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NONUNIFORM POLYNOMIAL DICHOTOMY WITH LYAPUNOV TYPE NORMS*

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Dedicated to Dr. Vasile Drăgan on the occasion of his 70$^{\text{th}}$ anniversary

Abstract

The paper considers a general concept of polynomial dichotomy which includes as particular cases some well-known dichotomy concepts. The main objective is to obtain some characterizations of the nonuniform polynomial dichotomy behavior with respect to a family of norms compatible with the projection families.
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1 Introduction

In the literature devoted to dynamical systems and evolution equations, the uniform exponential dichotomy concept introduced by O. Perron [10] is considered one of the most important asymptotic properties. However, there are some cases, such as the nonautonomous settings, where this concept is

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too restrictive, so we need to look for a more general behavior. In this sense we refer to the nonuniform polynomial dichotomy notion which was firstly mentioned in 2009 by L. Barreira and C. Valls [1] for the continuous case, A. Bento and C. Silva [3] for discrete-time systems and then it was studied in the works of M. Megan, T. Ceauşu and M.L.Rămneanţu. [7], [11]. Also, another approaches of the dichotomy concept are discussed in the papers [9], [8].

The aim of this paper is to obtain some characterizations of the nonuniform polynomial dichotomy behavior using some families of norms which are compatible with the projection families, because in this way we reduce our theory to the uniform case. The idea of using such a sequence of norms comes from the work of L. Barreira, D. Dragičević and C. Valls [2] who introduced in 2015 the notion of exponential dichotomy with respect to a family of norms and which was generalized by M. Megan and V. Crai [5].

2 Preliminaries

Let X be a real or complex Banach space and $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators acting on X. The norms on X and on $\mathcal{B}(X)$ will be denoted by $\|.\|$. The identity operator on X is denoted by I. We also denote by

$$\Delta = \{(t,s) \in \mathbb{R}^2_+ : t \geq s\}, \qquad T = \{(t,s,t_0) \in \mathbb{R}^3_+ : t \geq s \geq t_0\}$$

Definition 1. An application $U : \Delta \to \mathcal{B}(X)$ is called *evolution operator* on X if

(e_1) $U(t,t) = I$ for every $t \geq 0$.

(e_2) $U(t,s)U(s,t_0) = U(t,t_0)$ for all $(t,s,t_0) \in T$.

Definition 2. An evolution operator $U : \Delta \to \mathcal{B}(X)$ is said to be *strongly measurable* if for all $(s,x) \in \mathbb{R}_+ \times X$, the mapping $t \mapsto \|U(t,s)x\|$ is measurable on $[s,\infty)$.

Definition 3. An application $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is said to be a *projection family* on X if $P^2(t) = P(t)$, for all $t \geq 0$.

Remark 1. *If $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is a projection family on X, then the mapping $Q : \mathbb{R}_+ \to \mathcal{B}(X), Q(t) = I - P(t)$ is also a projection family on X,* *which is called the complementary projection of P. In what follows, we will denote*

$$U_P(t,s) = U(t,s)P(s), \qquad U_Q(t,s) = U(t,s)Q(s), \quad \textit{for all } (t,s) \in \Delta.$$

Remark 2. *It is obvious that the following relations take place:*

$$U_P(t,t) = P(t), \qquad U_Q(t,t) = Q(t), \ \textit{for all } (t,s) \in \Delta.$$
$$U_P(t,t_0) = U_P(t,s)U_P(s,t_0), \quad U_Q(t,t_0) = U_Q(t,s)U_Q(s,t_0), \forall (t,s,t_0) \in T.$$

Definition 4. A projection family $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is said to be

(i) *strongly continuous* if for all $x \in X$ the mapping $t \mapsto P(t)x$ is continuous on $\mathbb{R}_+$.

(ii) *invariant* to the evolution operator $U : \Delta \to \mathcal{B}(X)$ if $U(t,s)P(s) = P(t)U(t,s)$, for all $(t,s) \in \Delta$.

(iii) *strongly invariant* to the evolution operator $U : \Delta \to \mathcal{B}(X)$ if it is invariant to the operator U and for all $(t,s) \in \Delta$, $U(t,s) : KerP(s) = RangeQ(s) \to KerP(t) = RangeQ(t)$ is invertible.

Remark 3. *If the projection family $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is strongly invariant to the evolution operator $U : \Delta \to \mathcal{B}(X)$ and $Q : \mathbb{R}_+ \to \mathcal{B}(X)$ is the complementary projection family of P, then for all $(t,s) \in \Delta$ there exists the application $V : \Delta \to \mathcal{B}(X)$, $V(t,s) : RangeQ(t) \to RangeQ(s)$ with the following properties:*

(v_1) $U(t,s)V(t,s)Q(t)x = Q(t)x$

(v_2) $V(t,s)U(t,s)Q(s)x = Q(s)x$

(v_3) $V(t,t_0) = V(s,t_0)V(t,s)$

(v_4) $V(t,s)Q(t) = Q(s)V(t,s)Q(t)$,

for all $(t,s,t_0) \in T$. We will denote $V_Q(t,s) = V(t,s)Q(t)$.

Proof. See [6]. □

In what follows, if $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is an invariant projection family to the evolution operator $U : \Delta \to \mathcal{B}(X)$, we will say that (U,P) is a dichotomic pair.

3 Polynomial growth concepts

Definition 5. We say that the pair (U, P) *has nonuniform polynomial growth (n.p.g.)* if there are a nondecreasing function $M : \mathbb{R}_+ \to [1, \infty)$ and a positive constant $\omega > 0$ such that

(npg_1) $(s+1)^\omega \|U_P(t,s)x\| \le M(s)(t+1)^\omega \|P(s)x\|$

(npg_2) $(s+1)^\omega \|Q(s)x\| \le M(t)(t+1)^\omega \|U_Q(t,s)x\|$

for all $(t, s, x) \in \Delta \times X$.

Theorem 1. *If the projection family $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is strongly invariant to the evolution operator $U : \Delta \to \mathcal{B}(X)$, the the pair (U, P) has nonuniform polynomial growth if and only if there exist a nondecreasing function $M : \mathbb{R}_+ \to [1, \infty)$ and a positive constant $\omega > 0$ such that:*

$(npg1'')$ $(s+1)^\omega \|U_P(t,s)x\| \le M(s)(t+1)^\omega \|P(s)x\|$

$(npg2'')$ $(s+1)^\omega \|V_Q(t,s)x\| \le M(t)(t+1)^\omega \|Q(t)x\|$

for all $(t, s, x) \in \Delta \times X$.

Proof. We need to proof that relations (npg_2) and (npg_2'') are equivalent. We have

$$\begin{aligned}\|V_Q(t,s)x\| = \|V(t,s)Q(t)x\| = \|Q(s)V(t,s)Q(t)x\| \le \\ \le M(t)\left(\frac{t+1}{s+1}\right)^\omega \|U(t,s)V(t,s)Q(t)x\| = \\ = M(t)\left(\frac{t+1}{s+1}\right)^\omega \|Q(t)x\|.\end{aligned}$$

$$\begin{aligned}\|Q(s)x\| &= \|V(t,s)U(t,s)Q(s)x\| = \|V(t,s)Q(t)U(t,s)Q(s)x\| = \\ &= \|V_Q(t,s)U_Q(t,s)x\| \le M(t)\left(\frac{t+1}{s+1}\right)^\omega \|Q(t)U(t,s)Q(s)x\| = \\ &= M(t)\left(\frac{t+1}{s+1}\right)^\omega \|U_Q(t,s)x\|.\end{aligned}$$

□

Definition 6. We say that a family of norms $\mathcal{N} = \{\|\cdot\|_t,\ t \geq 0\}$ is *compatible with the projection family* $P : \mathbb{R}_+ \to \mathcal{B}(X)$ if there exists a nondecreasing function $C : \mathbb{R}_+ \to [1, \infty)$ such that:

$$\|x\| \leq \|x\|_t \leq C(t)\left(\|P(t)x\| + \|Q(t)x\|\right)$$

for all $(t, x) \in \mathbb{R}_+ \times X$.

Definition 7. Let $\mathcal{N} = \{\|\cdot\|_t,\ t \geq 0\}$ be a family of norms on X, compatible with the projection family $P : \mathbb{R}_+ \to \mathcal{B}(X)$ and P strongly invariant to U. We say that the pair (U, P) has *polynomial growth with respect to the family of norms* $\mathcal{N}$ ($\mathcal{N}$.p.g.) if there exist the constants $M > 1$ and $\omega > 0$ such that:

($\mathcal{N}pg_1$) $(s+1)^\omega \|U_P(t,s)x\|_t \leq M(t+1)^\omega \|P(s)x\|_s$

($\mathcal{N}pg_2$) $(s+1)^\omega \|V_Q(t,s)x\|_s \leq M(t+1)^\omega \|Q(t)x\|_t$

for all $(t, s, x) \in \Delta \times X$.

Theorem 2. *Let (U, P) be a dichotomic pair with P strongly invariant to U. Then (U, P) has nonuniform polynomial growth if and only if there exists $\mathcal{N} = \{\|\cdot\|_t,\ t \geq 0\}$ a family of norms on X compatible with P such that (U, P) has polynomial growth with respect to $\mathcal{N}$.*

Proof. Necessity.
We suppose that (U, P) has n.p.g. We define $\mathcal{N} = \{\|\cdot\|_t,\ t \geq 0\}$ the family of norms by

$$\|x\|_t = \sup_{\tau \geq t} \left(\frac{t+1}{\tau+1}\right)^\omega \|U_P(\tau, t)x\| + \sup_{s \leq t} \left(\frac{s+1}{t+1}\right)^\omega \|V_Q(t, s)x\|,$$

where $\omega > 0$ is given by Definition 5. We have that $\mathcal{N}$ is a family of norms compatible with P. Indeed,

1. $\|x\|_t \geq \|P(t)x\| + \|Q(t)x\| \geq \|P(t)x + Q(t)x\| = \|x\|$

2. $\|P(t)x\|_t = \sup_{\tau \geq t} \left(\frac{t+1}{\tau+1}\right)^\omega \|U_P(\tau, t)x\| \leq M(t)\|P(t)x\|$

3. $\|Q(t)x\|_t = \sup_{s \leq t} \left(\frac{s+1}{t+1}\right)^\omega \|V_Q(t, s)x\| \leq M(t)\|Q(t)x\|$

4. $\|x\|_t = \|P(t)x + Q(t)x\|_t \leq \|P(t)x\|_t + \|Q(t)x\|_t$

$$\leq N(t)\left(\|P(t)x\| + \|Q(t)x\|\right).$$

We prove now the relations $(\mathcal{N}pg_1)$ and $(\mathcal{N}pg_2)$.

$$\|U_P(t,s)x\|_t = \|U(t,s)P(s)x\|_t = \sup_{\tau\geq t}\left(\frac{t+1}{\tau+1}\right)^{\omega}\|U(\tau,t)P(t)U(t,s)P(s)x\| =$$
$$= \sup_{\tau\geq t}\left(\frac{t+1}{\tau+1}\right)^{\omega}\|U(\tau,t)U(t,s)P(s)x\| = \sup_{\tau\geq t}\left(\frac{t+1}{\tau+1}\right)^{\omega}\|U_P(\tau,s)x\| =$$
$$= \sup_{\tau\geq t}\left(\frac{t+1}{s+1}\right)^{\omega}\left(\frac{s+1}{\tau+1}\right)^{\omega}\|U_P(\tau,s)x\| =$$
$$= \left(\frac{t+1}{s+1}\right)^{\omega}\sup_{\tau\geq t}\left(\frac{s+1}{\tau+1}\right)^{\omega}\|U_P(\tau,s)x\| \leq$$
$$\leq \left(\frac{t+1}{s+1}\right)^{\omega}\sup_{\tau\geq s}\left(\frac{s+1}{\tau+1}\right)^{\omega}\|U_P(\tau,s)x\| = \left(\frac{t+1}{s+1}\right)^{\omega}\|P(s)x\|_s \leq$$
$$\leq 2\left(\frac{t+1}{s+1}\right)^{\omega}\|P(s)x\|_s = M\left(\frac{t+1}{s+1}\right)^{\omega}\|P(s)x\|_s.$$

For $(\mathcal{N}pg_2)$ we have

$$\|V_Q(t,s)x\|_s = \|V(t,s)Q(t)x\|_s = \sup_{\tau\leq s}\left(\frac{\tau+1}{s+1}\right)^{\omega}\|V(s,\tau)Q(s)V(t,s)Q(t)x\|$$
$$= \sup_{\tau\leq s}\left(\frac{\tau+1}{s+1}\right)^{\omega}\|V(s,\tau)V(t,s)Q(t)x\| = \sup_{\tau\leq s}\left(\frac{\tau+1}{s+1}\right)^{\omega}\|V_Q(t,\tau)x\| =$$
$$= \sup_{\tau\leq s}\left(\frac{t+1}{s+1}\right)^{\omega}\left(\frac{\tau+1}{t+1}\right)^{\omega}\|V_Q(t,\tau)x\|$$
$$= \left(\frac{t+1}{s+1}\right)^{\omega}\sup_{\tau\leq s}\left(\frac{\tau+1}{t+1}\right)^{\omega}\|V_Q(t,\tau)x\| \leq$$
$$\leq \left(\frac{t+1}{s+1}\right)^{\omega}\sup_{\tau\leq t}\left(\frac{\tau+1}{t+1}\right)^{\omega}\|V_Q(t,\tau)x\| = \left(\frac{t+1}{s+1}\right)^{\omega}\|Q(t)x\|_t \leq$$
$$\leq 2\left(\frac{t+1}{s+1}\right)^{\omega}\|Q(t)x\|_t = M\left(\frac{t+1}{s+1}\right)^{\omega}\|Q(t)x\|_t.$$

Sufficiency.
We suppose that there exists $\mathcal{N} = \{\|\cdot\|_t,\ t\geq 0\}$ a family of norms compatible with P such that $(\mathcal{N}pg_1)$ and $(\mathcal{N}pg_2)$ take place. We prove relations (npg_1) and (npg_2).

$$\|U_P(t,s)x\| \leq \|U_P(t,s)x\|_t \leq M\left(\frac{t+1}{s+1}\right)^{\omega}\|P(s)x\|_s \leq$$
$$\leq MC(s)\left(\frac{t+1}{s+1}\right)^{\omega}\|P(s)x\| = M(s)\left(\frac{t+1}{s+1}\right)^{\omega}\|P(s)x\|,$$

so (npg_1) holds. For (npg_2) we have

$$\|V_Q(t,s)x\| \leq \|V_Q(t,s)x\|_s \leq M\left(\frac{t+1}{s+1}\right)^{\omega}\|Q(t)x\|_t \leq$$
$$\leq MC(t)\left(\frac{t+1}{s+1}\right)^{\omega}\|Q(t)x\| =$$
$$= M(t)\left(\frac{t+1}{s+1}\right)^{\omega}\|Q(t)x\|.$$

□

4 Polynomial dichotomy concepts

Let (U,P) be a dichotomic pair.

Definition 8. We say that the pair (U,P) is *nonuniformly polynomially dichotomic (n.p.d.)* if there are a nondecreasing function $N:\mathbb{R}_+ \to [1,\infty)$ and a positive constant $\nu > 0$ such that:

(npd_1) $(t+1)^{\nu}\|U_P(t,s)x\| \leq N(s)(s+1)^{\nu}\|P(s)x\|$

(npd_2) $(t+1)^{\nu}\|Q(s)x\| \leq N(t)(s+1)^{\nu}\|U_Q(t,s)x\|$

for all $(t,s,x) \in \Delta \times X$.

Remark 4. *If (U,P) is n.p.d., then (U,P) has n.p.g., but the converse implication is not true.*

Theorem 3. *If $P:\mathbb{R}_+ \to \mathcal{B}(X)$ is a projection family strongly invariant to the evolution operator $U:\Delta \to \mathcal{B}(X)$, then the pair (U,P) is nonuniformly polynomially dichotomic if and only if there exist a nondecreasing function $N:\mathbb{R}_+ \to [1,\infty)$ and a positive constant $\nu > 0$ with*

(npd_1'') $(t+1)^{\nu}\|U_P(t,s)x\| \leq N(s)(s+1)^{\nu}\|P(s)x\|$

(npd_2'') $(t+1)^{\nu}\|V_Q(t,s)x\| \leq N(t)(s+1)^{\nu}\|Q(t)x\|,$

for all $(t,s,x) \in \Delta \times X$.

Proof. It is obvious that we have to prove the equivalence between (npd_2) and (npd_2''). Firstly we suppose that (npd_2) holds. Then we have:

$$\left(\frac{t+1}{s+1}\right)^{\nu} \|V_Q(t,s)x\| = \left(\frac{t+1}{s+1}\right)^{\nu} \|V(t,s)Q(t)x\| =$$
$$= \left(\frac{t+1}{s+1}\right)^{\nu} \|Q(s)V(t,s)Q(t)x\| \leq$$

$$\leq \left(\frac{t+1}{s+1}\right)^{\nu} \cdot N(t) \cdot \left(\frac{s+1}{t+1}\right)^{\nu} \|U(t,s)Q(s)V(t,s)Q(t)x\| =$$
$$= N(t)\|U(t,s)V(t,s)Q(t)x\| = N(t)\|Q(t)x\|, \text{ so } (npd_2'') \text{ is proved .}$$

Conversely we obtain

$$\left(\frac{t+1}{s+1}\right)^{\nu} \|Q(s)x\| = \left(\frac{t+1}{s+1}\right)^{\nu} \|V(t,s)U(t,s)Q(s)x\| =$$
$$= \left(\frac{t+1}{s+1}\right)^{\nu} \|V(t,s)Q(t)U(t,s)Q(s)x\| =$$
$$= \left(\frac{t+1}{s+1}\right)^{\nu} \|V_Q(t,s)U(t,s)Q(s)x\| \leq$$
$$\leq \left(\frac{t+1}{s+1}\right)^{\nu} \cdot N(t) \cdot \left(\frac{s+1}{t+1}\right)^{\nu} \|Q(t)U(t,s)Q(s)x\| =$$
$$= N(t)\|U(t,s)Q(s)x\| = N(t)\|U_Q(t,s)x\|$$

□

Now, let us consider (U,P) a dichotomic pair with $P : \mathbb{R}_+ \to \mathcal{B}(X)$ a projection family strongly invariant to the evolution operator $U : \Delta \to \mathcal{B}(X)$ and $\mathcal{N} = \{\|\cdot\|_t,\ t \geq 0\}$ a family of norms compatible with P.

Definition 9. We say that the pair (U,P) is *polynomially dichotomic with respect to the family of norms* $\mathcal{N} = \{\|\cdot\|_t,\ t \geq 0\}$ $(\mathcal{N}pd)$ if there exist the constants $N > 1$ and $\nu > 0$ such that

$(\mathcal{N}pd_1)$ $(t+1)^{\nu}\|U_P(t,s)x\|_t \leq N(s+1)^{\nu}\|P(s)x\|_s$

$(\mathcal{N}pd_2)$ $(t+1)^{\nu}\|V_Q(t,s)x\|_s \leq N(s+1)^{\nu}\|Q(t)x\|_t$,

for all $(t,s,x) \in \Delta \times X$.

Remark 5. *We have that* ($\mathcal{N}pd$) *implies* ($\mathcal{N}pg$).

Theorem 4. *Let* (U, P) *be a dichotomic pair with* P *strongly invariant to* U. *Then* (U, P) *is nonuniformly polynomially dichotomic if and only if there exists a family of norms* $\mathcal{N} = \{\|\cdot\|_t,\ t \geq 0\}$ *compatible with the projection family* $P : \mathbb{R}_+ \to \mathcal{B}(X)$ *such that* (U, P) *is polynomially dichotomic with respect to the family of norms* $\mathcal{N}$.

Proof. Necessity.
We suppose that (U, P) is n.p.d. We consider $\mathcal{N} = \{\|\cdot\|_t,\ t \geq 0\}$ a family of norms defined by

$$\|x\|_t = \sup_{\tau \geq t} \left(\frac{\tau + 1}{t + 1}\right)^{\nu} \|U_P(\tau, t)x\| + \sup_{s \leq t} \left(\frac{t + 1}{s + 1}\right)^{\nu} \|V_Q(t, s)x\|,$$

where $\nu > 0$ is given by Definition 8. We have that $\mathcal{N}$ is a family of norms compatible with P. Indeed, if (U, P) is u.p.d. we have

(1.) $\|x\|_t \geq \|P(t)x\| + \|Q(t)x\| \geq \|P(t)x + Q(t)x\| = \|x\|$

(2.) $\|P(t)x\|_t = \sup_{\tau \geq t} \left(\frac{\tau + 1}{t + 1}\right)^{\nu} \|U_P(\tau, t)x\| \leq N(t)\|P(t)x\|$

(3.) $\|Q(t)x\|_t = \sup_{s \leq t} \left(\frac{t + 1}{s + 1}\right)^{\nu} \|V_Q(t, s)x\| \leq N(t)\|Q(t)x\|$

(4.)

$$\begin{aligned} \|x\|_t =& \|P(t)x + Q(t)x\|_t \leq \|P(t)x\|_t + \|Q(t)x\|_t \\ & \leq N(t)\left(\|P(t)x\| + \|Q(t)x\|\right). \end{aligned}$$

Now we have to prove ($\mathcal{N}pd_1$) and ($\mathcal{N}pd_2$).

$$\begin{aligned} &\left(\frac{t + 1}{s + 1}\right)^{\nu} \|U_P(t, s)x\|_t = \left(\frac{t + 1}{s + 1}\right)^{\nu} \|U(t, s)P(s)x\|_t = \\ &= \left(\frac{t + 1}{s + 1}\right)^{\nu} \|P(t)U(t, s)P(s)x\|_t = \\ &= \left(\frac{t + 1}{s + 1}\right)^{\nu} \sup_{\tau \geq t} \left(\frac{\tau + 1}{t + 1}\right)^{\nu} \|U(\tau, t)P(t)U(t, s)P(s)x\| = \\ &= \left(\frac{t + 1}{s + 1}\right)^{\nu} \sup_{\tau \geq t} \left(\frac{\tau + 1}{t + 1}\right)^{\nu} \|U(\tau, t)U(t, s)P(s)x\| = \end{aligned}$$

$$= \left(\frac{t+1}{s+1}\right)^{\nu} \sup_{\tau \geq t} \left(\frac{\tau+1}{t+1}\right)^{\nu} \|U(\tau,s)P(s)x\| \leq$$
$$\leq \left(\frac{t+1}{s+1}\right)^{\nu} \sup_{\tau \geq s} \left(\frac{\tau+1}{t+1}\right)^{\nu} \|U_P(\tau,s)x\| =$$
$$= \sup_{\tau \geq s} \left(\frac{\tau+1}{s+1}\right)^{\nu} \|U_P(\tau,s)x\| = \|P(s)x\|_s \leq 2\|P(s)x\|_s =$$
$$= N\|P(s)x\|_s, \text{ where } N = 2 > 1.$$

For $(\mathcal{N}pd_2)$ we have

$$\left(\frac{t+1}{s+1}\right)^{\nu} \|V_Q(t,s)x\|_s = \left(\frac{t+1}{s+1}\right)^{\nu} \|V(t,s)Q(t)x\|_s =$$
$$= \left(\frac{t+1}{s+1}\right)^{\nu} \|Q(s)V(t,s)Q(t)x\|_s =$$
$$= \left(\frac{t+1}{s+1}\right)^{\nu} \sup_{\tau \leq s} \left(\frac{s+1}{\tau+1}\right)^{\nu} \|V(s,\tau)Q(s)V(t,s)Q(t)x\| =$$
$$= \left(\frac{t+1}{s+1}\right)^{\nu} \sup_{\tau \leq s} \left(\frac{s+1}{\tau+1}\right)^{\nu} \|V(s,\tau)V(t,s)Q(t)x\| =$$
$$= \left(\frac{t+1}{s+1}\right)^{\nu} \sup_{\tau \leq s} \left(\frac{s+1}{\tau+1}\right)^{\nu} \|V_Q(t,\tau)x\| \leq$$
$$\leq \left(\frac{t+1}{s+1}\right)^{\nu} \sup_{\tau \leq t} \left(\frac{s+1}{\tau+1}\right)^{\nu} \|V_Q(t,\tau)x\| =$$
$$= \sup_{\tau \leq t} \left(\frac{t+1}{\tau+1}\right)^{\nu} \|V_Q(t,\tau)x\| = \|Q(t)x\|_t \leq 2\|Q(t)x\|_t =$$
$$= N\|Q(t)x\|_t, \text{ where } N = 2 > 1.$$

Sufficiency.
We suppose that there exists $\mathcal{N} = \{\|\cdot\|_t,\ t \geq 0\}$ a family of norms compatible with P such that $(\mathcal{N}pd_1)$ and $(\mathcal{N}pd_2)$ take place. Firstly we prove relation (npd_1).

$$\left(\frac{t+1}{s+1}\right)^{\nu} \|U_P(t,s)x\| \leq \left(\frac{t+1}{s+1}\right)^{\nu} \|U_P(t,s)x\|_t \leq$$
$$\leq N \left(\frac{t+1}{s+1}\right)^{\nu} \left(\frac{s+1}{t+1}\right)^{\nu} \|P(s)x\|_s =$$
$$= N\|P(s)x\|_s \leq NC(s)\|P(s)x\| = N(s)\|P(s)x\|,$$

so (npd_1) holds.

Further we show (npd_2).

$$\begin{aligned}\left(\frac{t+1}{s+1}\right)^{\nu}\|V_Q(t,s)x\| &\leq \left(\frac{t+1}{s+1}\right)^{\nu}\|V_Q(t,s)x\|_s \leq \\ &\leq \left(\frac{t+1}{s+1}\right)^{\nu}\left(\frac{s+1}{t+1}\right)^{\nu}\|Q(t)x\|_t = \\ &= N\|Q(t)x\|_t \leq NC(t)\|Q(t)x\| = N(t)\|Q(t)x\|.\end{aligned}$$

□

In what follows, we consider $U : \Delta \to \mathcal{B}(X)$ a strongly measurable evolution operator, $P : \mathbb{R}_+ \to \mathcal{B}(X)$ a projection family strongly continuous and strongly invariant to U and $\mathcal{N} = \{\|\cdot\|_t,\ t \geq 0\}$ a family of norms compatible with P with $t \mapsto \|\cdot\|_t$continuous.

Theorem 5. *The pair (U, P) is polynomially dichotomic with respect to a family of norms if and only if (U, P) has polynomial growth with respect to the family of norms and there exist the constants $D > 1$ and $d > 0$ such that*

$$(\mathcal{N}pD_1)\quad \int_s^{\infty}\left(\frac{\tau+1}{s+1}\right)^{d}\cdot\frac{1}{\tau+1}\|U_P(\tau,s)x\|_\tau d\tau \leq D\|P(s)x\|_s$$

$$(\mathcal{N}pD_2)\quad \int_s^{t}\left(\frac{t+1}{\tau+1}\right)^{d}\cdot\frac{1}{\tau+1}\|V_Q(t,\tau)x\|_\tau d\tau \leq D\|Q(t)x\|_t,$$

for all $(t, s, x) \in \Delta \times X$.

Proof. Necessity. We suppose that (U, P) is $\mathcal{N}pd$. Then, from Remark 5 we have that (U, P) has $\mathcal{N}pg$. Let $d \in (0, \nu)$. We prove $(\mathcal{N}pD_1)$.

$$\begin{aligned}&\int_s^{\infty}\left(\frac{\tau+1}{s+1}\right)^{d}\frac{1}{\tau+1}\|U(\tau,s)P(s)x\|_\tau d\tau \leq \\ &\leq \int_s^{\infty}\left(\frac{\tau+1}{s+1}\right)^{d}\frac{1}{\tau+1}N\left(\frac{s+1}{\tau+1}\right)^{\nu}\|P(s)x\|_s d\tau =\end{aligned}$$

$$= N(s+1)^{\nu-d}\|P(s)x\|_s \int_s^{\infty} (\tau+1)^{d-\nu-1} d\tau \leq$$

$$\leq \frac{N}{\nu-d}(s+1)^{\nu-d}(s+1)^{d-\nu}\|P(s)x\|_s =$$

$$= \frac{N}{\nu-d}\|P(s)x\|_s \leq D\|P(s)x\|_s, \text{ where } D = 1+\frac{N}{\nu-d}.$$

For $(\mathcal{N}pD_2)$ we have

$$\int_s^t \left(\frac{t+1}{\tau+1}\right)^d \frac{1}{\tau+1}\|V_Q(t,\tau)x\|_\tau d\tau \leq$$

$$\leq N\int_s^t \left(\frac{t+1}{\tau+1}\right)^d \frac{1}{\tau+1}\left(\frac{\tau+1}{t+1}\right)^\nu \|Q(t)x\|_t d\tau =$$

$$= N(t+1)^{d-\nu}\|Q(t)x\|_t \int_s^t (\tau+1)^{\nu-d-1} d\tau \leq$$

$$\leq \frac{N}{\nu-d}(t+1)^{d-\nu}(t+1)^{\nu-d}\|Q(t)x\|_t =$$

$$= \frac{N}{\nu-d}\|Q(t)x\|_t \leq D\|Q(t)x\|_t, \text{ where } D = 1+\frac{N}{\nu-d}.$$

Sufficiency. Now we suppose that (U,P) has polynomial growth with respect to a family of norms and there exist the constants $D>1$ and $d>0$ such that $(\mathcal{N}pD_1)$ and $(\mathcal{N}pD_2)$ are satisfied. We prove that (U,P) is polynomially dichotomic with respect to that family of norms.
For $(\mathcal{N}pd_1)$ we consider firstly $t \geq 2s+1$.

$$\left(\frac{t+1}{s+1}\right)^d \|U_P(t,s)x\|_t = \frac{2}{t+1}\int_{\frac{t-1}{2}}^t \left(\frac{t+1}{s+1}\right)^d \|U(t,\tau)U(\tau,s)P(s)x\|_t d\tau =$$

$$= \frac{2}{t+1}\int_{\frac{t-1}{2}}^t \left(\frac{t+1}{s+1}\right)^d M\left(\frac{t+1}{\tau+1}\right)^\omega \|U(\tau,s)P(s)x\|_\tau d\tau \leq$$

$$\leq 2M\int_{\frac{t-1}{2}}^t \frac{1}{\tau+1}\left(\frac{\tau+1}{s+1}\right)^d \left(\frac{t+1}{\tau+1}\right)^\omega \left(\frac{t+1}{\tau+1}\right)^d \|U_P(\tau,s)x\|_\tau d\tau =$$

$$= 2M \int_{\frac{t-1}{2}}^{t} \frac{1}{\tau+1} \left(\frac{\tau+1}{s+1}\right)^{d} \left(\frac{t+1}{\tau+1}\right)^{\omega+d} \|U_P(\tau,s)x\|_\tau d\tau \leq$$

$$\leq 2^{\omega+d+1} M \int_{s}^{\infty} \frac{1}{\tau+1} \left(\frac{\tau+1}{s+1}\right)^{d} \|U_P(\tau,s)\|_\tau d\tau \leq$$

$$\leq 2^{\omega+d+1} MD\|P(s)x\|_s.$$

If $t \in [s, 2s+1)$ we have

$$\left(\frac{t+1}{s+1}\right)^{d} \|U_P(t,s)x\|_t \leq M \left(\frac{t+1}{s+1}\right)^{d+\omega} \|P(s)x\|_s \leq 2^{d+\omega}\|P(s)x\|_s.$$

So, we obtained

$$(t+1)^d \|U_P(t,s)x\|_t \leq N(s+1)d\|P(s)x\|_s, \text{ for all } (t,s,x) \in \Delta \times X,$$

where $N = 1 + M \cdot 2^{d+\omega+1}$. In order to prove $(\mathcal{N}pd_2)$ we do a similar computation. For $t \geq 2s+1$ we have

$$\left(\frac{t+1}{s+1}\right)^{d} \|V_Q(t,s)x\|_s = \frac{1}{s+1} \int_{s}^{2s+1} \left(\frac{t+1}{s+1}\right)^{d} \|V(t,s)Q(t)x\|_s d\tau =$$

$$= \frac{1}{s+1} \int_{s}^{2s+1} \left(\frac{t+1}{s+1}\right)^{d} \|V(\tau,s)V(t,\tau)Q(t)x\|_s d\tau \leq$$

$$\leq M \int_{s}^{2s+1} \frac{1}{s+1} \left(\frac{t+1}{s+1}\right)^{d} \left(\frac{\tau+1}{s+1}\right)^{\omega} \|Q(\tau)V(t,\tau)Q(t)x\|_\tau d\tau =$$

$$= M \int_{s}^{2s+1} \frac{1}{\tau+1} \left(\frac{t+1}{\tau+1}\right)^{d} \left(\frac{\tau+1}{s+1}\right)^{\omega+d+1} \|V_Q(t,\tau)x\|_\tau d\tau \leq$$

$$\leq M \cdot 2^{\omega+d+1} \int_{s}^{t} \left(\frac{t+1}{\tau+1}\right)^{d} \frac{1}{\tau+1} \|V_Q(t,\tau)x\|_\tau d\tau \leq$$

$$\leq MD \cdot 2^{\omega+d+1} \|Q(t)x\|_t.$$

Now, for $t \in [s, 2s+1)$ it results

$$\left(\frac{t+1}{s+1}\right)^{d} \|V_Q(t,s)x\|_s \leq M \left(\frac{t+1}{s+1}\right)^{\omega+d} \|Q(t)x\|_t \leq M \cdot 2^{\omega+d}\|Q(t)x\|_t.$$

So, we obtained

$$(t+1)^d \|V_Q(t,s)x\|_s \leq N(t+1)^d \|Q(t)x\|_t, \text{ for all } (t,s,x) \in \Delta \times X,$$

where $N = 1 + MD \cdot 2^{d+\omega+1}$. Finally, we obtained that conditions $(\mathcal{N}pd_1)$ and $(\mathcal{N}pd_2)$ are fulfilled, from where it results the conclusion. $\square$

References

[1] L. Barreira, C. Valls. Polynomial growth rates. *Nonlinear Anal.* 71:5208-5219, (2009).

[2] L. Barreira, D. Dragičević and C. Valls, Characterization of strong exponential dichotomies, *Bull. Braz. Math. Soc., New Series.* 46(1):81-103, (2015).

[3] A. Bento, C. Silva. Stable manifolds for nonuniform polynomial dichotomies. *J. Funct. Anal.* 257:122-148, (2009).

[4] R. Boruga, M. Megan. Nonuniform polynomial dichotomy with invariant projections families. *Proceedings of the XV^{th} ICMA, University "Politehnica" of Timişoara.* ISSN 1224-6069:39-46, (2019).

[5] V. Crai, M. Megan. (h; k)-Dichotomy and Lyapunov Type Norms. *Annals of West University of Timisoara - Mathematics and Computer Science.* LVI 2:115-130, (2018).

[6] N.Lupa, M.Megan. Exponential dichotomies of evolution operators in Banach Spaces. *Monatshefte für Mathematik.* 174(2):265-284, (2014).

[7] M. Megan, T. Ceauşu, M. L. Rămneanţu. On Nonuniform Polynomial Stability for Evolution Operators on the Half-line. *Theory and Applications of Mathematics and Computer Science.* 4(2):202-210, (2014).

[8] M. Megan, C. Stoica. Concepts of dichotomy for skew-evolution semiflows on Banach spaces. *Annals of the Academy of the Romanian Scientists, Series on Mathematics and Applications*, vol. 2:125-140, (2010).

[9] C.L.Mihiţ. On some nonuniform dichotomic behaviors of discrete skew-product semiflows. *Theory and Applications of Mathematics and Computer Science.* 9:1-7, (2019).

[10] O. Perron. Die Stabilitätsfrage bei Differentialgleichungen. *Math. Z.* 32:703-728, (1930).

[11] M. L. Rămneanţu, T. Ceauşu, M.Megan. On nonuniform polynomial dichotomy of evolution operators in Banach spaces. *International Journal of Pure and Applied Mathematics.* 75(3):305-318, (2012).