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NEW RESULTS ON WATER HAMMER STABILITY*

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Dedicated to Dr. Vasile Drăgan on the occasion of his 70$^{\text{th}}$ anniversary

Abstract

The present paper starts from a model with distributed parameters (i.e. described by hyperbolic partial differential equations with non-standard (derivative) boundary conditions) of a hydroelectric power plant with tunnel, surge tank and penstock. The association of a system of functional differential equations of neutral type and the one-to-one correspondence between the solutions of the two mathematical objects is given. Further, it is given the deduction - *via* singular perturbations - of the nonlinear ordinary differential equations for modeling the surge tank in order to discuss its stability under constant power delivery of the hydraulic turbine. Some other unsolved problems are pointed out.
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1 Introduction. State of the art

In the analysis of hydroelectric plant dynamics two basic phenomena are observable: water hammer and frequency/megawatt control [1, 2]. From

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the mathematical and engineering points of view, the water hammer is better described by distributed parameters (water wave propagation) while the second one is described by lumped parameters i.e. by ordinary differential equations. It may happen to these types of models to interact - there exist control models for turbine control dynamics which integrate the "water column" [3] - a standard distributed model - in the turbine dynamics.

A. Usually the two aspects of the modeling do not interact since water hammer is a research problem for civil engineers while frequency/megawatt control concerns power and control engineers. Both aspects can be followed having in mind the hydroelectric plant structure of Figure 1

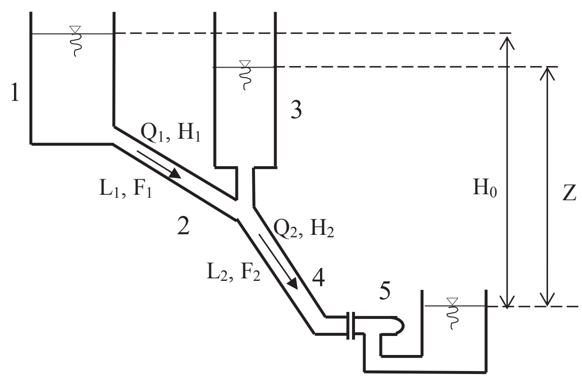

Figure 1: Hydroelectric plant structure. 1. Lake. 2. Tunnel. 3. Surge tank. 4. Penstock. 5. Hydraulic turbine.

This structure is common for the hydroelectric power plants throughout the world: such examples as *"Bicaz"* and *"Someş Mărişelu"* in Romania [1] or *"Tanzmühle"* in Germany [4, 5] illustrate this assertion.

The hydraulic turbine dynamics for control is a long term interest: this fact follows from the state of the art studies on turbine dynamics models which are made and published by the IEEE (Institute of Electrical and Electronic Engineers) at intervals around 20 years [6, 7, 8]. Such studies contain also a "bunch" of models for mechanical, hydraulic or electro-hydraulic controllers. Models of hydraulic turbines and their solutions for controllers can be found in [9].

On the other hand, the water hammer models are almost unchanged along several decades since they strongly rely on the Saint Venant equations: one can meet them in such references as [10, 11, 1]. It is true that the computational approaches to water hammer have progressed since the basics of Joukowsky and Allievi, due to the development of computing methods and computers themselves (both hardware and software). Nevertheless, an evergreen problem continues to exist - the surge tank role and design. The surge tank appeared as "smoother" of the water hammer, having thus a stabilizer role for water mass oscillations [11, 1]. Since it accumulates a huge mass of water, the surge tank may be viewed as a filtering capacitor in some electronic filter but unlike the electric capacitor, the surge tank is a construction hence its design must avoid errors. This fact explains the continued interest for surge tanks along decades [11, 1]. As it appears from some recent papers [12, 13], in the water hammer process, when the hydraulic turbine with its controller are decoupled from the upstream structure (Figure 1), the surge tank remains the only stabilizer of the water hammer process. On the other hand, some recent studies show a certain influence of the surge tank on the control process of the hydraulic turbine [14, 15] thus pointing towards an interaction of the upstream and turbine dynamics.

B. The considerations presented previously indicate the usefulness of a unified model for the two phenomena - water hammer and frequency/megawatt control. We can mention here the pioneering papers [16, 17, 18] where distributed parameters are in pair with the lumped ones. These papers deal however with relatively small power plants without surge tanks, with the tunnel uniting directly the lake and the turbine. An interesting, even pioneering model, can be met in [19] where the structure of Figure 1 is modeled *via* the Saint Venant equations and a surge tank with throttling; since the authors are interested in water hammer, the turbine models are missing. The main contributions of the paper are the model with several time scales and the stability analysis of an associated system of functional differential equations of neutral type. Further studies [12, 13] have completed the water hammer stability analysis of [19].

C. The present paper continues the aforementioned papers by completing the model of the surge tank [19]with the mass oscillation dynamics [11] in the context of the water hammer model with distributed parameters and by using a more recent formula for hydraulic turbine active torque [2] instead of the oldest one in [17]. On the new model we shall discuss the time scales of the model and possible simplifying assumptions. The role of the singular perturbations and, consequently, the importance of the approaches of [20] will be thus pointed out. Other issues of the analysis will be concerned with steady states and inherent stability of the surge tank itself.

2 The equations of the mathematical model and its parameters

The basic equations are those of all textbooks of the field, with additional terms accounting for various local head losses, borrowed from [19, 1], also from [11, 21, 22, 23]

$$
\begin{aligned}
&\partial_{x_i}\left(H_i + \frac{V_i^2}{2g}\right) + \frac{1}{g}\partial_t V_i + \frac{\lambda_i}{2D_i}V_i|V_i| = 0 \\
&\partial_t H_i + \frac{a_i^2}{2g}\partial_{x_i} V_i = 0\ ;\ t > 0\ ,\ 0 < x_i < L_i\ ;\ i = 1,2 \\
&Q_i = F_i V_i\ ;\ H_1(0,t) = H_0 \\
&H_1(L_1,t) + \frac{V_1^2(L_1,t)}{2g} = Z(t) + \lambda_s \frac{\mathrm{d}Z}{\mathrm{d}t} = H_2(0,t) + \frac{V_2^2(0,t)}{2g} \\
&F_s \frac{\mathrm{d}Z}{\mathrm{d}t} = Q_1(L_1,t) - Q_2(0,t) \\
&Q_2(L_2,t) = (1-k)\alpha_g F_\theta(t)\sqrt{H_2(L_2,t)} + k\bar{Q}\Omega_c\Omega(t) \\
&J\Omega_c \frac{\mathrm{d}\Omega}{\mathrm{d}t} = \eta \frac{\gamma}{2g} Q_2(L_2,t) H_2(L_2,t) - N_g
\end{aligned}
\tag{1}
$$

where $i = 1$ accounts for the tunnel and $i = 2$ for the penstock; the significance of the notations for state variables and technical parameters is given in the Appendix.

In comparison to other models we used e.g. [12, 13], this basic model contains the turbine speed correction factor $k \in (0,1)$ in the formula for the wicket gate flow $Q_2(L_2,t)$, according to [11], and the more recent formula for the turbine active power [2] in the differential equation of Ω, also the dynamic heads in the boundary conditions corresponding to the surge tank throttle.

For the mathematical treatment as well as for the numerical simulations, equations (1) undergo some transformations which can be defined as *introduction of the p.u. (per unit) variables and of the rated conduit lengths* as follows. The flows $Q_i = F_i V_i$ are rated to the maximally available flow $\bar{Q} = \alpha_q F_{\theta max}\sqrt{H_0}$; the piezometric heads H_i, Z are rated to the piezometric head H_0 of the lake (Figure 1) and the rotating speed Ω is rated to the synchronous speed Ω_c equal to 3000 rpm in Europe and 3600 rpm in USA.

In the following we shall denote by lower case letters the rated variables q_i, h_i, z; let $\varphi := \Omega/\Omega_c$ be the rated rotating speed and f_θ - the rated cross section area of the turbine wicket gates. The p.u. lengths of the two water conduits are defined as $\xi_i = x_i/L_i$, $i = 1, 2$. The available power of the hydraulic turbine is rated to the value

$$\bar{N}_a = \eta \frac{\gamma}{2g} \bar{Q} H_0 \;\Rightarrow\; \nu_g = N_g/\bar{N}_a$$

In the process of introducing the p.u. variables, the model coefficients are transformed and the following time constants can be introduced for each of the two conduits ($i = 1, 2$):

- the water starting time constant $T_{wi} := \frac{L_i}{H_0} \cdot \frac{1}{g} \cdot \frac{\bar{Q}}{F_i}$;
- the fill up time constant $T_i := \frac{L_i F_i}{\bar{Q}}$;
- the wave propagation time constant $T_{pi} := \frac{L_i}{a_i}$.

For the surge tank there is introduced the fill up time constant T_s and for the hydraulic turbine the time constant of the turning masses T_a as follows

$$T_s := \frac{F_s H_0}{\bar{Q}} \;,\; T_a := \frac{J \Omega_c^2}{\bar{N}_a}$$

With these new parameters - time constants - thus introduced, we can now introduce the p.u. time variable by rating t to the largest of the time constants defined above - the fill up time constant T_1 of the tunnel - $\tau = t/T_1$. After all aforementioned transformations we make the additional notations

$$\begin{aligned} &T_{pi}/T_{wi} = \delta_i \;,\; T_{wi}/T_1 = \theta_{wi} \;,\; T_s/T_1 = \theta_s \;,\; T_{pi}/T_1 = \theta_{pi} \\ &T_i/T_1 = \theta_i(\theta_1 = 1) \;,\; T_{wi}/T_i = \gamma_i \;,\; \lambda_s' = \lambda_s/T_1 \end{aligned}$$

to introduce the following working model

$$\begin{aligned} &\theta_{wi}\partial_\tau q_i + \partial_{\xi_i}(h_i + \frac{1}{2}\gamma_i q_i^2) + \frac{1}{2}(\lambda_i g)\frac{L_i}{D_i}\gamma_i q_i|q_i| = 0 \\ &\delta_i^2 \theta_{wi} \partial_\tau h_i + \partial_{\xi_i} q_i = 0 \;;\; t > 0 \;,\; 0 < \xi_i < 1 \;;\; i = 1, 2 \\ &h_1(0,\tau) = 1 \;;\; \theta_s \frac{\mathrm{d}z}{\mathrm{d}\tau} = q_1(1,\tau) - q_2(0,\tau) \\ &h_1(1,\tau) + \frac{1}{2}\gamma_1 q_1^2(1,\tau) = z(\tau) + \lambda_s' \frac{\mathrm{d}z}{\mathrm{d}\tau} = h_2(0,\tau) + \frac{1}{2}\gamma_2 q_2^2(0,\tau) \\ &q_2(1,\tau) = f_\theta(\tau)(k\varphi + (1-k)\mathcal{F}(h_2(1,\tau))) \\ &\theta_a \frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = q_2(1,\tau) h_2(1,\tau) - \nu_g \end{aligned} \tag{2}$$

where the function $\mathcal{F}(X)$ introduced in (2) by

$$\mathcal{F}(X) = \begin{cases} 0 \ , \ X \leq 0 \\ \sqrt{X} \ , \ X \geq 0 \end{cases}$$

takes into account that "there is no flow from a lower piezometric head to a bigger one" ("water does not flow from downstream to upstream").

3 Model properties

In what follows we shall consider equations (2). These equations define a nonlinear boundary value problem for nonlinear partial differential equations. In the absence of the terms describing the distributed Darcy Weisbach losses the equations are *hyperbolic conservation laws*.

3.1 Several time scales

We shall use the numerical data of two hydroelectric plants of Romania - "Bicaz" and "Someş-Mărişelu" [1] to obtain the following values for the coefficients and the time constants

1^o For "Bicaz" hydroelectric plant:

$$\begin{aligned} &\theta_{w1} = T_{w1}/T_1 = 0.0146 \ , \ \delta_1 = T_{p1}/T_{w1} = 0.26 \ , \ \delta_1^2 \theta_{w1} \approx 10^{-4} \\ &\theta_s = T_s/T_1 = 0.5 \ , \ \gamma_2 = T_{w2}/T_2 = 8.64 \times 10^{-3} \ , \\ &\theta_{w2} = T_{w2}/T_1 = 3.8 \times 10^{-4} \ , \ \delta_2 = T_{p2}/T_{w2} = 0.374 \ , \\ &\delta_2^2 \theta_{w2} = 0.53 \times 10^{-4} \ , \ \theta_a = T_a/T_1 \approx 8 \times 10^{-3} \end{aligned}$$

2^o For "Someş-Mărişelu" hydroelectric plant:

$$\begin{aligned} &\theta_{w1} = 0.022 \ , \ \delta_1 = 1.15 \ , \ \delta_1^2 \theta_{w1} \approx 0.03 \\ &\theta_s = 1.011 \ , \ \gamma_2 = 6.5 \times 10^{-3} \ , \ \theta_{w2} = 3.4 \times 10^{-4} \\ &\delta_2 = 0.74 \ , \ \delta_2^2 \theta_{w2} = 2 \times 10^{-4} \ , \ \theta_a = 0.038 \end{aligned}$$

To point out the time scales, we have to compare the coefficients multiplying the time derivatives in (2). It appears that in both cases mentioned above, the largest time scales are of the surge tank dynamics, followed by the time scales of the tunnels. These facts explain the engineering approaches for hydroelectric plant dynamics. The *dynamics of frequency/megawatt control* integrates the turbine dynamics and the penstock dynamics; moreover, the penstock dynamics is assimilated to a lumped parameter one. The *dynamics of the water hammer* integrates the slow dynamics of the tunnel and of the surge tank; due to the interconnection of the two conduits through the boundary conditions, the faster dynamics of the penstock is also taken into account either with distributed parameters or with lumped parameters. Combining various situations can send to a "bunch" of models; here we shall discuss some of them.

3.2 Basic theory for the overall model

By basic theory it is understood existence, uniqueness and "good" data (parameters and initial conditions) dependence i.e. what is called *well posedness in the sense of Hadamard*. In our case we deal with quasilinear partial differential equations having *nonstandard* nonlinear boundary conditions. We call the boundary conditions *nonstandard* since they contain ordinary differential equations coupled to the standard boundary conditions, be they of Dirichlet or Neumann type.

The basic theory for such boundary value problems for hyperbolic partial differential equations started to be constructed since the papers of A. D. Myshkis and his co-workers [24, 25, 26, 27]; simultaneously K. L. Cooke published a similar research [28, 29] but his proofs were not complete; the fully proven result is to be found in [30]. On the other hand, as already mentioned, the partial differential equations of (2) arise from conservation laws; for them the basic theory displays additional difficulties. Results are known mainly for classical solutions [31, 32], sometimes for generalized ones e.g. [33]. Fortunately, in the case of water flow control, the classical solutions are the more adequate ones.

In this paper we shall follow the approach of [29, 30], already applied in [12, 13], by neglecting the dynamic head and the Darcy Weisbach losses, followed by the association of a system of neutral functional differential equations to the modified equations (2) (through the aforementioned neglecting).

Neglecting of the dynamic head $(1/2)\gamma_i q_i^2$ is based on experimental data showing that its space variation (with respect to ξ_i) is negligible in comparison to the same variation of the piezometric head h_i (some numerical data, gathered from several hundreds of hydroelectric plants in former USSR, can be found in [11]). This assertion is made for absolute values, not for p.u. but the fact that in (2), for the considered cases of Romania, the ratios γ_i range from 0.02 to 10^{-4} can be also an argument.

Neglecting of the Darcy Weisbach losses terms is considered to be covering from the engineering point of view concerning stability: since stability is improved by energy dissipation, additional losses can only improve the stability properties obtained by neglecting them. Therefore the first two lines of (2) will be modified and the whole boundary value problem becomes

$$\begin{aligned}
&\theta_{wi}\partial_\tau q_i + \partial_{\xi_i} h_i = 0\ ,\ \delta_i^2\theta_{wi}\partial_\tau h_i + \partial_{\xi_i} q_i = 0\\
&h_1(0,\tau) = 1\ ,\ h_1(1,\tau) = z(\tau) + \lambda_s'\frac{\mathrm{d}z}{\mathrm{d}\tau} = h_2(0,\tau)\\
&\theta_s\frac{\mathrm{d}z}{\mathrm{d}\tau} = q_1(1,\tau) - q_2(0,\tau)\ ;\ q_2(1,\tau) = f_\theta(\tau)(k\varphi + (1-k)\mathcal{F}(h_2(1,\tau)))\\
&\theta_a\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = q_2(1,\tau)h_2(1,\tau) - \nu_g
\end{aligned} \tag{3}$$

These equations define a problem for the linear lossless wave equation with nonstandard nonlinear boundary conditions. In order to apply the method of [29, 30], we introduce first the Riemann invariants by

$$\begin{aligned}
r_i^\pm(\xi_i,\tau) &= h_i(\xi_i,\tau) \pm \frac{1}{\delta_i}q_i(\xi_i,\tau)\\
h_i(\xi_i,\tau) &= \frac{1}{2}(r_i^+(\xi_i,\tau) + r_i^-(\xi_i,\tau))\ ;\\
q_i(\xi_i,\tau) &= \frac{1}{2}\delta_i(r_i^+(\xi_i,\tau) - r_i^-(\xi_i,\tau))
\end{aligned} \tag{4}$$

to obtain the boundary value problem (3) written in the Riemann invariants

$$\begin{aligned}
&\theta_{pi}\partial_\tau r_i^\pm \pm \partial_{\xi_i} r_i^\pm = 0\ ,\ t>0\ ,\ 0<\xi_i<1\ ;\ i=1,2\\
&r_1^+(0,\tau) + r_1^-(0,\tau) = 2\\
&\frac{1}{2}(r_1^+(1,\tau) + r_1^-(1,\tau)) = z(\tau) + \lambda_s'\frac{\mathrm{d}z}{\mathrm{d}\tau} = \frac{1}{2}(r_2^+(0,\tau) + r_2^-(0,\tau))\\
&\theta_s\frac{\mathrm{d}z}{\mathrm{d}\tau} = \frac{1}{2}\delta_1(r_1^+(1,\tau) - r_1^-(1,\tau)) - \frac{1}{2}\delta_2(r_2^+(0,\tau) - r_2^-(0,\tau))\\
&\frac{1}{2}\delta_2(r_2^+(1,\tau) - r_2^-(1,\tau)) = f_\theta(\tau)\left[k\varphi + \frac{1-k}{\sqrt{2}}\mathcal{F}(r_2^+(1,\tau) + r_2^-(1,\tau))\right]\\
&\theta_a\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = \frac{1}{4}\delta_2(r_2^+(1,\tau) - r_2^-(1,\tau))(r_2^+(1,\tau) + r_2^-(1,\tau)) - \nu_g
\end{aligned} \tag{5}$$

with θ_{pi} already defined as T_{pi}/T_1; observe that $\theta_{pi} = \delta_i\theta_{wi}$.

The characteristic lines of (5) are given by

$$\tau_i^\pm(\sigma;\xi_i,\tau) = \tau \pm \theta_{pi}(\sigma - \xi_i)\ ,\ i=1,2 \tag{6}$$

defining for each wave equation ($i = 1, 2$) the two characteristic lines crossing some point $(\xi_i, \tau) \in [0, 1] \times \mathbb{R}_+$. The Riemann invariants are constant along the characteristic lines: $r_i^+(\cdot; \xi_i, \tau)$ along τ_i^+ and $r_i^-(\cdot; \xi_i, \tau)$ along τ_i^-. From here we obtain the representation formulae

$$\begin{aligned} r_i^+(\xi_i, \tau) &= r_i^+(1, \tau + \theta_{pi}(1 - \xi_i)) \\ r_i^-(\xi_i, \tau) &= r_i^-(0, \tau + \theta_{pi}\xi_i)\ ,\ i = 1, 2 \end{aligned} \tag{7}$$

expressing the Riemann invariants by their values at the boundaries. If (ξ_i, τ) are such that the characteristic lines can be extended to the left up to $\xi_i = 0$ (the τ_i^+ one) and to the right up to $\xi_i = 1$ (the τ_i^- one) without crossing firstly the line $\tau = 0$ in the strip $(0, 1) \times \mathbb{R}_+$, then we can write down

$$r_i^+(0, \tau) = r_i^+(1, \tau + \theta_{pi})\ ,\ r_i^-(1, \tau) = r_i^-(0, \tau + \theta_{pi}) \tag{8}$$

Defining the functions

$$y_i^+(\tau) := r_i^+(1, \tau)\ ,\ y_i^-(\tau) := r_i^-(0, \tau) \tag{9}$$

and taking into account (8), we substitute them in the boundary conditions of (5) to obtain the following system

$$\begin{aligned} & y_1^+(\tau + \theta_{p1}) + y_1^-(\tau) = 2 \\ & \frac{1}{2}(y_1^+(\tau) + y_1^-(\tau + \theta_{p1})) = z(\tau) + \lambda_s' \frac{\mathrm{d}z}{\mathrm{d}\tau} = \frac{1}{2}(y_2^+(\tau + \theta_{p2}) + y_2^-(\tau)) \\ & \theta_s \frac{\mathrm{d}z}{\mathrm{d}\tau} = \frac{1}{2}\delta_1(y_1^+(\tau) - y_1^-(\tau + \theta_{p1})) - \frac{1}{2}\delta_2(y_2^+(\tau + \theta_{p2}) - y_2^-(\tau)) \\ & \frac{1}{2}\delta_2(y_2^+(\tau) - y_2^-(\tau + \theta_{p2})) = f_\theta(\tau)\left[k\varphi + \frac{1-k}{\sqrt{2}}\mathcal{F}(y_2^+(\tau) + y_2^-(\tau + \theta_{p2}))\right] \\ & \theta_a \frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = \frac{1}{4}\delta_2(y_2^+(\tau) - y_2^-(\tau + \theta_{p2}))(y_2^+(\tau) + y_2^-(\tau + \theta_{p2})) - \nu_g \end{aligned} \tag{10}$$

The next step of the development will be to give system (10) a form allowing the construction by steps of the solution. This step will require some tedious manipulation. We denote $w_i^\pm(\tau) := y_i^\pm(\tau + \theta_{pi})$ and re-write the *difference subsystem* as follows

$$\begin{aligned} & w_1^+(\tau) + w_1^-(\tau - \theta_{p1}) = 2 \\ & w_1^-(\tau) + w_1^+(\tau - \theta_{p1}) = 2z(\tau) + 2\lambda_s' \frac{\mathrm{d}z}{\mathrm{d}\tau} = w_2^+(\tau) + w_2^-(\tau - \theta_{p2}) \\ & \frac{1}{2}\delta_2(w_2^+(\tau - \theta_{p2}) - w_2^-(\tau)) = f_\theta(\tau)\left[k\varphi + \frac{1-k}{\sqrt{2}}\mathcal{F}(w_2^-(\tau) + w_2^+(\tau - \theta_{p2}))\right] \end{aligned}$$

where $\mathcal{F}(\cdot)$ has been already defined. The first three equations above are linear and can be given the form allowing the construction by steps. We need however for this to consider the first (linear) differential equation

$$2\theta_s \frac{\mathrm{d}z}{\mathrm{d}\tau} = -(\delta_1 w_1^-(\tau) + \delta_2 w_2^+(\tau)) + (\delta_1 w_1^+(\tau - \theta_{p1}) + \delta_2 w_2^-(\tau - \theta_{p2}))$$

and substitute $\mathrm{d}z/\mathrm{d}\tau$ in the difference system to obtain, after the inversion of a 2×2 matrix

$$\begin{aligned}
&w_1^+(\tau) = 2 - w_1^-(\tau - \theta_{p1}) \\
&w_1^-(\tau) = \frac{1}{1 + (\delta_1 + \delta_2)\lambda_s'/\theta_s}[2z(\tau) - (1 + (\delta_2 - \delta_1)\lambda_s'/\theta_s)w_1^+(\tau - \theta_{p1}) + \\
&\qquad +2\delta_2(\lambda_s'/\theta_s)w_2^-(\tau - \theta_{p2})] \\
&w_2^+(\tau) = \frac{1}{1 + (\delta_1 + \delta_2)\lambda_s'/\theta_s}[2z(\tau) + 2\delta_1(\lambda_s'/\theta_s)w_1^+(\tau - \theta_{p1}) - \\
&\qquad -(1 + (\delta_1 - \delta_2)\lambda_s'/\theta_s)w_2^-(\tau - \theta_{p2})]
\end{aligned}$$

The last, nonlinear difference equation, can be given the form

$$\begin{aligned}
&\frac{1}{2}\delta_2(w_2^-(\tau) + w_2^+(\tau - \theta_{p2})) + \frac{1-k}{\sqrt{2}}\mathcal{F}(w_2^-(\tau) + w_2^+(\tau - \theta_{p2})) = \\
&\qquad = -kf_\theta(\tau)\varphi(\tau) + \delta_2 w_2^+(\tau - \theta_{p2})
\end{aligned}$$

Let τ be fixed and consider the function $(1/2)\delta_2 X + ((1-k)/\sqrt{2})\mathcal{F}(X)$ which is strictly increasing for all $X \in \mathbb{R}$ hence invertible. Observe that the sign of the left hand side of the nonlinear difference equation under its new form is tested by the right hand side. If this last sign is negative then $\mathcal{F}(X) = 0$ and the difference equation results

$$w_2^-(\tau) = w_2^+(\tau - \theta_{p2}) - \frac{2k}{\delta_2}\varphi(\tau)$$

If the sign is positive, then $\mathcal{F}(X) = \sqrt{X}$ and, with $Y := \sqrt{X}$, the following quadratic equation is obtained

$$\frac{1}{2}\delta_2 Y^2 + \frac{1-k}{\sqrt{2}}f_\theta Y - (\delta_2 w_2^+(\tau - \theta_{p2}) - kf_\theta\varphi) = 0$$

Let Y^+ be the positive root of this equation, $X := (Y^+)^2$ and the corresponding recurrence is obtained. Unifying the two cases, we obtain the following nonlinear recurrence

$$w_2^-(\tau) = w_2^+(\tau-\theta_{p2}) - \frac{2k}{\delta_2}\varphi(\tau) + \left(\frac{1-k}{\delta_2}\right)^2 f_\theta \mathcal{G}(\delta_2 w_2^+(\tau-\theta_{p2}) - kf_\theta\varphi)$$

where

$$\mathcal{G}(X) = \begin{cases} f_\theta - \sqrt{f_\theta^2 + \dfrac{4\delta_2}{(1-k)^2}X}\ ,\ X \geq 0 \\ 0\ ,\ X \leq 0 \end{cases}$$

is continuous at $X = 0$.

We can now turn to the differential equations of (10) where we substitute $y_i^\pm$ by $w_i^\pm$ and take into account the newly obtained difference equations written as recurrences. Summarizing, we have associated to system (3), *via* (4), (10), the following nonlinear system of coupled delay differential and difference equations

$$\begin{aligned}
&(\theta_s + (\delta_1+\delta_2)\lambda_s')\frac{\mathrm{d}z}{\mathrm{d}\tau} = -(\delta_1+\delta_2)z(\tau) + \delta_1 w_1^+(\tau-\theta_{p1}) + \delta_2 w_2^-(\tau-\theta_{p1}) \\
&\theta_a\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = \left[kf_\theta\varphi - \frac{(1-k)^2}{2\delta_2}\mathcal{G}(\delta_2 w_2^+(\tau-\theta_{p2}) - kf_\theta\varphi)\right]\times \\
&\times\left[w_2^+(\tau-\theta_{p2}) - \frac{k}{\delta_2}f_\theta\varphi + \frac{1}{2}\left(\frac{1-k}{\delta_2}\right)^2\mathcal{G}(\delta_2 w_2^+(\tau-\theta_{p2}) - kf_\theta\varphi)\right] - \nu_g \\
&w_1^+(\tau) = 2 - w_1^-(\tau-\theta_{p1}) \\
&w_1^-(\tau) = \frac{1}{1+(\delta_1+\delta_2)\lambda_s'/\theta_s}[2z(\tau) - (1+(\delta_2-\delta_1)\lambda_s'/\theta_s)w_1^+(\tau-\theta_{p1}) + \\
&\qquad +2\delta_2(\lambda_s'/\theta_s)w_2^-(\tau-\theta_{p2})] \\
&w_2^+(\tau) = \frac{1}{1+(\delta_1+\delta_2)\lambda_s'/\theta_s}[2z(\tau) + 2\delta_1(\lambda_s'/\theta_s)w_1^+(\tau-\theta_{p1}) - \\
&\qquad -(1+(\delta_1-\delta_2)\lambda_s'/\theta_s)w_2^-(\tau-\theta_{p2})] \\
&w_2^-(\tau) = w_2^+(\tau-\theta_{p2}) - \frac{2k}{\delta_2}f_\theta\varphi + \left(\frac{1-k}{\delta_2}\right)^2\mathcal{G}(\delta_2 w_2^+(\tau-\theta_{p2}) - kf_\theta\varphi)
\end{aligned} \tag{11}$$

The solution of this system can be constructed *by steps* as follows: if $w_i^{\pm}(\cdot)$ are given on the initial intervals $[-\theta_{pi}, 0)$, then the solutions of the two differential equations of (11) can be constructed on $(0, \theta_{pi})$ provided $f_\theta(\tau)$ is known and the initial conditions $z(0), \varphi(0)$ are given. Since $z(\tau)$ and $\varphi(\tau)$ are now known on $(0, \theta_{pi})$, $w_i^{\pm}(\tau)$ can be obtained from the difference equations on the aforementioned intervals. The process is then iterated on the next intervals. The resulting solution appears to be continuous and piecewise differentiable (the variables z, φ) while the variables $w_i^{\pm}$ have the smoothness of their initial conditions and, in general, have finite discontinuities ("jumps") in $\tau = m_1\theta_{p1} + m_2\theta_{p2}$, where m_i are integers. It is also clear that the solution of (11) can be constructed also backwards.

The only "missing link" of (11) with (3) is given by the initial conditions. Clearly $(z(0), \varphi(0)$ "migrate" from (3) to (11), therefore we can focus on the initial conditions for $w_i^{\pm}(\cdot)$ on $(-\theta_{pi}, 0)$. Starting from the initial conditions of (3) namely $(q_i^{\mathrm{o}}(\xi_i), h_i^{\mathrm{o}}(\xi_i))$ given on $0 \leq \xi_i \leq 1$, we use (4) to obtain $r_{i\mathrm{o}}^{\pm}(\xi_i)$ on $0 \leq \xi_i \leq 1$. Next we use again the representation formulae as follows.

Consider those points (ξ_i, τ) which are such that the characteristics $\tau_i^+(\sigma; \xi_i, \tau) = \tau + \theta_{pi}(\sigma - \xi_i)$ cannot be extended "to the left" up to $\xi_i = 0$ but only to the point where $\tau + \theta_{pi}(\sigma - \xi_i)$ i.e. up to $\sigma = \xi_i - \tau/\theta_{pi}$. It follows that

$$r_i^+(\xi_i - \tau/\theta_{pi}, 0) = r_i^+(1, \tau + \theta_{pi}(1 - \xi_i)) = w_i^+(\tau - \theta_{pi}\xi_i)$$

Since $0 \leq \xi_i - \tau/\theta_{pi} \leq 1$, it follows that $w_{i\mathrm{o}}^+(\theta) = r_{i\mathrm{o}}^+(-\theta/\theta_{pi})$ for $-\theta_{pi} \leq \theta 0$. In the same way, using those characteristic lines $\tau_i^-(\sigma; \xi_i, \tau)$ which cannot be extended to $\sigma = 1$ but only to the point where $\tau - \theta_{pi}(\sigma - \xi_i) = 0$ i.e. to $\sigma = \xi_i + \theta/\theta_{pi}$, the following initial condition is obtained

$$r_i^-(\xi_i + \tau/\theta_{pi}, 0) = r_i^-(0, \tau + \theta_{pi}\xi_i) = w_i^-(\tau + \theta_{pi}(\xi_i - 1))$$

hence $w_{i\mathrm{o}}^-(\theta) = r_i^-(1 + \theta/\theta_{pi})$ for $-\theta_{pi} \leq \theta 0$.

Consider now the converse: let $\{z(0), \varphi(0), w_{i\mathrm{o}}^{\pm}(\theta), -\theta_{pi} \leq \theta 0\}$ be a set of initial conditions for (11). Define

$$\begin{aligned} r_i^+(\xi_i, \tau) &= w_i^+(\tau - \theta_{pi}\xi_i)\ ,\ r_i^-(\xi_i, \tau) = w_i^-(\tau + \theta_{pi}(\xi_i - 1)) \\ h_i(\xi_i, \tau) &= \frac{1}{2}[r_i^+(\xi_i, \tau) + r_i^-(\xi_i, \tau)] = \\ &= \frac{1}{2}[w_i^+(\tau - \theta_{pi}\xi_i) + w_i^-(\tau + \theta_{pi}(\xi_i - 1))] \\ q_i(\xi_i, \tau) &= \frac{1}{2}\delta_i[r_i^+(\xi_i, \tau) - r_i^-(\xi_i, \tau)] = \\ &= \frac{1}{2}\delta_i[w_i^+(\tau - \theta_{pi}\xi_i) - w_i^-(\tau + \theta_{pi}(\xi_i - 1))] \end{aligned} \tag{12}$$

Then, if $w_{io}^{\pm}(\theta)$ are sufficiently smooth, the set of functions $\{z(\tau), \varphi(\tau); h_i(\xi_i, \tau), q_i(\xi_i, \tau)\}$ is a (possibly discontinuous) classical solution of (3) with the initial conditions $\{\{z(0), \varphi(0); h_i(\xi_i, 0), q_i(\xi_i, 0)\}$. Summarizing, we have obtained and proven the following result

Theorem 1. *Consider the boundary value problem defined by (3) and a set of initial conditions* $\{z(0), \varphi(0); h_{io}(\xi_i), q_{io}(\xi_i), 0 \leq \xi_i \leq 1, i = 1, 2\}$ *with* $\{h_{io}, q_{io}\}$ *to define a classical solution for (3). Let* $r_i^{\pm}(\xi_i, \tau)$ *defined by (4) be the corresponding Riemann invariants of this solution. Let* $y_i^{\pm}(\tau)$ *be defined by (9) and* $w_i^{\pm}(\tau) := y_i^{\pm}(\tau + \theta_{pi})$. *Then* $\{z(\tau), \varphi(\tau); w_i^{\pm}(\tau)\}$ *is a solution of (11) constructed by steps, starting from the initial conditions* $\{z(0), \varphi(0); w_{io}(\tau), -\theta_{pi} \leq \tau \leq 0\}$ *where*

$$\begin{aligned} w_{io}^{+}(\tau) &= r_{io}^{+}(-\tau/\theta_{pi}) = h_{io}(-\tau/\theta_{pi}) + q_{io}(-\tau/\theta_{pi})/\delta_i \\ w_{io}^{-}(\tau) &= r_{io}^{-}(1 + \tau/\theta_{pi}) = h_{io}(1 + \tau/\theta_{pi}) + q_{io}(1 + \tau/\theta_{pi})/\delta_i \end{aligned} \tag{13}$$

Conversely, let $\{z(\tau), \varphi(\tau); w_i^{\pm}(\tau)\}$ *be a solution of (11) defined by the initial conditions* $\{z(0), \varphi(0); w_{io}(\tau), -\theta_{pi} \leq \tau \leq 0\}$ *with* $w_{io}(\cdot)$ *sufficiently smooth e.g. continuously differentiable. Then the set of functions* $\{z(\tau), \varphi(\tau); h_i(\xi_i, \tau), q_i(\xi_i, \tau)\}$ *with* $\{h_i(\xi_i, \tau), q_i(\xi_i, \tau)\}$ *defined by (12) is a (possibly discontinuous) solution of (3) with the initial conditions resulting by taking* $\tau = 0$ *in the aforementioned set of functions.*

In the following we shall discuss briefly the significance of Theorem 1. Its main result - in the spirit of [25, 29, 30] - establishes a one-to-one correspondence between the solutions of two mathematical objects - the boundary value problem for hyperbolic partial differential equations defined by (3) with a set of initial conditions and the initial value problem for system (11) - a system of coupled delay differential and difference equations.

This one-to-one correspondence is indeed far going since any property obtained for one mathematical object is automatically projected back on the other one. For instance, the basic theory (existence, uniqueness and data dependence) for the coupled delay differential and difference equations is well studied since it was established that this system is with deviated argument of neutral type [34, 35, 36]. In this way the basic theory is projected back on a boundary value problem with nonlinear and non standard (i.e. containing ordinary differential equations) which is less studied by the usual approaches in the field of partial differential equations.

The stability properties as established *via* Lyapunov functionals can be better tackled using the *energy identity* which is well defined for systems described by partial differential equations with various boundary conditions.

However, as known from the pioneers of the Lyapunov method [37, 38], the energy Lyapunov function(al) is a "weak" one i.e. is only non-increasing along system's solutions. Therefore asymptotic stability follows by applying the Barbashin Krasovskii LaSalle invariance principle which is proven for equations with deviated argument of delayed and neutral type but is not considered for partial differential equations unless we discuss special cases [39].

In what is left of the paper we shall deal with some of these considerations.

3.3 Equilibria - steady state constant solutions

It is known that usually the steady state solutions are solutions defined on the whole real axis $\mathbb{R}$ and *not* by some initial conditions. The steady state solutions can be constant or *recurrent* (periodic, quasi-periodic, almost periodic or other) and have a certain physical/engineering significance. In particular, the constant solutions (the equilibria) signify that certain technical quantities have to be constant in order to ensure a proper and profitable operation. In the following we shall discuss the equilibria of (2): being constant (with respect to τ) solutions, they are obtained by letting the time derivatives to 0 and solving the resulting system of algebraic and differential equations. We shall have

$$\begin{aligned}
&\bar{q}_i(\xi_i) \equiv \text{const} \ , \ \frac{\mathrm{d}\bar{h}_i}{\mathrm{d}\xi_i} + \frac{1}{2}(\lambda_i g)\frac{L_i}{D_i}\gamma_i \bar{q}_i|\bar{q}_i| = 0 \ , i = 1, 2 \\
&\bar{h}_1(0) = 1 \ ; \ \bar{h}_1(1) + \frac{1}{2}\gamma_1 \bar{q}_1^2 = \bar{z} = \bar{h}_2(0) + \frac{1}{2}\gamma_2 \bar{q}_2^2 \ ; \ \bar{q}_1 = \bar{q}_2 = \bar{q} \\
&\bar{q}_2 = \bar{f}_\theta[k\bar{\varphi} + (1-k)\mathcal{F}(\bar{h}_2(1))] \ , \ \bar{q}_2 \bar{h}_2(1) = \nu_g
\end{aligned} \tag{14}$$

Since $\bar{q}_1 = \bar{q}_2 = \bar{q}$, it follows that

$$\begin{aligned}
&\bar{h}_1(\xi_1) = 1 - \frac{1}{2}(\lambda_1 g)\frac{L_1}{D_1}\gamma_1 \bar{q}|\bar{q}|\xi_1 \ , \ \bar{z} = 1 - \frac{1}{2}(\lambda_1 g)\frac{L_1}{D_1}\gamma_1 \bar{q}|\bar{q}| - \frac{1}{2}\gamma_1 \bar{q}^2 \\
&\bar{h}_2(0) = 1 - \frac{1}{2}(\lambda_1 g)\frac{L_1}{D_1}\gamma_1 \bar{q}|\bar{q}| - \frac{1}{2}(\gamma_1 + \gamma_2)\bar{q}^2 \ , \\
&\bar{h}_2(\xi_2) = \bar{h}_2(0) - \frac{1}{2}(\lambda_2 g)\frac{L_2}{D_2}\gamma_2 \bar{q}|\bar{q}|\xi_2 \\
&\bar{h}_2(1) = 1 - \frac{1}{2}\left[(\lambda_1 g)\frac{L_1}{D_1}\gamma_1 + (\lambda_2 g)\frac{L_2}{D_2}\gamma_2\right]\bar{q}|\bar{q}| - \frac{1}{2}(\gamma_1 + \gamma_2)\bar{q}^2
\end{aligned} \tag{15}$$

But the flow and the piezometric head at the hydraulic turbine are imposed by the frequency/megawatt Grid condition i.e. by ν_g, $\bar{\varphi}$ in connection with the wicket gates position $\bar{f}_\theta$. We have therefore to consider the steady state conditions at the turbine

$$\bar{q} = \bar{f}_\theta[k\bar{\varphi} + (1-k)\mathcal{F}(\bar{h}_2(1))] \ , \ \bar{q}\bar{h}_2(1) = \nu_g$$

and clearly $\bar{q} > 0$ requires $\bar{h}_2(1) > 0$. Consequently the necessary piezometric head $\bar{h}_2(1)$ follows from

$$\nu_g/\bar{f}_\theta = \bar{h}_2(1)[k\bar{\varphi} + (1-k)\mathcal{F}(\bar{h}_2(1))]$$

and we have to consider the cubic equation

$$(1-k)Y^3 + (k\bar{\varphi})Y^2 - \nu_g/\bar{f}_\theta = 0 \tag{16}$$

This cubic equation has three real roots, among which a single one is strictly positive. As (15) shows, this root has to be less than 1; this condition is ensured by

$$\nu_g/\bar{f}_\theta < 1 - k + k\bar{\varphi} \tag{17}$$

Let Y_+ be the aforementioned positive root of (16) hence $\bar{h}_2(1) = (Y_+)^2$. We turn now to the last expression for $\bar{h}_2(1)$ in (15). Let

$$A_0 = \left[(\lambda_1 g)\frac{L_1}{D_1} + 1\right]\gamma_1 + \left[(\lambda_2 g)\frac{L_2}{D_2} + 1\right]\gamma_2$$

Numerical data show that $0 < A_0 < 2$. With this notation

$$\bar{h}_2(1) = 1 - \frac{1}{2}A_0\bar{q}^2 \ \Rightarrow \ \bar{q}^2 = 2\frac{1-\bar{h}_2(1)}{A_0}$$

and $\bar{q} < 1$ is also a necessary condition. This would require $Y_+ > \sqrt{1 - A_0/2}$. Considering again (16) and (17) the following requirement has to be fulfilled

$$(1-k)(1-A_0/2)^{3/2} + (k(1-A_0/2)\bar{\varphi}) < \nu_g/\bar{f}_\theta < 1 - k + k\bar{\varphi} \tag{18}$$

which holds for $0 < k < 1$, $0 < A_0 < 2$, $0 < \bar{\varphi} < 1$.

We end here the general discussion concerning the equilibrium. Particular cases are obtained by taking $k = 0$ as it is customary in hydraulic turbine dynamics and/or by neglecting the Darcy Weisbach losses. In this cases the formulae for the steady state are more explicit.

4 Stability problems

The aim of this section is twofold. Following [30] we consider the so called *model augmented validation* which integrates standard validation i.e. well posedness in the sense of Hadamard - see [40] - together with existence and inherent stability of the steady states, as suggested by the *Stability Postulate* of N. G. Četaev [41, 42]. On the other hand stability of the steady states - viewed as operating points of a technical system - is a necessary condition for a safe and profitable operation of the aforementioned technical system.

Consequently we shall consider in the following the stability problems for system (3). In order to define them we remind here that two kinds of operating regimes exists for hydroelectric power plants: *the normal and the abnormal*. The normal regimes are concerned with the frequency/megawatt control of the Electrical Grid. The system considered here contains the dynamics which is located downstream with respect to the surge tank: the accepted contemporary models of the water turbines [6, 7, 8] normally integrate the penstock dynamics (called *water column dynamics*) in the turbine dynamics - see also [3, 9]. The turbine is considered together with the speed controller; the controller normally receives only the rotating speed signal and acts on the cross section area of the turbine's wicket gates. There exist various structures of the controller - which can be mechanical, hydraulic or electrohydraulic - and a lot of recommendations for controller tuning. However, the newest and rapidly establishing approach is now predictive control [2], turned equally useful for steam turbines under the new advent of the renewable energies.

The abnormal regimes are concerned mainly with the water hammer occurring when the hydraulic turbine is *shut down*. In this case the involved dynamics is the entire dynamics located upstream with respect to the turbine. The analysis of this case was made e.g. in [19] (being reproduced in [1]) and developed in [12, 13]; the development takes into account the advances in the study of the difference operator of the neutral functional differential equations as appear in [36].

In connection to water hammer analysis another problem occurs - the *inherent stability of the surge tank*. The inherent stability of the surge tanks is an old date problem see e.g. [11, 4, 5, 1] which is important as long as one is concerned with nonlinear second order models of the surge tank. As long as the overall model of the water hammer was concerned with a first order system [19, 1, 12, 13], the problem was integrated in the water hammer stability.

4.1 The inherent stability of the surge tank

The inherent stability of the surge tank is discussed since the beginning of the XXth century - the by now classical paper of Thoma. We let aside the history of the problem, sending to such basic monographs as [11, 1, 43], but mention only the framework. The initial framework had been suggested by the size of the hydraulic plants of the time: the very short penstock (the surge tank was built closely to the turbine building), small or medium power and relatively short tunnels. In the contemporary settings these considerations amount to *several time scales*. Starting from the basic model (1) and taking into account the values of the time constants, the following can be stated.

1° The dynamics of the tunnel, corresponding to $i = 1$ in (2) is discussed by assuming $\delta_1^2\theta_{w1} \approx 0$ what implies $q_1(\xi_1, \tau) \equiv q_1(\tau)$; integrating the first equation with respect to ξ_1 from 0 to 1 and taking into account the boundary conditions, the so called *equation of the water column dynamics* is obtained

$$\theta_{w1}\frac{\mathrm{d}q_1}{\mathrm{d}\tau} + z - 1 + \frac{1}{2}(\lambda_1 g)\frac{L_1}{D_1}\gamma_1 q_1|q_1| + \frac{\lambda_s'}{\theta_s}(q_1 - q_2(0,\tau)) = 0$$

When the surge tank has no throttling i.e. $\lambda_s' = 0$, the equation above looks like the standard one e.g. [1, 11, 22].

2° For the downstream dynamics - the penstock and the hydraulic turbine - the entire dynamics and the Darcy Weisbach losses are neglected. Therefore

$$h_2(\xi_2, \tau) \equiv h_2(\tau)\ ,\ q_2(\xi_2, \tau) \equiv q_2(\tau)$$

From the boundary conditions we shall have

$$h_2 = \frac{\nu_g}{q_2}\ ,\ h_2 = z + \frac{\lambda_s'}{\theta_s}(q_1 - q_2)$$

3° In the case without throttling, using the continuity equation for the surge tank, the standard surge tank equations are obtained [1, 22]

$$\begin{aligned} &\theta_{w1}\frac{\mathrm{d}q_1}{\mathrm{d}\tau} + z - 1 + \frac{1}{2}(\lambda_1 g)\frac{L_1}{D_1}\gamma_1 q_1|q_1| = 0 \\ &\theta_s\frac{\mathrm{d}z}{\mathrm{d}\tau} = q_1 - \frac{\nu_g}{z} \end{aligned} \tag{19}$$

When $\lambda'_s \neq 0$, combining the aforementioned boundary conditions will lead to the quadratic equation

$$\frac{\lambda'_s}{\theta_s} q_2^2 - \left(z + \frac{\lambda'_s}{\theta_s} q_1 \right) q_2 + \nu_g = 0$$

giving

$$q_2 = \frac{z + (\lambda'_s/\theta_s) q_1 \pm \sqrt{(z + (\lambda'_s/\theta_s) q_1)^2 - 4(\lambda'_s/\theta_s)\nu_g}}{2\lambda'_s/\theta_s}$$

and only the root with the "minus" obeys the natural condition $q_1 - q_2 > 0$. Another checking of this choice is to take $\lim_{\lambda'_s/\theta_s \to 0} q_2$ to find $q_2 \to \nu_g/z$

Finally the following equations are obtained

$$\begin{aligned} & 2\theta_{w1} \frac{\mathrm{d}q_1}{\mathrm{d}\tau} + z - 2 + (\lambda'_s/\theta_s) q_1 + \sqrt{(z + (\lambda'_s/\theta_s) q_1)^2 - 4(\lambda'_s/\theta_s)\nu_g} + \\ & \qquad + \frac{1}{2}(\lambda_1 g) \frac{L_1}{D_1} \gamma_1 q_1 |q_1| = 0 \\ & 2\lambda'_s \frac{\mathrm{d}z}{\mathrm{d}\tau} = (\lambda'_s/\theta_s) q_1 - z + \sqrt{(z + (\lambda'_s/\theta_s) q_1)^2 - 4(\lambda'_s/\theta_s)\nu_g} \end{aligned} \tag{20}$$

For the nonlinear second order system (20) it is necessary to compute the steady state, introduce the system in deviations and discuss at least stability by the first approximation. With respect to this we shall adapt the results of [1, 44] to our case of surge tank with throttling. Our first remark is that the effect of the throttling does not apply to the steady state: in the equation of q_1 the throttling appears from the boundary condition

$$h_1(1, \tau) = z(\tau) + \lambda'_s \frac{\mathrm{d}z}{\mathrm{d}\tau}$$

and $\mathrm{d}z/\mathrm{d}\tau = 0$ at the steady state hence the steady state equations are

$$\bar{z} - 1 + \frac{1}{2}(\lambda_1 g) \frac{L_1}{D_1} \gamma_1 \bar{q}_1 |\bar{q}_1| = 0 \ ; \ \bar{q}_1 = \bar{q}_2 \ , \ \bar{q}_2 = \nu_g/\bar{z} \tag{21}$$

Let $\zeta(\tau) := z(\tau) - 1$ be the deviation of the surge tank water level with respect to the lake level (piezometric head which equals 1 when rated). We deduce the equation for the steady state of the surge tank level - a cubic equation

$$\zeta(1 + \zeta)^2 + \frac{1}{2}(\lambda_1 g) \frac{L_1}{D_1} \gamma_1 \nu_g^2 = 0 \tag{22}$$

Its “discriminant” reads

$$\Delta = -\left(\frac{8}{9}\right)^2 + \frac{1}{4}\left(A_0 - \frac{2}{27}\right)^2$$

with $A_0 := (1/2)(L_1/D_1)\gamma_1\nu_g^2$. In practical cases e.g the plants previously discusses, $\Delta > 0$ hence equation (22) has two complex conjugate roots and a real negative root located in the interval$(-1/3, 0)$. Consequently $\bar{z}$ is located in the interval $(2/3, 1)$.

In order to study the stability of the equilibrium of (20) we shall follow the approach of [44], Section 2.4, pp. 35-38. We introduce as state variables (in deviations) $\eta := \zeta - \bar{\zeta}$, $\upsilon = \mathrm{d}\zeta/\mathrm{d}\tau$. Some straightforward manipulation will give the following state equations

$$\begin{aligned}
&\frac{\mathrm{d}\eta}{\mathrm{d}\tau} = \upsilon \\
&\left(1 - \lambda_s' \frac{\nu_g/\theta_s}{(1+\bar{\zeta}+\eta+\lambda_s'\upsilon)^2}\right)\theta_s\frac{\mathrm{d}\upsilon}{\mathrm{d}\tau} = \frac{\nu_g}{(1+\bar{\zeta}+\eta+\lambda_s'\upsilon)^2} - \\
&-\frac{1}{\theta_{w1}}\left[1+\bar{\zeta}+\eta+\lambda_s'\upsilon+\gamma_1\left(\theta_s\upsilon+\frac{\nu_g}{1+\bar{\zeta}+\eta+\lambda_s'\upsilon}\right)^2\right]
\end{aligned} \tag{23}$$

Equations (23) point out quite clearly the contribution of the throttling to the surge tank dynamics. By letting $\lambda_s' = 0$, equations (23) become very much alike to the basic equations (2.44) of [44] but without local hydraulic losses ($R' = 0$ in the aforementioned equations). Even before taking the Lyapunov function following the same reference, the state space restriction suggested by (23) namely

$$\lambda_s' \frac{\nu_g/\theta_s}{(1+\bar{\zeta}+\eta+\lambda_s'\upsilon)^2} < 1 \tag{24}$$

will introduce some restrictions on λ_s' i.e. on the throttling and on its design.

4.2 Other stability development for surge tanks

We shall only sketch here some possible stability problems for surge tanks with throttling. The first one has been already mentioned in the previous subsection: tanks with throttling versus tanks without throttling. Following [44] we mention here the case of a surge tank fed by several tunnels and the case of several tunnels with intermediary intake shafts feeding the surge tank. For these structures there is discussed only the case of the surge tank without throttling hence the case with throttling will be a new case analysis. Suggestions for the associated Lyapunov functionals can be found in [44]. The way of connecting the surge tank to the tunnels can suggest the corresponding boundary conditions in a possible generalization of the results of [19]. Those results, dealing with "arithmetic properties", deserve attention in the context of the recent (more or less) results on the difference operators attached to neutral functional differential equations [36, 45]. There exist also other possible developments motivated by hydraulic power plant dynamics - still "accepted" in the search for clean and sustainable energy sources.

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