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CONTROLLABILITY AND GRAMIANS OF 2D CONTINUOUS TIME LINEAR SYSTEMS*

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Dedicated to Dr. Vasile Drăgan on the occasion of his 70th anniversary

Abstract

The controllability of a class of 2D linear time varying continuous time control systems is studied. The state space representation is provided and the formulas of the states and the input-output map of these systems are derived. The fundamental concepts of controllability and reachability are analysed and suitable controllability and reachability Gramians are constructed to characterize the controllable and the reachable time varying systems. In the case of time invariant 2D systems, some algorithms are developed to calculate different controllability Gramians as solutions of adequate Lyapunov type equations. Corresponding Matlab programs are implemented to solve these Lyapunov equations.
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1 Introduction

A distinct and important branch of the systems theory is represented by the two-dimensional (2D) systems. The reasons for the growing interest in this domain are on the one hand the productivity in application fields and on the other hand the richness and significance of the theoretical approaches. The application fields cover various areas, from signal and image processing, computer tomography, gravity and magnetic field mapping, seismology, control of multipass processes, to pollution modelling or iterative learning control synthesis. The domain of 2D systems needs a specific theoretical approach, since many aspects of the 1D systems do not generalize and there are many 2D systems phenomena which have no 1D systems counterparts.

The concept of controllability which is fundamental to control theory was introduced for 1D systems by Kalman [4], being imposed by the engineering problems of time optimal control. They were extended to 2D systems for Roesser [10], Fornasini and Marchesini [2], and Attasi [1] models. In the case of 2D systems some notions such as local and global controllability were introduced, but they are not satisfactory from the point of view of minimality. In order to maintain their relationship with minimality, new concepts of modal controllability and modal observability were introduced in [5]. But these notions do not allow the richness of the 1D notions characterizations.

Controllability is an important subject in the study of multidimensional hybrid (continuous-discrete) systems [3], [6], [7], [8], [9].

In the present paper a class of 2D linear systems is studied from the point of view of controllability. This class is the continuous counterpart of the Attasi type 2D discrete time systems and has the advantage of allowing many of the characteristics of the 1D continuous time systems.

In Section 2 the state space representation of these systems is given, as well as the formulas of the state and of the input-output map.

Section 3 introduces the notions of controllability and reachability of the states and the complete controllability and complete reachability of the systems. Suitable controllability and reachability Gramians are constructed and they are used to obtain necessary and sufficient rank conditions of controllability and reachability for time varying systems.

Section 5 is devoted to time invariant 2D systems. The expression of the controllability Gramian defined in Section 3 is derived for the time invariant case and three other controllability Gramians are provided. It is shown that these Gramians can be calculated as solutions of four Lyapunov type equations. Simpler forms of the Lyaponov type equations are deduced for the time interval $(0,\infty)\times(0,\infty)$ for some positions of the spectra of the drift matrices A_1 and A_2 in the complex plane.

Some Matlab programs are implemented to solve these Lyapunov equations and to determine the corresponding Gramians.

2 A class of 2D time varying continuous time systems

A system is characterized by the linear spaces $X=\mathbf{R}^n$, $U=\mathbf{R}^m$ and $Y=\mathbf{R}^p$, called respectively the *state*, *input* and *output spaces*. The time set of the 2D system is $T=\mathbf{R}\times\mathbf{R}$.

Definition 1 *A two-dimensional time varying continuous time linear system is an ensemble* $\Sigma=(A_1(t_1),A_2(t_2),B(t_1,t_2),C(t_1,t_2),D(t_1,t_2))\in\mathbf{R}^{n\times n}\times\mathbf{R}^{n\times n}\times\mathbf{R}^{n\times m}\times\mathbf{R}^{p\times n}\times\mathbf{R}^{p\times m}$ *with* $A_1(t_1)A_2(t_2)=A_2(t_2)A_1(t_2)$ $\forall(t_1,t_2)\in T$, *where all matrices are continuous with respect to* t_1, t_2 *and* $(t_1,t_2)\in T$ *respectively; the state space representation of* Σ *consists in the following state and output equations:*

$$\begin{aligned}\frac{\partial^2 x}{\partial t_1\partial t_2}(t_1,t_2) = & A_1(t_1)\frac{\partial x}{\partial t_2}(t_1,t_2)+A_2(t_2)\frac{\partial x}{\partial t_1}(t_1,t_2)\\ & -A_1(t_1)A_2(t_2)x(t_1,t_2)+B(t_1,t_2)u(t_1,t_2) \qquad (1)\\ y(t_1,t_2) = & \, C(t_1,t_2)x(t_1,t_2)+D(t_1,t_2)u(t_1,t_2). \qquad (2)\end{aligned}$$

We denote by $\Phi_1(t_1,t_1^0)$ the fundamental matrix of $A_1(t_1)$ and by $\Phi_2(t_2,t_2^0)$ the fundamental matrix of $A_2(t_2)$.

The matrix $\Phi_1(t_1,t_1^0$ has the following properties, for any $t_1,t_1^0 s_1\in\mathbf{R}$:

$$\begin{aligned}
&i) && \frac{\mathrm{d}}{\mathrm{d}t_1}\Phi_1(t_1,t_1^0)=A_1(t_1)\Phi_1(t_1,t_1^0),\\
&ii) && \Phi_1(t_1^0,t_1^0)=I_n,\\
&iii) && \Phi_1(t_1,s_1)\Phi_1(s_1,t_1^0)=\Phi_1(t_1,t_1^0),\\
&iv) && \Phi_1(t_1,t_1^0)^{-1}=\Phi_1(t_1^0,t_1), \qquad (3)\\
&v) && \Phi_1(t_1,t_1^0)=I+\sum_{l=1}^{\infty}\int_{t_1^0}^{t_1}A_1(s_1)\int_{t_1^0}^{s_1}A_1(s_2)\cdots\int_{t_1^0}^{s_{l-1}}A_1(s_l)\mathrm{d}s_l\cdots\mathrm{d}s_2\mathrm{d}s_1.
\end{aligned}$$

vi) If A_1 is a constant matrix, then $\Phi_1(t_1,t_1^0)=\sum_{k=0}^{\infty}\frac{A_1^k(t_1-t_1^0)^k}{k!}=e^{A_1(t_1-t_1^0)}$.

The matrix $\Phi_2(t_2,t_2^0)$ has similar properties.

Remark 1 *Since $A_1(t_1)$ and $A_2(t_2)$ are commutative matrices, it follows by v) that $\Phi_1(t_1,t_1^0)$ and $\Phi_2(t_2,t_2^0)$ are commutative matrices for any $t_1,t_1^0,t_2,t_2^0\in$ $\mathbf{R}$, as well as $A_1(t_1)$ and $\Phi_2(t_2,t_2^0)$ or $A_2(t_2)$ and $\Phi_1(t_1,t_1^0)$.*

By $(s,l)<(t,k)$ for $(s,l),(t,k)\in T$ we mean $s\leq t,\ l\leq k$ and $(s,l)\neq(t,k)$; $(s,l)\leq(t,k)$ means $s\leq t,\ l\leq k$.

Definition 2 *A vector $x_0\in X$ is called the initial state of Σ at the moment $(t_1^0,t_2^0)\in T$ if, for any $(t_1,t_2)\in T$ with $(t_1^0,t_2^0)\leq(t_1,t_2)$, the following conditions hold:*

$$x(t_1,t_2^0)=\Phi_1(t_1,t_1^0)x_0,\ x(t_1^0,t_2)=\Phi_2(t_2,t_2^0)x_0. \tag{4}$$

This definition indicates that the behaviour of the 2D system on the boundary of the time set T is the behaviour of two 1D systems, which are compatible with the 2D one.

For $(t_1^0,t_2^0)\leq(t_1,t_2)$ we denote by $[t_1^0,t_1;t_2^0,t_2]$ the set $[t_1^0,t_1]\times[t_2^0,t_2]$.

Proposition 1 *The state of the system Σ at the moment $(t_1,t_2)\in T$ determined by the control $u(\cdot,\cdot):[t_1^0,t_1;t_2^0,t_2]\to U$ and by the initial state $x_0\in X$ is*

$$x(t_1,t_2)=\Phi_1(t_1,t_1^0)\Phi_2(t_2,t_2^0)x_0+$$
$$\int_{t_1^0}^{t_1}\int_{t_2^0}^{t_2}\Phi_1(t_1,s_1)\Phi_2(t_2,s_2)B(s_1,s_2)u(s_1,s_2)\mathrm{d}s_1\mathrm{d}s_2. \tag{5}$$

Proof: One writes equation (1) in the form

$$\frac{\partial}{\partial t_2}\left(\frac{\partial x}{\partial t_1}(t_1,t_2)-A_1(t_1)x(t_1,t_2)\right)=$$
$$A_2(t_2)\left(\frac{\partial x}{\partial t_1}(t_1,t_2)-A_1(t_1)x(t_1,t_2)\right)+B(t_1,t_2)u(t_1,t_2). \tag{6}$$

Let us denote

$$z(t_1,t_2)=\frac{\partial x}{\partial t_1}(t_1,t_2)-A_1(t_1)x(t_1,t_2). \tag{7}$$

Then $z(t_1,t_2)$ is the solution of the differential equation

$$\frac{\partial z}{\partial t_2}(t_1,t_2)=A_2(t_2)z(t_1,t_2)+B(t_1,t_2)u(t_1,t_2),$$

with the following initial condition, obtained by (4) and (3 i).

$$z(t_1, t_2^0) = \frac{\partial x}{\partial t_1}(t_1, t_2^0) - A_1(t_1)x(t_1, t_2^0) = 0.$$

By the Variation of Parameters Formula, one obtains

$$z(t_1, t_2) = \int_{t_2^0}^{t_2} \Phi_2(t_2, s_2)B(t_1, s_2)u(t_1, s_2)\mathrm{d}s_2. \tag{8}$$

Equation (7) can be written as the differential equation

$$\frac{\partial x}{\partial t_1}(t_1, t_2) = A_1(t_1)x(t_1, t_2) + z(t_1, t_2).$$

Again by the Variation of Parameters Formula, one obtains

$$x(t_1, t_2) = \Phi_1(t_1, t_1^0)x(t_1^0, t_2) + \int_{t_1^0}^{t_1} \Phi_1(t_1, s_1)z(s_1, t_2)\mathrm{d}s_1.$$

Using the initial condition (4) $x(t_1^0,t_2) = \Phi_2(t_2,t_2^0)x_0$, one gets the formula (5) of the states of the system Σ.

By replacing the state $x(t_1, t_2)$ given by (5) in the output equation (2) we get the following general response of the system.

Proposition 2 *The input-output map of the system Σ is given by the formula*

$$\begin{aligned} y(t_1, t_2) &= C(t_1, t_2)\Phi_1(t_1, t_1^0)\Phi_2(t_2, t_2^0)x_0 + \\ &\int_{t_1^0}^{t_1}\int_{t_2^0}^{t_2} C(t_1, t_2)\Phi_1(t_1, s_1)\Phi_2(t_2, s_2)B(s_1, s_2)u(s_1, s_2)\mathrm{d}s_1\mathrm{d}s_2 + \\ &D(t_1,t_2)u(t_1,t_2). \end{aligned} \tag{9}$$

3 Controllability and reachability of time varying 2D continuous time systems

A triplet $(t_1, t_2, x) \in \mathbf{R} \times \mathbf{R} \times X$ is said to be a *phase* of Σ if x is the state of Σ at the moment (t_1, t_2) (i.e. $x = x(t_1, t_2)$, where $x(t_1, t_2)$ is given by (5)). In this case one says that the control u from (5) transfers the phase (t_1^0, t_2^0, x_0) to (t_1, t_2, x).

Definition 3 *A state $x \in X$ is said to be controllable on $[t_1^0, t_1; t_2^0, t_2]$ if there exists a control $u(\cdot,\cdot)$ which transfers the phase (t_1^0, t_2^0, x) to $(t_1, t_2, 0)$.*

A state $x \in X$ is said to be reachable on $[t_1^0, t_1; t_2^0, t_2]$ if there exists a control $u(\cdot,\cdot)$ which transfers the phase $(t_0, t_0, 0)$ to (t_1, t_2, x).

If any state $x \in X$ is controllable/reachable on $[t_1^0, t_1; t_2^0, t_2]$, the system Σ is said to be completely controllable/completely reachable on $[t_1^0, t_1; t_2^0, t_2]$.

In order to verify these properties of the 2D systems, one introduces suitable Gramians.

Definition 4 *For $(t_1^0, t_2^0) \leq (t_1, t_2)$, the matrix*

$$\mathcal{R}_\Sigma(t_1^0, t_1; t_2^0, t_2) = \int_{t_1^0}^{t_1} \int_{t_2^0}^{t_2} \Phi_1(t_1, s_1)\Phi_2(t_2, s_2) \times$$
$$\times B(s_1, s_2)B(s_1, s_2)^T \Phi_2(t_2, s_2)^T \Phi_1(t_1, s_1)^T \mathrm{d}s_1 \mathrm{d}s_2, \tag{10}$$

is called the 2D reachability Gramian of Σ.

Proposition 3 *The set of the states of the system Σ which are reachable on $[t_1^0, t_1; t_2^0, t_2]$ is the subspace*

$$X_r = Im\ \mathcal{R}_\Sigma(t_1^0, t_1; t_2^0, t_2).$$

Proof: Sufficiency. By replacing in (5) $x(t_1, t_2)$ and x_0 respectively by x and 0, one obtains that the state x is reachable on $[t_1^0, t_1; t_2^0, t_2]$ if and only if there exists a control $u(\cdot,\cdot)$ such that

$$x = \int_{t_1^0}^{t_1} \int_{t_2^0}^{t_2} \Phi_1(t_1, s_1)\Phi_2(t_2, s_2)B(s_1, s_2)u(s_1, s_2) ds_1 ds_2. \tag{11}$$

Assume that $x \in X_r$. Then $\exists\ v \in \mathbf{R}^n$ such that $x = \int_{t_1^0}^{t_1} \int_{t_2^0}^{t_2} \Phi_1(t_1, s_1) \times$

$$\times \Phi_2(t_2, s_2)B(s_1, s_2)B(s_1, s_2)^T \Phi_2(t_2, s_2)^T \Phi_1(t_1, s_1)^T v \mathrm{d}s_1 \mathrm{d}s_2.$$

By considering the control $u(s_1, s_2) = B(s_1, s_2)^T \Phi_2(t_2, s_2)^T \Phi_1(t_1, s_1)^T v$ one obtains (11), hence the state x is reachable on $[t_1^0, t_1; t_2^0, t_2]$.

Necessity can be obtained in a similar way to the proof of Proposition 4.

Theorem 1 *The system Σ is completely reachable on $[t_1^0, t_1; t_2^0, t_2]$ if and only if*

$$rank\ \mathcal{R}_\Sigma(t_1^0, t_1; t_2^0, t_2) = n. \tag{12}$$

Proof: By Proposition 3, the system Σ is completely reachable on $[t_1^0, t_1; t_2^0, t_2]$ if and only if $\mathrm{Im}\mathcal{R}_\Sigma(t_1^0, t_1; t_2^0, t_2) = \mathbf{R}^n$. Since the dimension of the subspace $\mathrm{Im}\mathcal{R}_\Sigma(t_1^0, t_1; t_2^0, t_2)$ of $\mathbf{R}^n$ is equal to $\mathrm{rank}\mathcal{R}_\Sigma(t_1^0, t_1; t_2^0, t_2)$, this equality is equivalent to (12).

Definition 5 *For* $(t_1^0, t_2^0) \leq (t_1, t_2)$, *the matrix*

$$\mathcal{C}_\Sigma(t_1^0, t_1; t_2^0, t_2) = \int_{t_1^0}^{t_1} \int_{t_2^0}^{t_2} \Phi_1(t_1^0, s_1)\Phi_2(t_2^0, s_2) \times$$

$$\times B(s_1, s_2)B(s_1, s_2)^T \Phi_2(t_2^0, s_2)^T \Phi_1(t_1^0, s_1)^T \mathrm{d}s_1 \mathrm{d}s_2, \tag{13}$$

is called the <u>*2D controllability Gramian*</u> *of* Σ.

Proposition 4 *The set of the states of the system* Σ *which are controllable on* $[t_1^0, t_1; t_2^0, t_2]$ *is the subspace*

$$X_c = Im\ \mathcal{C}_\Sigma(t_1^0, t_1; t_2^0, t_2).$$

Proof: Sufficiency. By replacing in (5) $x(t_1, t_2)$ and x_0 respectively by 0 and x, one obtains that the state x is controllable on $[t_1^0, t_1; t_2^0, t_2]$ if and only if there exists a control $u(\cdot, \cdot)$ such that

$$\Phi_1(t_1, t_1^0)\Phi_2(t_2, t_2^0)x =$$

$$-\int_{t_1^0}^{t_1} \int_{t_2^0}^{t_2} \Phi_1(t_1, s_1)\Phi_2(t_2, s_2)B(s_1, s_2)u(s_1, s_2)\mathrm{d}s_1\mathrm{d}s_2. \tag{14}$$

By premultiplying (14) by $\Phi_1(t_1^0, t_1)\Phi_2(t_2^0, t_2)$ one obtains (see (3) iv)) that the state x is controllable on $[t_1^0, t_1; t_2^0, t_2]$ if and only if

$$x = -\int_{t_1^0}^{t_1} \int_{t_2^0}^{t_2} \Phi_1(t_1^0, s_1)\Phi_2(t_2^0, s_2)B(s_1, s_2)u(s_1, s_2)ds_1ds_2. \tag{15}$$

Assume that $x \in X_c$. Then $\exists\ v \in \mathbf{R}^n$ such that $x = \int_{t_1^0}^{t_1} \int_{t_2^0}^{t_2} \Phi_1(t_1^0, s_1) \times$

$$\times \Phi_2(t_2^0, s_2)B(s_1, s_2)B(s_1, s_2)^T \Phi_2(t_2^0, s_2)^T \Phi_1(t_1, s_1)^T v \mathrm{d}s_1\mathrm{d}s_2.$$

By considering the control $u(s_1, s_2) = -B(s_1, s_2)^T \Phi_2(t_2^0, s_2)^T \Phi_1(t_1, s_1)^T v$ one obtains (15), hence the state x is controllable on $[t_1^0, t_1; t_2^0, t_2]$.

Necessity. Let $x \in X$ be a controllable state. Assume by absurd that $x \notin \mathrm{Im}\ \mathcal{C}_\Sigma(t_1^0, t_1; t_2^0, t_2)$. Since $\mathcal{C}_\Sigma(t_1^0, t_1; t_2^0, t_2)$ is a symmetrical matrix, the state space has the direct sum decomposition $\mathbf{R}^n = \mathrm{Im}\ \mathcal{C}_\Sigma(t_1^0, t_1; t_2^0, t_2) \oplus \mathrm{Ker}\ \mathcal{C}_\Sigma(t_1^0, t_1; t_2^0, t_2)$, hence $\exists!\ x_1 \in \mathrm{Im}\ \mathcal{C}_\Sigma(t_1^0, t_1; t_2^0, t_2), x_2 \in \mathrm{Ker}\ \mathcal{C}_\Sigma(t_1^0, t_1; t_2^0, t_2)$ such that $x = x_1 + x_2$ and by assumption, $x_2 \neq 0$. Then $x_2^T \mathcal{C}_\Sigma(t_1^0, t_1; t_2^0, t_2) x_2 = 0$, equality which can be written as $0 = \int_{t_1^0}^{t_1} \int_{t_2^0}^{t_2} w(s_1, s_2)^T w(s_1, s_2) ds_1 ds_2 = \int_{t_1^0}^{t_1} \int_{t_2^0}^{t_2} \|w(s_1, s_2)\|^2 \mathrm{d}s_1 \mathrm{d}s_2$, hence

$$w(s_1, s_2)^T := x_2^T \Phi_1(t_1^0, s_1) \Phi_2(t_2^0, s_2) B(s_1, s_2) = 0. \tag{16}$$

By the sufficiency part, x_1 is controllable. It follows that the states x and x_1 have the representation (15) with some controls u and u_1 respectively. Since $x_2 = x - x_1$, x_2 has the representation (15) with the control $u_2 = u - u_1$. Then one gets by (15) $\|x_2\|^2 = x_2^T x_2 =$

$$-\int_{t_1^0}^{t_1} \int_{t_2^0}^{t_2} x_2^T \Phi_1(t_1^0, s_1) \Phi_2(t_2^0, s_2) B(s_1, s_2) u_2(s_1, s_2) \mathrm{d}s_1 \mathrm{d}s_2 = 0,$$

hence $x_2 = 0$, contradiction.

Theorem 2 *The system Σ is completely controlable on $[t_1^0, t_1; t_2^0, t_2]$ if and only if*

$$\mathit{rank}\ \mathcal{C}_\Sigma(t_1^0, t_1; t_2^0, t_2) = n. \tag{17}$$

The proof is similar to the proof of Theorem 1.

4 Gramians of the 2D time invariant continuous time systems

Since in the controllability theory the output equation (2) is not used, we can consider the system as a triplet $\Sigma = (A_1, A_2, B) \in \mathbf{R}^{n \times n} \times \mathbf{R}^{n \times n} \times \mathbf{R}^{n \times m}$.

The system is linear time invariant (LTI) if A_1, A_2 and B are constant matrices. In this case, by (3) vi, the fundamental matrices of the matrices $A_i,\ i = 1, 2$ have the form $\Phi_1(t_1, t_1^0) = e^{A_1(t_1 - t_1^0)}$ and $\Phi_2(t_2, t_2^0) = e^{A_2(t_2 - t_2^0)}$ and the state formula (5) becomes

$$x(t_1, t_2) = e^{A_1(t_1 - t_1^0)} e^{A_2(t_2 - t_2^0)} x_0 + \int_{t_1^0}^{t_1} \int_{t_2^0}^{t_2} e^{A_1(t_1 - s_1)} e^{A_2(t_2 - s_2)} B u(s_1, s_2) \mathrm{d}s_1 \mathrm{d}s_2. \tag{18}$$

The 2D controllability Gramians (13) becomes

$$\mathcal{C}_{\Sigma}(t_1^0,t_1;t_2^0,t_2)=\int_{t_1^0}^{t_1}\int_{t_2^0}^{t_2} e^{A_1(t_1^0-s_1)}e^{A_2(t_2^0-s_2)}BB^Te^{A_2^T(t_2-s_2)}e^{A_1^T(t_1^0-s_1)}\mathrm{d}s_1\mathrm{d}s_2. \quad (19)$$

In order to obtain in a simple way a formula for the computation of the 2D controllability Gramian (10), we will consider the corresponding 1D LTI system Σ_1

$$\dot{x} = Ax(t) + Bu(t), \quad (20)$$

with the 1D controllability Gramian

$$\mathcal{C}_{\Sigma_1}(t^0,t) = \int_{t^0}^{t} e^{A(t^0-s)}BB^Te^{A^T(t^0-s)}\mathrm{d}s. \quad (21)$$

Theorem 3 *The controllability Gramian* $\mathcal{C}_{\Sigma_1}(t^0,t)$ *is the solution of the Lyapunov equation*

$$AP + PA^T = -e^{-A(t-t^0)}BB^Te^{-A^T(t-t^0)} + BB^T. \quad (22)$$

Proof: We will replace P by $\mathcal{C}_{\Sigma_1}(t^0,t)$ in the Lyapunov equation (22) and finally, using the formula $\frac{\mathrm{d}}{\mathrm{d}s}e^{A(t^0-s)} = -Ae^{A(t^0-s)}$, we will write the integrand in the form of the derivative of a product.

$$A\mathcal{C}_{\Sigma_1}(t^0,t) + \mathcal{C}_{\Sigma_1}(t^0,t)A^T =$$

$$\int_{t^0}^{t}(Ae^{A(t^0-s)}BB^Te^{A^T(t^0-s)} + e^{A(t^0-s)}BB^Te^{A^T(t^0-s)}A^T)\mathrm{d}s =$$

$$\int_{t^0}^{t}(\frac{\mathrm{d}}{\mathrm{d}s}(e^{A(t^0-s)}B)B^Te^{A^T(t^0-s)} + e^{A(t^0-s)}B\frac{\mathrm{d}}{\mathrm{d}s}(B^Te^{A^T(t^0-s)}A^T))\mathrm{d}s =$$

$$\int_{t^0}^{t}\frac{\mathrm{d}}{\mathrm{d}s}(e^{A(t^0-s)}BB^Te^{A^T(t^0-s)})\mathrm{d}s = -e^{A(t^0-s)}BB^Te^{A^T(t^0-s)}\mid_{t_0}^{t}=$$

$$-e^{-A(t-t^0)}BB^Te^{A^T(t-t^0)} + BB^T.$$

Theorem 4 *The 2D controllability Gramian* $\mathcal{C}_{\Sigma}(t_1^0,t_1;t_2^0,t_2)$ *is the solution of the 2D Lyapunov equation*

$$\begin{gathered} A_1A_2P + PA_2^TA_1^T + A_1PA_2^T + A_2PA_1^T = \\ e^{-A_1(t_1-t_1^0)}e^{-A_2(t_2-t_2^0)}BB^Te^{-A_2^T(t_2-t_2^0)}e^{-A_1^T(t_1-t_1^0)} - \\ e^{-A_1(t_1-t_1^0)}BB^Te^{-A_1^T(t_1-t_1^0)} - e^{-A_2(t_2-t_2^0)}BB^Te^{-A_2^T(t_2-t_2^0)} + BB^T. \end{gathered} \quad (23)$$

Proof: The left-hand member of the 2D Lyapunov equation (23) can be written as $A_1(A_2P + PA_2^T) + (A_2P + PA_2^T)A_1^T$.

We replace P by $\mathcal{C}_\Sigma(t_1^0, t_1; t_2^0, t_2)$ and we obtain $A_2\mathcal{C}_\Sigma(t_1^0, t_1; t_2^0, t_2) + \mathcal{C}_\Sigma(t_1^0, t_1; t_2^0, t_2)A_2^T$. Using the commutativity of A_1, A_2 and their exponential matrices, this expression can be written as $\int_{t_1^0}^{t_1} e^{A_1(t_1^0 - s_1)}(A_2P_1 + P_1A_2^T)e^{A_1^T(t_1^0 - s_1)}\mathrm{d}s_1$, where $P_1 = \int_{t_2^0}^{t_2} e^{A_2(t_2^0 - s_2)}BB^Te^{A_2^T(t_2^0 - s_2)}\mathrm{d}s_2$. By applying Theorem 3 (i.e. (21) and (22)), one gets

$$A_2P_1 + P_1A_2^T = -e^{-A_2(t_2 - t_2^0)}BB^Te^{-A_2^T(t_2 - t_2^0)} + BB^T.$$

Then, again by (22) and (23), with BB^T replaced by

$$N := -e^{-A_2(t_2 - t_2^0)}BB^Te^{-A_2^T(t_2 - t_2^0)} + BB^T,$$

A by A_1 and $P = \mathcal{C}_{\Sigma_1}(t^0, t)$ by $\int_{t_1^0}^{t_1} e^{A_1(t_1^0 - s_1)}Ne^{A_1^T(t_1^0 - s_1)}\mathrm{d}s_1$, one obtains (23).

Theorem 5 *The following assertions are equivalent:*
i. The system Σ is not completely controllable on $[t_1^0, t_1; t_2^0, t_2]$

ii. rank $\mathcal{C}_\Sigma(t_1^0, t_1; t_2^0, t_2) < n$.

iii. There exists $v \in \mathbf{R}^n$, $v \neq 0$, such that

$$v^TA_1^iA_2^jB = 0, \ \forall\, i, j \in \{0, 1, \ldots, n-1\}. \tag{24}$$

Proof: The equivalence of i and ii follows from Theorem 3 by negation. Condition ii holds if and only if there exists $v \in \mathbf{R}^n$ $v \neq 0$ such that $v^T\mathcal{C}_\Sigma(t_1^0, t_1; t_2^0, t_2) = 0$. Then $v^T\mathcal{C}_\Sigma(t_1^0, t_1; t_2^0, t_2)v = 0$. As in the proof of Proposition 4, this implies that $v^Te^{A_1(t_1^0 - s_1)}e^{A_2(t_2^0 - s_2)}B = 0$. This is equivalent to the equality to 0 of the coefficients of the (2D) Taylor series obtained using the exponential series, i.e. to $v^TA_1^iA_2^jB = 0$, $\forall\, i, j \in \mathbf{N}$, which is equivalent to iii by Theorem Hamilton-Cayley. By reversing the implications one proves that iii implies ii.

Remark 2 *As in the proof of theorem 5, one can prove that condition (24) is equivalent to the fact that the following 2D Gramians have the rank less than n.*

$$\mathcal{G}_\Sigma^1(t_1^0, t_1; t_2^0, t_2) = \int_{t_1^0}^{t_1}\int_{t_2^0}^{t_2} e^{A_1(s_1 - t_1^0)}e^{A_2(s_2 - t_2^0)}BB^Te^{A_2^T(s_2 - t_2^0)}e^{A_1^T(s_1 - t_1^0)}\mathrm{d}s_1\mathrm{d}s_2. \tag{25}$$

$$\mathcal{G}^2_\Sigma(t_1^0,t_1;t_2^0,t_2)=\int_{t_1^0}^{t_1}\int_{t_2^0}^{t_2} e^{A_1(t_1^0-s_1)}e^{A_2(s_2-t_2^0)}BB^Te^{A_2^T(s_2-t_2^0)}e^{A_1^T(t_1^0-s_1)}\mathrm{d}s_1\mathrm{d}s_2. \quad (26)$$

$$\mathcal{G}^3_\Sigma(t_1^0,t_1;t_2^0,t_2)=\int_{t_1^0}^{t_1}\int_{t_2^0}^{t_2} e^{A_1(s_1-t_1^0)}e^{A_2(t_2^0-s_2)}BB^Te^{A_2^T(t_2^0-s_2)}e^{A_1^T(s_1-t_1^0)}\mathrm{d}s_1\mathrm{d}s_2. \quad (27)$$

By negation and Theorem 5 one obtains the following results.

Theorem 6 *The system Σ is completely controllable on $[t_1^0,t_1;t_2^0,t_2]$ if and only if the four Gramians $\mathcal{C}_\Sigma(t_1^0,t_1;t_2^0,t_2)$ and $\mathcal{G}^i_\Sigma(t_1^0,t_1;t_2^0,t_2)$, $i=1,2,3$ have the rank n.*

Let us consider the operator $\mathcal{L}$ defined by

$$\mathcal{L}(A_1,A_2,B)=A_1A_2P+PA_2^TA_1^T+A_1PA_2^T+A_2PA_1^T.$$

Theorem 7 *The 2D Gramian $\mathcal{G}^1_\Sigma(t_1^0,t_1;t_2^0,t_2)$ is the solution of the 2D Lyapunov equation*

$$\begin{aligned}\mathcal{L}(A_1,A_2,B) &= e^{A_1(t_1-t_1^0)}e^{A_2(t_2-t_2^0)}BB^Te^{A_2^T(t_2-t_2^0)}e^{A_1^T(t_1-t_1^0)}-\\ &-e^{A_1(t_1-t_1^0)}BB^Te^{A_1^T(t_1-t_1^0)}-e^{A_2(t_2-t_2^0)}BB^Te^{A_2^T(t_2-t_2^0)}+BB^T.\end{aligned} \quad (28)$$

Theorem 8 *The 2D controllability Gramian $\mathcal{G}^2_\Sigma(t_1^0,t_1;t_2^0,t_2)$ is the solution of the 2D Lyapunov equation*

$$\begin{aligned}\mathcal{L}(A_1,A_2,B) &= -e^{A_1(t_1-t_1^0)}e^{-A_2(t_2-t_2^0)}BB^Te^{-A_2^T(t_2-t_2^0)}e^{A_1^T(t_1-t_1^0)}+\\ &+e^{A_1(t_1-t_1^0)}BB^Te^{A_1^T(t_1-t_1^0)}+e^{-A_2(t_2-t_2^0)}BB^Te^{-A_2^T(t_2-t_2^0)}-BB^T.\end{aligned} \quad (29)$$

Theorem 9 *The 2D controllability Gramian $\mathcal{G}^3_\Sigma(t_1^0,t_1;t_2^0,t_2)$ is the solution of the 2D Lyapunov equation*

$$\begin{aligned}\mathcal{L}(A_1,A_2,B) &= -e^{-A_1(t_1-t_1^0)}e^{A_2(t_2-t_2^0)}BB^Te^{A_2^T(t_2-t_2^0)}e^{-A_1^T(t_1-t_1^0)}+\\ &+e^{-A_1(t_1-t_1^0)}BB^Te^{-A_1^T(t_1-t_1^0)}+e^{A_2(t_2-t_2^0)}BB^Te^{A_2^T(t_2-t_2^0)}-BB^T.\end{aligned} \quad (30)$$

Remark 3 *By (24), we can prove that, for A_1, A_2 and B constant matrices, if the system Σ is completely controllable on an interval $[t_1^0,t_1;t_2^0,t_2]$, then it is completely controllable on any such interval. Therefore we can consider for instance the initial moment $(t_1^0,t_2^0)=(0,0)$ and $t_1=\infty$ and/or $t_2=\infty$.*

In some particular cases, the Lyapunov equations (23), (28), (29) and (30) have simpler forms when one takes $t_1 = \infty$ and $t_2 = \infty$.

If the right-hand member contains the exponential matrix of an $n \times n$ matrix A, then A must be a stable matrix, i.e. all eigenvalues λ of A verify $\mathrm{Re}\lambda < 0$, in other words $\sigma(A) \subset \mathbf{C}^-$, where $\sigma(A)$ denotes the spectrum of A. In this case $\lim_{t\to\infty} e^{At} = 0$ (where 0 is the null $n \times n$ matrix.

If the right-hand member contains the exponential matrix of $-A$, then $-A$ must be a stable matrix, therefore the stability condition is $\sigma(A) \subset \mathbf{C}^+$ and one gets $\lim_{t\to\infty} e^{-At} = 0$. In these cases, by taking the limit as $t \to \infty$, Theorems 4, 6, 7, 8 and 9 give the following Lyapunov equations.

Theorem 10 *i. If $\sigma(A_1) \subset \mathbf{C}^+$ and $\sigma(A_2) \subset \mathbf{C}^+$, then the 2D controllability Gramian $\mathcal{C}_\Sigma(0,\infty;0,\infty)$ is the solution of the 2D Lyapunov equation $\mathcal{L}(A_1, A_2, B) = BB^T$.*

ii. If $\sigma(A_1) \subset \mathbf{C}^-$ and $\sigma(A_2) \subset \mathbf{C}^-$, then the 2D Gramian $\mathcal{G}^1_\Sigma(0,\infty;0,\infty)$ is the solution of the 2D Lyapunov equation $\mathcal{L}(A_1, A_2, B) = BB^T$.

iii. If $\sigma(A_1) \subset \mathbf{C}^-$ and $\sigma(A_2) \subset \mathbf{C}^+$, then the 2D Gramian $\mathcal{G}^2_\Sigma(0,\infty;0,\infty)$ is the solution of the 2D Lyapunov equation $\mathcal{L}(A_1, A_2, B) = -BB^T$.

iv. If $\sigma(A_1) \subset \mathbf{C}^+$ and $\sigma(A_2) \subset \mathbf{C}^-$, then the 2D Gramian $\mathcal{G}^3_\Sigma(0,\infty;0,\infty)$ is the solution of the 2D Lyapunov equation $\mathcal{L}(A_1, A_2, B) = -BB^T$.

In Matlab, the command gram(sys,'c') provides the 1D version of the 2D Gramian $\mathcal{G}^1_\Sigma(0,\infty;0,\infty)$ as the controllability Gramian of the 1D system sys with the state equation (20).

5 Applications in Matlab

In **Matlab**, the function ***lyap(A,Q)*** provides the solution for the classical Lyapunov equation $AP + PA^T = -Q$, so according to Theorem 3 an M-function could be defined to provide the 1D controllability Gramian as follows :

```
function [ CG1D ] = Cont_gram( A, B, t, t0)
%The function calculates the controllability Gramian
% over [t0, t}by solving a Lyapunov equation
Q=expm(-A*(t-t0))*B*transp(B)*expm(-transp(A)*(t-t0)) ...
    -B*transp(B);
CG1D = lyap(A,Q); end
```

By taking advantage of the re-writing of the left term in the equation (23) as in the beginning of the proof of Theorem 4 it is to be noticed that the 2D controllability Gramian could be found as the solution of the Lyapunov equation $A_1P+PA_1^T = P_1$ where P_1 is the solution of the Lyapunov equation $A_2P + PA_2^T = -Q$ with

$$Q = -E_1E_2BB^TE_2^TE_1^T + E_1BB^TE_1^T + E_2^TBB^TE_2^T - BB^T.$$

where $E_1 = e^{-A_1(t_1-t_1^0)}$ and $E_2 = e^{-A_2(t_2-t_2^0)}$.

This allows the coding of an M-function that provides the 2D controllability Gramian as folows:

```
function [ CG2D ] = Cont_gram2D( A1, A2, B, t10, t1, t20, t2 )
%The function calculates the 2D controllability Gramian
% over [t10,t1]x[t20, t2] by succesively solving
% two Lyapunov equations
E1= expm(-A1*(t1-t10)); E2= expm(-A2*(t2-t20));
Q=-E1*E2*B*transp(E1*E2*B)+ E1*B*transp(E1*B)+...
    E2*B*transp(E2*B)-B*transp(B);
P1 = lyap(A1,Q); CG2D=lyap(A2, -P1); end
```

For the sake of brevity the verifications (dimension compatibility, matrix commutativity...) were not included in the previous M-functions..

By applying the above function to the system $\Sigma = (A_1, A_2, B)$ with $A_1 = \begin{bmatrix} 5 & -4 & 10 \\ -1 & 2 & 2 \\ 1 & -1 & -1 \end{bmatrix}$, $A_2 = \begin{bmatrix} -9 & 8 & 20 \\ 2 & -3 & -4 \\ -2 & 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $A_1A_2 = A_2A_1$ we obtain in Matlab:

```
>> A1=[5 -4 -10; -1 2 2;1 -1 -1];
>> A2=[-9 8 20; 2 -3 -4; -2 2 3];
>> B= [1;1;1];
>> CGA1A2B = Cont_gram2D(A1, A2, B, 0, 1, 0, 1)
CGA1A2B =   1.0e+04 *    1.9607   -0.6132    0.5909
                        -0.6132    0.1918   -0.1848
                         0.5909   -0.1848    0.1783
>> det(CGA1A2B)
ans = 2.6211e+04
```

so the system is completely controllable on $[0, 1; 0, 1]$, hence according to Remark 3 on any interval.

Similar M-functions could be written for the Gramians from Theorems 7, 8, 9 by adjusting the matrices E_1, E_2 and some signs.

For $\mathcal{G}_{\Sigma}^{1}$ in Theorem 7:

```
function [ G_1 ] = CGram_G1( A1, A2, B, t10, t1, t20, t2 )
%The function calculates the 2D controllability Gramian G1
% over [t10,t1]x[t20, t2] by succesively solving
% two Lyapunov equations
E1= expm(A1*(t1-t10)); E2= expm(A2*(t2-t20));
Q=-E1*E2*B*transp(E1*E2*B)+ E1*B*transp(E1*B)+...
    E2*B*transp(E2*B)-B*transp(B);
P1 = lyap(A1,Q); G_1=lyap(A2, -P1); end
```

Keeping the system from the previous example we obtain again its controllability, but for B changed to $B_1 = [1 \quad 1 \quad 0]^T$ the Gramian has the rank 1, so the system is uncontrollable:

```
>> A1=[5 -4 -10; -1 2 2;1 -1 -1];
>> A2=[-9 8 20; 2 -3 -4; -2 2 3];
>> B= [1;1;1];
>> G1 = CGram_G1(A1, A2, B, 0, 1, 0, 1)
G1 =   130.6080  -40.0438   30.4506
       -40.0438   14.8626   -8.4626
        30.4506   -8.4626    7.5869
>> % det(G1) = 64.9710
>> B=[1; 1; 0];
>> G1 = CGram_G1(A1, A2, B, 0, 1, 0, 1)
G1 =    1.3811    1.3811    0.0000
        1.3811    1.3811   -0.0000
        0.0000   -0.0000    0.0000
```

For $\mathcal{G}_{\Sigma}^{2}$ in Theorem 8:

```
function [ G_2 ] = CGram_G2( A1, A2, B, t10, t1, t20, t2 )
%The function calculates the 2D controllability Gramian G2
% over [t10,t1]x[t20, t2] by succesively solving two
% Lyapunov equations
E1= expm(A1*(t1-t10)); E2= expm(-A2*(t2-t20));
Q=E1*E2*B*transp(E1*E2*B)- E1*B*transp(E1*B)-...
    E2*B*transp(E2*B)+B*transp(B);
P1 = lyap(A1,Q); G_2=lyap(A2, -P1); end
```

For the previously considered examples the results match the above conclusions:

```
>> G2 = CGram_G2(A1, A2, B, 0, 1, 0, 1)
G2 =   1.0e+07 * 1.0412   -0.3369    0.3354
                -0.3369    0.1090   -0.1085
                 0.3354   -0.1085    0.1080

>>% det(G2)=4.2660e+09
>> G2 = CGram_G2(A1, A2, B1, 0, 1, 0, 1)
G2 =   10.2050   10.2050   -0.0000
       10.2050   10.2050   -0.0000
       -0.0000   -0.0000   -0.0000
```

Similarly for $\mathcal{G}^3_\Sigma$ in Theorem 9 the computations are consistent regardless the matrices A_1 and $-A_2$ are stable and the results are again concordant with the ones obtained by using the other Gramians:

```
function [ G_3 ] = CGram_G3( A1, A2, B, t10, t1, t20, t2 )
%The function calculates the 2D controllability Gramian G3
% over [t10,t1]x[t20, t2] by succesively solving two
% Lyapunov equations
E1= expm(-A1*(t1-t10)); E2= expm(A2*(t2-t20));
Q=E1*E2*B*transp(E1*E2*B)- E1*B*transp(E1*B)-...
    E2*B*transp(E2*B)+B*transp(B);
P1 = lyap(A1,Q); G_3=lyap(A2, -P1); end

>> G3 = CGram_G3(A1, A2, B, 0, 1, 0, 1)
G3 =    2.1496    0.6148    0.6422
        0.6148    0.2002    0.1704
        0.6422    0.1704    0.2067
>>% det(G3) =  3.9920e-04
>> G3 = CGram_G3(A1, A2, B1, 0, 1, 0, 1)
G3 =    0.1869    0.1869    0.0000
        0.1869    0.1869    0.0000
        0.0000    0.0000    0.0000
```

In **Matlab**, the command ***gram(sys,'c')*** provides the 1D version of the 2D Gramian $\mathcal{G}^1_\Sigma(0,\infty;0,\infty)$ as the controllability Gramian of the 1D system sys of the form (20). The command ***gram(sys,'c',opt)*** allows the computation of the 1D controllability Gramian on an interval, but the limitation is that the matrix A must be stable, otherwise an error message is returned.

The previuos functions do not impose the stability of the matrices A_i or $-A_i$, but, as in Theorem 10 we must check the stability if we need a $(0, \infty; 0, \infty)$ Gramian to be finite. The following M-script illustrates that, taking into account that the right member of the equation(s) becomes BB_T for $\mathcal{C}_\Sigma$ and $\mathcal{G}^1_\Sigma$, respectively $-BB_T$ for $\mathcal{G}^2_\Sigma$ and $\mathcal{G}^3_\Sigma$.

```
%Given A1, A2 commuting and B the appropriate (0, In;, 0 Inf)
% controllability Gramian is calculated according to the
% stability of the pairs (-A1, -A2), (A1, A2),
% (A1, -A2) or (-A1, A2)
if (all(eig(A1)> 0) &  all(eig(A2)>0) )
    disp('the Gramian is C')
    Q = - B*transp(B);
    P1 = lyap(A1,Q);
    C=lyap(A2, -P1);
    disp(C);
    elseif (all(eig(A1)< 0) & all(eig(A2)<0) )
            disp('the Gramian is G1')
            Q = - B*transp(B);
            P1 = lyap(A1,Q);
            G1=lyap(A2, -P1);
            disp(G1);
        elseif (all(eig(A1)< 0) & all(eig(A2)>0) )
                disp('the Gramian is G2')
                Q = B*transp(B);
                P1 = lyap(A1,Q);
                G2=lyap(A2, -P1);
                disp(G2);
            elseif (all(eig(A1)> 0) & all(eig(A2)<0) )
                 disp('the Gramian is G3')
                 Q = B*transp(B);
                 P1 = lyap(A1,Q);
                 G3=lyap(A2, -P1);
                 disp(G3);
else
     disp('None of the conditions are fulfilled.')
end
```

The matrices $-A_1$ and A_2 considered above are stable since $\sigma(A_1) = \{1, 2, 3\}$

and $\sigma(A_2) = \{-5, -3, -1\}$. By changing alternatively the signs and eventually replacing A_1 such that both A_1 and $-A_1$ be unstable we pass the script through all cases:

```
>> A1=[5 -4 -10; -1 2 2;1 -1 -1];
>> A2=[-9 8 20; 2 -3 -4; -2 2 3];
>> CG_0Inf_0Inf
the Gramian is G3
    2.5833    0.8833    0.7000
    0.8833    0.3833    0.2000
    0.7000    0.2000    0.2167
>> A1 = -A1;
>> CG_0Inf_0Inf
the Gramian is G1
    2.5833    0.8833    0.7000
    0.8833    0.3833    0.2000
    0.7000    0.2000    0.2167
>> A2=-A2;
>> CG_0Inf_0Inf
the Gramian is G2
    2.5833    0.8833    0.7000
    0.8833    0.3833    0.2000
    0.7000    0.2000    0.2167
>> A1=-A1;
>> CG_0Inf_0Inf
the Gramian is C
    2.5833    0.8833    0.7000
    0.8833    0.3833    0.2000
    0.7000    0.2000    0.2167
>> A1=diag([1, 2, -1]);
>> CG_0Inf_0Inf
None of the conditions are fulfilled.
```

So, based upon Theorem 10 the function ***gram(sys,'c')*** can be extended for the 2D case.

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