H}^n \triangleq \mathbb{H}^{n,n}$ and $\mathbb{H}^{n+} \triangleq \{V \in \mathbb{H}^n; V_i \geq 0,\, i = 1, \ldots, N\}$. For $P, V \in \mathbb{H}^{n+}$, we write that $P > V$ if $P_i > V_i$ for each $i = 1, \ldots, N$.

3 Preliminaries

In a probability space $(\Omega, \mathfrak{F}, \mathrm{Prob})$ equipped with filtration $\{\mathfrak{F}_k\}$, we consider the following Markov jump system

$$\mathcal{G} : \begin{cases} x(k+1) &= A_{\theta(k)}x(k) + B_{\theta(k)}u(k) + J_{\theta(k)}w(k) \\ \quad y(k) &= L_{\theta(k)}x(k) \\ \quad z(k) &= C_{\theta(k)}x(k) + D_{\theta(k)}u(k) + E_{\theta(k)}w(k) \end{cases} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state, $u(k) \in \mathbb{R}^m$ is the control input, $w(k) \in \mathbb{R}^r$ is the noise process, $z(k) \in \mathbb{R}^q$ is the controlled output, and $y(k) \in \mathbb{R}^p$ is the measured output, with $n \geq p$. The variable $\theta(k)$ is a homogeneous Markov chain with state space $\mathbb{N}$, $\mathrm{Prob}(\theta(k+1) = j \mid \mathfrak{F}_k) = \mathrm{Prob}(\theta(k+1) = j \mid \theta(k)) = p_{\theta(k)j} \geq 0,\ j \in \mathbb{N}$. We set $\mathbb{N}_i \triangleq \{j \in \mathbb{N}; p_{ij} > 0\}$ and $p_i \triangleq \{p_{ij}; j \in \mathbb{N}_i\}$

for all $i \in \mathbb{N}$. The system (1) evolves from $x(0) = x_0$ and $\theta(0) = \theta_0$, a random variable taking values in $\mathbb{N}$. It is assumed that $w(k)$ is a wide-sense white noise sequence with $\mathbf{E}(w(k)) = 0$ and $\mathbf{E}(w(k)w(k)') = I_{r_w}$, $k \geq 0$, independent of $\theta(k)$.

Assumption 1 *L_i has full row rank p for all $i \in \mathbb{N}$.* ■

We assume that $\theta(k)$ cannot be measured, but instead there is a detector whose output depends on the Markov chain in the following fashion:

$$\text{Prob}(\hat{\theta}(k) = \ell \mid \hat{\mathfrak{F}}_k) = \text{Prob}(\hat{\theta}(k) = \ell \mid \theta(k)) = \alpha_{\theta(k)\ell}$$

where $\hat{\mathfrak{F}}_k$ is the $\sigma-$algebra generated by $\{x(i), u(i), w(i), \theta(i), \hat{\theta}(i), 0 \leq i < k-1\} \cup \{x(k), u(k), w(k), \theta(k)\}$, for $k > 0$, and $\{x(0), u(0), w(0), \theta(0)\}$, for $k = 0$. We assume that $\mathbb{M}$ is the output map of $\hat{\theta}(k)$, $\alpha_{i\ell} \geq 0$ for all $i \in \mathbb{N}$ and $\ell \in \mathbb{M}$, and define the set of possible outcomes of $\hat{\theta}(k)$ given that $\theta(k) = i$ as $\mathbb{M}_i \triangleq \{\ell \in \mathbb{M}; \alpha_{i\ell} > 0\}$, and set $\alpha_i \triangleq \{\alpha_{i\ell}; \ell \in \mathbb{M}_i\}$, $i \in \mathbb{N}$. As discussed in [34], the process $(\theta, \hat{\theta})$ is known as a hidden Markov chain.

We aim at designing the following static output feedback controller,

$$u(k) = K_{\hat{\theta}(k)} y(k) \tag{2}$$

that depends on the measured outputs $y(k)$ and $\hat{\theta}(k)$, for $K \triangleq (K_1, \ldots, K_M)$. By connecting (1) and (2), we get the following closed-loop system

$$\mathcal{G}_K : \begin{cases} x(k+1) &= \bar{A}_{\theta(k)\hat{\theta}(k)} x(k) + J_{\theta(k)} w(k) \\ z(k) &= \bar{C}_{\theta(k)\hat{\theta}(k)} x(k) + E_{\theta(k)} w(k) \end{cases} \tag{3}$$

for $\bar{A}_{\theta(k)\hat{\theta}(k)} \triangleq A_{\theta(k)} + B_{\theta(k)} K_{\hat{\theta}(k)} L_{\theta(k)}$ and $\bar{C}_{\theta(k)\hat{\theta}(k)} \triangleq C_{\theta(k)} + D_{\theta(k)} K_{\hat{\theta}(k)} L_{\theta(k)}$.

Remark 1 *As discussed in [7], the pair $(\theta(k), \hat{\theta}(k))$ generalizes some observation models typically applied to MJLS.*

(i) The mode-dependent case, see, for instance, [5]. If we assume that $M = N$ and $\alpha_{ii} = 1$ for all $i \in \mathbb{N}$, we get that $\hat{\theta}(k) = \theta(k)$, that is, the detector provides perfect estimates of the Markov chain.

(ii) The cluster case, see, for instance, [42]. We consider that $M \leq N$ so that $\mathbb{N}$ can be written as the union of M disjoint sets (clusters) $\mathbb{N}_i$, that is, $\mathbb{N} = \cup_{i=1}^{M} \mathbb{N}_i$. By defining a function $g : \mathbb{N} \to \mathbb{M}$ such that $g(i) = j$ for all $i \in \mathbb{N}_j$, then $g(i)$ represents to which set (cluster) the state i belongs to. By considering $\mathbb{M}_i = \{g(i)\}$ and $\alpha_{ig(i)} = 1$, then $\hat{\theta}(k)$ would indicate the cluster in which $\theta(k)$ is.

(iii) The mode-independent case, see, for instance, [12]. By setting $M = 1$ and $\alpha_{i1} = 1$ for all $i \in \mathbb{N}$, we get that $\hat{\theta}(k)$ cannot provide any useful information on $\theta(k)$.

We present some basic definitions next.

Definition 1 *Given K, system (3) with $w \equiv 0$ is said to be stochastically stable (SS) if*

$$\|x\|_2^2 \triangleq \sum_{k=0}^{\infty} \mathbf{E}(\|x(k)\|^2) < \infty,$$

regardless of x_0, θ_0. ∎

The set of admissible controllers is given by $\mathfrak{K} \triangleq \{K \text{ such that (3) is SS}\}$. By defining the operators $\mathcal{L} \in \mathbb{H}^{n+}$ and $\mathcal{E} \in \mathbb{H}^{n+}$ for $S \in \mathbb{H}^{n+}$ such that

$$\mathcal{E}_i(S) \triangleq \sum_{j \in \mathbb{N}} p_{ij} S_j,$$
$$\mathcal{L}_i(S) \triangleq \sum_{\ell \in \mathbb{M}_i} \alpha_{i\ell} \bar{A}'_{i\ell} \mathcal{E}_i(S) \bar{A}_{i\ell},$$

we get from [7] that if there exists $P \in \mathbb{H}^{n+}$, $P > 0$ and K, such that

$$P - \mathcal{L}(P) > 0 \tag{4}$$

then $K \in \mathfrak{K}$.

Let us now recall some basic definitions regarding the H_2 norm of (3) and introduce a necessary assumption.

Assumption 2 *The Markov chain is ergodic, see, for instance, [34].* ∎

Assumption 2 is needed to ensure that there exists $\mu_i > 0$, $i \in \mathbb{N}$, $\sum_{i \in \mathbb{N}} \mu_i = 1$ such that as $k \to \infty$, $\mu_i(k) \to \mu_i$, where $\text{Prob}(\theta(k) = i) = \mu_i(k)$. Let us now define the stochastic version of the H_2 norm of (3) as discussed in [27].

Definition 2 *For $K \in \mathfrak{K}$, $\|\mathcal{G}_K\|_2^2 \triangleq \lim_{k \to \infty} \mathbf{E}\left(\|z(k)\|^2\right)$.*

We now discuss the conditions in which the limit in Definition 2 exists and recall how to calculate $\|\mathcal{G}_K\|_2^2$ through the next proposition based on the discussion presented in [27].

Proposition 1 *For $K \in \mathfrak{K}$ and considering that Assumption 2 holds, let $\bar{P} \in \mathbb{H}^{n+}$ be the unique solution of*

$$\bar{P} = \mathcal{L}(\bar{P}) + \mathbf{C}, \tag{5}$$

where $\mathbf{C} \in \mathbb{H}^{n+}$, $\mathbf{C}_i \triangleq \sum_{\ell \in \mathbb{M}_i} \alpha_{i\ell} C'_{i\ell} C_{i\ell}$. Then,

$$\|\mathcal{G}_K\|_2^2 = \sum_{i \in \mathbb{N}} \mu_i \, Tr(J'_i \mathcal{E}_i(\bar{P}) J_i + E'_i E_i). \tag{6}$$

where $\lim_{k \to \infty} Prob(\theta(k) = i) = \mu_i$, $i \in \mathbb{N}$.

We will need the following auxiliary result adapted from [5].

Proposition 2 *For $K \in \mathfrak{K}$ and $S \geq T \geq 0$ ($>$, respectively), let P, L be the unique solutions of $P - \mathcal{L}(P) = S$, and $L - \mathcal{L}(L) = T$, respectively. Then $P \geq L \geq 0$ (> 0).*

We get that if there exists $P \in \mathbb{H}^{n+}$, $P > 0$, such that the following inequality

$$P - \mathcal{L}(P) > \mathbf{C} \tag{7}$$

holds, then from (4), we get that $K \in \mathfrak{K}$. Furthermore, since there exists $V \in \mathbb{H}^{n+}$, $V > 0$ such that $P - \mathcal{L}(P) = \mathbf{C} + V$, then by considering Proposition 2, we get that $P > \bar{P}$. It follows from (6) that $\|\mathcal{G}_K\|_2^2 < \sum_{i \in \mathbb{N}} \mu_i \ \mathrm{Tr}(J'_i \mathcal{E}_i(P) J_i + E'_i E_i)$.

We are now able to state the main goal of this work, that is, finding $K \in \mathfrak{K}$ that minimizes γ such that $\|\mathcal{G}_K\|_2 < \gamma$. Formally,

$$\inf_{\gamma, K \in \mathfrak{K}, P} \gamma \tag{8}$$

such that (7) and

$$\sum_{i \in \mathbb{N}} \mu_i \mathrm{Tr}(J'_i \mathcal{E}_i(P) J_i + E'_i E_i) < \gamma^2 \tag{9}$$

hold. The problem in (8) is non-linear and hard to solve. As we are going to see in the next section, we propose a sub-optimal condition formulated in terms of LMIs and based on slack variables and rank restrictions that guarantees that $\|\mathcal{G}_K\|_2 < \gamma$. For the case in which all states are available, and also that $\hat{\theta} = \theta$, we are able to achieve the optimal H_2 state-feedback controller.

4 Main Result

In this section, we introduce the main design result in Theorem 1, discuss the special case of all states being available in Corollary 1, and recall the results in [10, 11] for comparison.

Considering that Assumption 1 is fulfilled, we get that there exists full rank real matrices $T_i \in \mathbb{B}(\mathbb{R}^n)$ for $i \in \mathbb{N}$ such that

$$L_i T_i = U \triangleq \begin{bmatrix} I_p & 0_{p\times n-p} \end{bmatrix}, \forall i \in \mathbb{N}. \tag{10}$$

Furthermore, we define $G_\ell \in \mathbb{B}(\mathbb{R}^n)$ for all $\ell \in \mathbb{M}_i$, $i \in \mathbb{N}$ with the following partition

$$G_\ell \triangleq \begin{bmatrix} G_{1\ell} & 0_{p\times n-p} \\ G_{2\ell} & G_{3\ell} \end{bmatrix} \tag{11}$$

for real matrices $G_{1\ell}$, $G_{2\ell}$, and $G_{3\ell}$ of compatible dimensions. For $s_{i\kappa} \geq 0$ such that $\sum_{\kappa\in\mathbb{K}} s_{i\kappa} = 1$, for all $i \in \mathbb{N}$, and $\mathbb{K} \subseteq \mathbb{N}$, we define $s_i \triangleq \{s_{i\kappa}; \kappa \in \mathbb{K}_i\}, i \in \mathbb{N}$, where $\mathbb{K}_i \triangleq \{\kappa \in \mathbb{K}; s_{i\kappa} > 0\}$, $\mathbb{K}_i \triangleq \{\underline{\kappa}_i, \ldots, \bar{\kappa}_i\}$, and

$$\mathcal{G}_i(s) \triangleq \begin{bmatrix} \sqrt{s_{i\underline{\kappa}_i}} I_n & \ldots, \sqrt{s_{i\bar{\kappa}_i}} I_n \end{bmatrix}'.$$

In what follows we will consider either $s_i = p_i$ and $\mathbb{K}_i = \mathbb{N}_i$ or $s_i = \alpha_i$ and $\mathbb{K}_i = \mathbb{M}_i$. For $M_i \triangleq \{M_{i\ell} \in \mathbb{B}(\mathbb{R}^n); \ell \in \mathbb{M}_i\}$, $i \in \mathbb{N}$, and $N \triangleq \{N_j \in \mathbb{B}(\mathbb{R}^n); j \in \mathbb{N}_i\}$, we set $\mathcal{P}_{i\alpha}(M) \triangleq \mathbf{diag}(M_{i\underline{\kappa}_i}, \ldots, M_{i\bar{\kappa}_i})$ and $\mathcal{P}_p(N) \triangleq \mathbf{diag}(N_{\underline{\kappa}_i}, \ldots, N_{\bar{\kappa}_i})$, respectively. We consider the following inequalities,

$$\sum_{i\in\mathbb{N}} \mu_i \mathrm{Tr}(W_i) < \upsilon \tag{12}$$

$$\begin{bmatrix} W_i & \bullet & \bullet \\ \mathcal{G}_i(p) J_i & \mathcal{P}_p(Q) & \bullet \\ E_i & 0 & I_q \end{bmatrix} > 0 \tag{13}$$

$$\begin{bmatrix} Q_i & \bullet \\ \mathcal{G}_i(\alpha) Q_i & \mathcal{P}_{i\alpha}(M) \end{bmatrix} > 0, \tag{14}$$

$$\begin{bmatrix} \mathrm{Her}(T_i G_\ell) - M_{i\ell} & \bullet & \bullet \\ \mathcal{G}_i(p)(A_i T_i G_\ell + B_i Y_\ell U) & \mathcal{P}_p(Q) & \bullet \\ C_i T_i G_\ell + D_i Y_\ell U & 0 & I_q \end{bmatrix} > 0 \tag{15}$$

for $i \in \mathbb{N}$, $\ell \in \mathbb{M}_i$. The next theorem provides a sub-optimal approach to (8).

Theorem 1 *Given $T_i \in \mathbb{B}(\mathbb{R}^n)$ as in (10), $i \in \mathbb{N}$, if there exists $\upsilon \in \mathbb{R}^+$, $W \in \mathbb{H}^{r+}$, $W > 0$, $Q \in \mathbb{H}^{n+}$, $Q > 0$, $M_{i\ell} > 0$, $i \in \mathbb{N}$, $\ell \in \mathbb{M}_i$, $M_{i\ell} \in \mathbb{B}(\mathbb{R}^n)$, $G_\ell \in \mathbb{B}(\mathbb{R}^n)$ partitioned as in (11), and $Y_\ell \in \mathbb{B}(\mathbb{R}^p, \mathbb{R}^m)$, $\ell \in \mathbb{M}$, such that (12)-(15) hold for all $i \in \mathbb{N}$, $\ell \in \mathbb{M}_i$, then by setting $K_\ell = Y_\ell G_{1\ell}^{-1}$ and $\gamma = \sqrt{\upsilon}$, we get that $K \in \mathfrak{K}$ and $\|\mathcal{G}_K\|_2 < \gamma$.* ■

Proof: First note that if (15) holds for all $i \in \mathbb{N}$, $\ell \in \mathbb{M}_i$, we get that $\mathrm{Her}(T_i G_\ell) - M_{i\ell} > 0$, and since $M_{i\ell} > 0$, then $T_i G_\ell$ is full rank. Recalling that T_i is full rank, we get that G_ℓ is non singular, see for instance, [40], and due to the lower triangular block structure of (11), we get that $G_{1\ell}$ is non singular as well. Thus it follows that

$$\begin{aligned} A_i T_i G_\ell + B_i Y_\ell U &= A_i T_i G_\ell + B_i K_\ell G_{1\ell} U \\ &= (A_i T_i + B_i K_\ell U) G_\ell \\ &= (A_i + B_i K_\ell L_i) T_i G_\ell = \bar{A}_{i\ell} T_i G_\ell \end{aligned}$$

where the second equality comes from the structure of G_ℓ in (11) and the last equality, from (10). Similarly $C_i T_i G_\ell + D_i Y_\ell U = (C_i T_i + D_i K_\ell U) G_\ell = \bar{C}_{i\ell} T_i G_\ell$. Recalling the results in [30] and [8], we get that $(T_i G_\ell)' M_{il}^{-1} (T_i G_\ell) \geq \mathrm{Her}(T_i G_\ell) - M_{i\ell}$, and thus from (15), we get that

$$\begin{bmatrix} (T_i G_\ell)' M_{il}^{-1} (T_i G_\ell) & \bullet & \bullet \\ \mathcal{G}_i(p) \bar{A}_{i\ell} T_i G_\ell & \mathcal{P}_p(Q) & \bullet \\ \bar{C}_{i\ell} T_i G_\ell & 0 & I_q \end{bmatrix} > 0. \tag{16}$$

By applying the congruence transformation $\mathbf{diag}((T_i G_i)^{-1}, I, I_q)$ to (16) and the Schur complement, we get that $M_{i\ell}^{-1} > \bar{A}'_{i\ell} \mathcal{E}_i(P) \bar{A}_{i\ell} + \bar{C}'_{i\ell} \bar{C}_{i\ell}$ where $P_i = Q_i^{-1}$ for all $i \in \mathbb{N}$, $\ell \in \mathbb{M}_i$. By applying the congruence transformation $\mathbf{diag}(Q_i^{-1}, I)$ to (14), and successively applying the Schur complement, we get that $P_i > \sum_{\ell \in \mathbb{N}_i} \alpha_{i\ell} M_{i\ell}^{-1} > \sum_{\ell \in \mathbb{N}_i} \alpha_{i\ell} [\bar{A}'_{i\ell} \mathcal{E}_i(P) \bar{A}_{i\ell} + \bar{C}'_{i\ell} \bar{C}_{i\ell}]$ for all $i \in \mathbb{N}$. Thus (7) holds and $K \in \mathfrak{K}$, and by Proposition 2, we get that $P > \bar{P}$, where $\bar{P}$ is the solution of (5). Furthermore, from (13), by applying similar steps as previously described, we get that $W_i > J_i' \mathcal{E}_i(P) J_i + E_i' E_i$, and thus by multiplying the last inequality by μ_i, summing up for all $i \in \mathbb{N}$, taking the trace operator, considering (12) and that $\bar{P} > P$, we get (9) so that $\|\mathcal{G}_K\|_2 < \gamma$, and thus the claim follows. ■

An approximation of the main goal in (8) can be now written as follows:

$$\inf_{\phi \in \Phi(T)} \{\upsilon : \text{ such that } (12) - (15) \text{ hold } \} \tag{17}$$

where $\phi = (\upsilon, G_\ell, Y_\ell, Q_i, M_{i\ell}, W_i, i \in \mathbb{N}, \ell \in \mathbb{M})$ and $\Phi(T)$ is the set of all solutions of (12)-(15) for a given $T = (T_1, \ldots, T_N)$.

We now analyze a by-product of Theorem 1, that is the state-feedback design. We show that, whenever we have perfect access to the system states, we retrieve the conditions of [28] for hidden MJLS. That is useful, as the conditions, for the mode-dependent case, leads to the optimal H_2 state-feedback control.

Corollary 1 *If $L_i = I_n$ for all $i \in \mathbb{N}$, then* (12)-(15) *are equivalent to Equations (14), (24)-(26) of [29].* ■

Proof: It follows by noting that $T_i = U = I_n$, $i \in \mathbb{N}$, if $L_i = I_n$, $\forall i \in \mathbb{N}$. ■

Next we revisit the result presented in [10, 11] that uses the two-step algorithm for solving (8), but adapted to our setting. For that, we consider the following inequalities

$$\begin{bmatrix} W_i & \bullet & \bullet \\ G_i J_i & \mathrm{Her}(G_i) - \mathcal{E}_i(P) & \bullet \\ E_i & 0 & I \end{bmatrix} > 0, \tag{18}$$

$$P_i > \sum_{\ell \in \mathbb{M}_i} \alpha_{i\ell} R_{i\ell}, \tag{19}$$

$$\begin{bmatrix} R_{i\ell} & \bullet & \bullet & \bullet \\ G_i(A_i + B_i H_\ell) & \mathrm{Her}(G_i) - \mathcal{E}_i(P) & \bullet & \bullet \\ C_i + D_i H_\ell & 0 & I & \bullet \\ X_\ell H_\ell - Y_\ell L_i & B_i' G_i' & D_i' & \mathrm{Her}(X_\ell) \end{bmatrix} > 0 \tag{20}$$

for all $i \in \mathbb{N}$, $\ell \in \mathbb{M}_i$, set $H = (H_1, \ldots, H_M)$, and define $\mathfrak{H}$ as the set of SS state-feedback controllers for (1) in the form $u(k) = F_{\hat{\theta}(k)} x(k)$. The following theorem is a direct adaptation of Theorem 2 of [11].

Theorem 2 *Given $H \in \mathfrak{H}$, if there exists $\upsilon \in \mathbb{R}^+$, $W \in \mathbb{H}^{r+}$, $W > 0$, $Z \in \mathbb{H}^n$, $G \in \mathbb{H}^n$, $P \in \mathbb{H}^{n+}$, $P > 0$, $R_{i\ell} \in \mathbb{B}(\mathbb{R}^n)$, $X_\ell \in \mathbb{B}(\mathbb{R}^m)$, $Y_\ell \in \mathbb{B}(\mathbb{R}^p, \mathbb{R}^m)$, such that* (12), (18)-(20) *hold for all $i \in \mathbb{N}$, $\ell \in \mathbb{M}_i$, then by setting $K_\ell = X_\ell^{-1} Y_\ell$ for all $\ell \in \mathbb{M}$ and $\gamma = \sqrt{\upsilon}$, we get that $K \in \mathfrak{K}$ and $\|\mathcal{G}_K\|_2 < \gamma$.*

Then, alternatively we rewrite (8) as follows

$$\inf_{\xi \in \Xi(H)} \{ \upsilon : \text{ such that } (12), (18) - (20) \text{ hold } \} \tag{21}$$

where $\xi \triangleq (\upsilon, W_i, P_i, G_i, R_{i\ell}, X_\ell, Y_\ell, i \in \mathbb{N}, \ell \in \mathbb{M})$ and $\Xi(H)$ is the set of solutions of (12), (18)-(20) for a given $H \in \mathfrak{H}$. The simple two-step procedure is illustrated by Algorithm 1. For improving the conservatism of the choice of $H \in \mathfrak{H}$ in the first step, a possible implementation is the iterative two-step procedure described in [11], which we do not pursue here.

Algorithm 1 The two-step procedure based on [31]

1: Calculate $H_\ell = Y_\ell G_{1\ell}^{-1}$ through Theorem 1 by setting $T_i = I_n$, $i \in \mathbb{N}$, and $U = I_n$;
2: Use $H \in \mathfrak{H}$ as an input to (21) and calculate $K \in \mathfrak{K}$.

The two methods of calculating the static output feedback control, described by the inequalities in (12)-(15) and (12), (18)-(20), have important similarities and differences. In the case of (12)-(15), we have to fix T, and this choice is strongly linked to the properties of the sensors being used, that is, to the matrices L. Regarding (12), (18)-(20), we note that if we consider H as a decision variable, the problem becomes bilinear and hard to solve. Then the solution is to fix H following some reasoning to find a suitable solution. We note the relationship of the existence of a solution of (12), (18)-(20) and the fact that system (1) must be stabilizable through state-feedback control, hence the assumption that $H \in \mathfrak{H}$. However both cases rely heavily on the choice of T or H and, if there are no solutions to (17) or (21), some workarounds should be adopted. For the former case, a change of basis of the states of (1) leads to a different T and may yield a feasible solution to (17). As for the latter case, another state-feedback controller must be calculated and tested in (21). Finally it is important to note that solving (21) through the two-step procedure described in Algorithm 1 is more computationally demanding than obtaining the solution from (17). This is evident as solving (17) (or some similar method) is required in the first step of Algorithm 1 in order to get the stabilizing state-feedback controller $H \in \mathfrak{H}$. Besides an important fact is that the matrices G_ℓ in (12)-(15) have a block fixed to a constant value. That may suggest that the bounds on the H_2 norm provided by Algorithm 1 may be less conservative when compared with the ones calculated through (12)-(15). In the next section, we will briefly expand this discussion through numerical simulations.

5 Illustrative Example

In this example, we briefly compare both methods employed in (17) and (21). For that we resort to the unstable lateral dynamics of the unmanned aircraft of [14]. Here we interchange the states of the original system through a simple coordinate transformation so that $x(t) = \begin{bmatrix}\Delta\beta(t) & \Delta\phi(t) & \Delta p(t) & \Delta r(t)\end{bmatrix}'$, where $\Delta\beta(t)$ is the variation in the sideslip angle, $\Delta\phi(t)$, the variation in the roll angle, $\Delta p(t)$, the variation in the roll rate, and $\Delta r(t)$, the variation in the yaw rate. The reason for doing so will be discussed shortly. We discretize the system through a zero-order hold with sampling time $T_s = 50$ ms so that

$$A_d \triangleq \begin{bmatrix} 0.9481 & 0.0159 & 0.0033 & -0.0450 \\ -0.0164 & 0.9999 & 0.0381 & 0.0073 \\ -0.6607 & -0.0062 & 0.5637 & 0.1133 \\ 1.0512 & 0.0089 & 0.0198 & 0.8368 \end{bmatrix},$$

$$B_d' \triangleq \begin{bmatrix} 0.0112 & 0.0812 & 2.9735 & -0.1175 \\ -0.0165 & -0.0006 & -0.0618 & 0.6414 \end{bmatrix}.$$

We assume that we can measure only the variations on the angles, namely $L_d \triangleq \begin{bmatrix} I_2 & 0_{2\times 2}\end{bmatrix}$, and that all states are subject to white noise, that is, $J_d \triangleq I_4$. The system is subject to faults on the rudder and we consider that the angle measurements can also be corrupted. In this case, we consider three possible states, the nominal mode of operation $\theta(k) = 1$, the case in which the rudder is faulty $\theta(k) = 2$, and the case in which the angle measurements are not correct $\theta(k) = 3$ so that $\mathbb{N} = \{1, 2, 3\}$. The transition probabilities are given as follows

$$[p_{ij}] = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.4 & 0.2 & 0.4 \\ 0.6 & 0.2 & 0.2 \end{bmatrix}$$

and then, we get that $\mu = (0.6222, 0.2000, 0.1778)$; $B_1 = B_3 = B_d$,

$$B_2 = B_d \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix};$$

$L_1 = L_2 = L_d$, $L_3 = 0.5L_d$; and finally $A_i = A_d$, $J_i = J_d$ for all $i \in \mathbb{N}$, along with

$$C_i = \begin{bmatrix} I_4 \\ 0_{2\times 4} \end{bmatrix}, D_i = \begin{bmatrix} 0_{4\times 2} \\ I_2 \end{bmatrix}$$

for all $i \in \mathbb{N}$. We assume that we cannot measure the exact state $\theta(k)$, but instead only estimation $\hat{\theta}(k)$ taken in the set $\mathbb{M} = \{1, 2, 3\}$ following the emission probabilities

$$[\alpha_{i\ell}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho & 1-\rho \\ 0 & 1-\rho & \rho \end{bmatrix}$$

where $\rho \in [0, 1]$ is the probability of correct detection, that is, $\mathrm{Prob}(\hat{\theta}(k) = i \mid \theta(k) = i) = \rho$ for $i \in \{2, 3\}$. We solve (17) for $\rho \in [0, 1]$ by setting $T_1 = I_4$ and $T_2 = \mathbf{diag}(2, 2, 1, 1)$. The minimum attained in (17) is represented by υ_1^*. The calculated υ_1^* and $\|\mathcal{G}_K\|_2^2$ are presented in Figure 1. We note that the

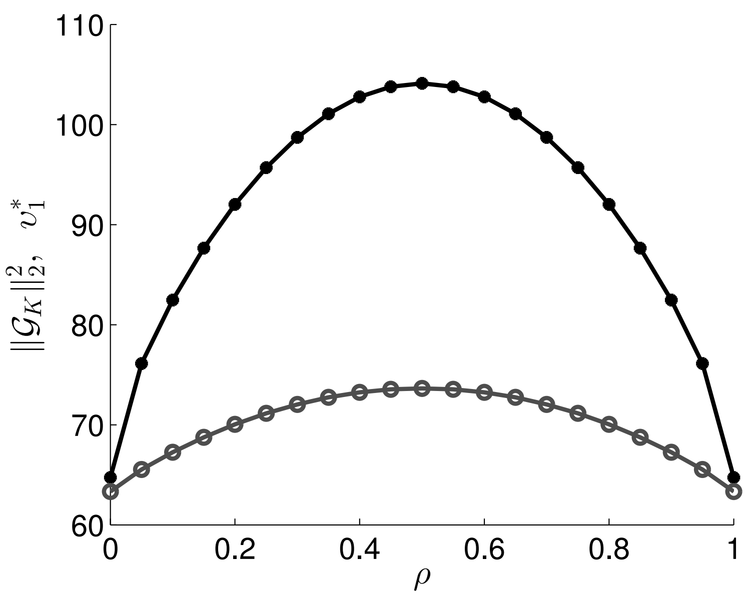

Figure 1: υ_1^* and $\|\mathcal{G}_K\|_2^2$ against ρ obtained through (17).

behavior of Figure 1 is consistent with the previous works on hidden MJLS, see, for instance [7, 11]. There is a maximum occurring in $\rho = 0.5$, so that the modes of operation $\theta(k) = 2$ and $\theta(k) = 3$ cannot be distinguished. In this case, there is a natural clusterization given by $\mathbb{N}^1 = \{1\}$ and $\mathbb{N}^2 = \{2, 3\}$ so that the controllers are given by

$$K_1 = \begin{bmatrix} 0.1170 & -0.2507 \\ -1.0436 & -0.0146 \end{bmatrix}, \quad K_2 = K_3 = \begin{bmatrix} 0.1499 & -0.3777 \\ -1.3429 & -0.0222 \end{bmatrix}$$

with $\upsilon_1^* = 104.1149$ and $\|\mathcal{G}_K\|_2^2 = 73.6410$, see Remark 1. Conversely, the minimum is attained for $\rho = 1$ which correspond to the mode-dependent case, see Remark 1, but also in $\rho = 0$: since we have three modes of operation and one of them can perfectly detected ($\theta(k) = 1$), then if we know for sure that the value of $\hat{\theta}(k)$ is wrong, we can always take the other one. It is noteworthy to point out that, even for the mode-dependent case, there is still a small degree of conservatism due to the sub-optimal approach used, since $\upsilon_1^* = 64.7260$ and $\|\mathcal{G}_K\|_2^2 = 63.3252$.

We can mention that the case in which $L_i = 0$ can be handled by this method if we set $B_i = 0$ and $D_i = 0$ in (15) and $T_i = I_n$. This is a common situation for modeling packet dropouts in NCS, see, for instance, the zeroing strategy discussed in [36]. Additionally, if the structure of L_i yields a permutation matrix T_i that moves the block $0_{n\times n-q}$ in (11) to the diagonal of (15), then it is not possible to find a solution to (17). However this limitation can be easily overcome by rearranging the states of the original system by a simple change of basis. This is the reason why the states of this example were changed with respect to the original source [14].

Let us move now to the two-step procedure of (21). As described in Algorithm 1, for calculating the stabilizing state-feedback controller $H_\ell = Y_\ell G_\ell^{-1}$ in the first step, we resort to Theorem 1 considering that $U = T_i = I_4$, $i \in \mathbb{N}$. Given H, we proceed in solving (21). The solutions of (21) for $\rho \in [0, 1]$ are represented by υ_2^* and shown in Figure 2. We note a similar behavior of υ_2^* and $\|\mathcal{G}_K\|_2^2$ against ρ compared to Figure 1, however the value of the costs in Figure 2 are smaller and less conservative in most of the interval of ρ. For the case $\rho = 0.5$, we get that $\upsilon_2^* = 69.7429$ and $\|\mathcal{G}_K\|_2^2 = 67.9100$, and for the mode-dependent case, $\upsilon_2^* = 63.9917$ and $\|\mathcal{G}_K\|_2^2 = 63.0027$. That is, in this example, the controllers obtained by (21) outperforms the ones given by (17) . That may be explained for the use of different state-feedback controllers calculated for each ρ as inputs for (21), whereas in (17) there is no such variation, and also that one of the blocks of G_ℓ in (11) is fixed to zero. Nonetheless, it is important to stress that using (21) amounts in solving two different optimization problems, one of them being (17) as illustrated in Algorithm 1, whereas in the first approach, we only have to solve (17).

6 Conclusion

We studied the H_2 static output feedback control for MJLS considering that the Markov chain cannot be perfectly measured and used in the control. In the place of θ, it is assumed that there is a variable $\hat{\theta}$ that could be viewed

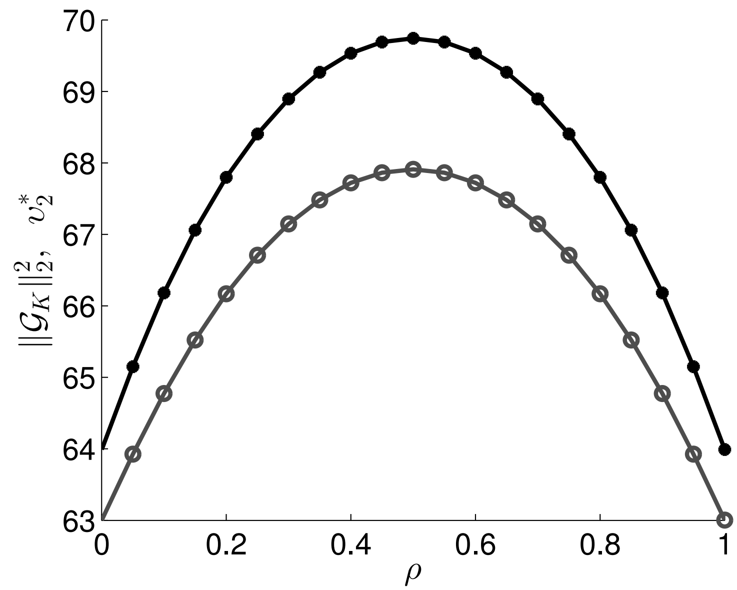

Figure 2: υ_2^* and $\|\mathcal{G}_K\|_2^2$ against ρ obtained through (21).

as a estimation of θ. Thus, the detector $\hat{\theta}$ is the only information that can be used by the controller, along with the measured output of the system, y. We introduce and discuss a new sub-optimal and convex result for obtaining the static output feedback controller depending only on $\hat{\theta}(k), y(k)$ that requires only simple assumptions concerning the rank of the sensor matrices. Finally we compare this design strategy with the two-step procedure of [11] in terms of the provided bounds on the H_2 norm of the closed-loop system through an academic application. This example indicates that the proposed technique can be a viable alternative with less computational effort when compared to the two-step procedure presented in [11], since the controller can be obtained in just one shot.

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