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EXACT SOLUTIONS FOR OSCILLATING MOTIONS OF SOME FLUIDS WITH POWER-LAW DEPENDENCE OF VISCOSITY ON THE PRESSURE*

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Dedicated to Dr. Vasile Drăgan on the occasion of his 70^{th} anniversary

Abstract

Analytical expressions for the steady-state components of the dimensionless starting solutions corresponding to some oscillatory motions through a horizontal rectangular channel of two classes of incompressible Newtonian fluids with power-law dependence of viscosity on the pressure are established in the simplest forms. The fluid motion is generated by the lower plate that oscillates in its plane. For validation, three limiting cases are considered and interesting graphical representations are provided. It is worth pointing out the fact that such solutions are important in practice for those who want to eliminate the transients from their experiments. In addition, the dimensionless steady shear stresses corresponding to the simple Couette flow of such fluids are constants on the whole flow domain although the adequate velocity fields are functions of y.

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keywords: Newtonian fluids, pressure-dependent viscosity, oscillatory motions, exact solutions.

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1 Introduction

The fact that the fluid viscosity depends on the pressure was already long time ago remarked by Stokes in his seminal work [1] and the adequate literature prior to 1931 can be found in the book of Bridgman [2]. Later, the dependence of the liquid viscosity on the pressure has been experimentally attested by Cutler et al. [3], Johnson and Cameron [4], Johnson and Tevaarwerk [5], Johnson and Greenwood [6], Bair and Winer [7], etc. An important physical situation in which the dependence of viscosity on the pressure cannot be ignored is the problem of elastohydrodynamics lubrication [8] in which the fluid viscosity strongly varies with the pressure.

Some theoretical studies regarding the existence and uniqueness of the flow of fluids with pressure-dependent viscosity were developed by Malek et al. [9] and exact steady solutions for such motions of fluids in discussion have been established by Hron et al. [10] and Rajagopal [11,12]. The first exact solutions for some unsteady flows of such fluids seem to be those of Rajagopal and Saccomandi [13]. A little later, Prusa [14] established some exact expressions in terms of Kelvin functions for the steady-state (permanent or long time) solutions corresponding to the modified Stokes problems for the fluids with linear or exponential dependence of viscosity on the pressure whose numerical solutions have been obtained by Srinivasan and Rajagopal [15]. Qualitative and uniqueness results as well as exact solutions for the same motion problems of fluids with power-law and exponential dependence of velocity on the pressure have been established by Rajagopal et al. [16] in terms of a suitable system of eigenvalues and eigenfunctions. Steady flows of these fluids in cylindrical or spherical domains have been studied by Kalagirou et al. [17], respectively Housiadas et al. [18] using a perturbation scheme for small values of the pressure-viscosity coefficient.

As it results from the previous presentation, there are very few exact solutions for motions of fluids with pressure-dependent viscosity although they play a very important role in the study of these fluids. In addition to provide solutions to problems with technical applications, they can be used as tests to verify numerical schemes that are developed to study more complex unsteady flow problems. Our purpose here is to provide simple analytical expressions for the steady-state solutions corresponding to oscillatory motions of two classes of incompressible Newtonian fluids with power-law dependence of viscosity on the pressure. As a check of their correctness it is shown, both analytically and graphically, that these solutions tend to those of ordinary Newtonian fluids performing the same motions when the dimensionless pressure-viscosity coefficient tend to zero. Furthermore, the dimensionless steady solutions corresponding to the simple Couette flow of the same fluids are also obtained as limiting cases of previous solutions.

2 Governing equations

Let us consider an incompressible Newtonian fluid with pressure-dependent viscosity at rest between two infinite horizontal parallel plates at the distance h apart. Its constitutive equation is

$$T = -pI + S = -pI + \eta(p)A, \tag{1}$$

where T is the Cauchy stress tensor, I is the unit tensor, S is the extra-stress tensor, A is the first Rivlin-Ericksen tensor, p is the hydrostatic pressure and $\eta(p)$ is the fluid viscosity which depends on the pressure.

In the following we study oscillatory motions of incompressible Newtonian fluids with power-law dependence of viscosity on the pressure of the forms

$$\eta(p) = \mu[\alpha(p - p_h) + 1]^{1/2} \ \text{ or } \ \eta(p) = \mu[\alpha(p - p_h) + 1]^2, \tag{2}$$

where μ is the fluid viscosity at the reference pressure p_h and α is a positive constant that is called the pressure-viscosity coefficient [18]. The case of ordinary incompressible Newtonian fluids is obtained making $\alpha \to 0$ in Eqs. (2). The gravity effects will be also taken into consideration.

At the moment $t = 0^+$ the lower plate begins to oscillate in its plane according to

$$\mathbf{v} = V\cos(\omega t)\mathbf{i} \ \text{ or } \ \mathbf{v} = V\sin(\omega t)\mathbf{i}, \tag{3}$$

where $\mathbf{v}$ is the velocity vector, V and ω are the amplitude, respectively the frequency of the oscillations and $\mathbf{i}$ is the unit vector along the x-direction of a suitable Cartesian coordinate system x, y and z whose y-axis is perpendicular to the plates. Owing to the shear the fluid is gradually moved and we are looking for a solution of the form [14,16]

$$\mathbf{v} = v(y,t)\mathbf{i}, p = p(y). \tag{4}$$

Substituting Eqs. (4) in (1) and the result in the balance of linear momentum

$$\operatorname{div} T + \rho\mathbf{b} = \rho\frac{d\mathbf{v}}{dt}, \tag{5}$$

we find the following relevant differential equations

$$\frac{\partial}{\partial y}\left[\eta(p)\frac{\partial v(y,t)}{\partial y}\right] = \rho\frac{\partial v(y,t)}{\partial t}, \ \frac{dp(y)}{dy} + \rho g = 0. \tag{6}$$

Into above relations ρ is the fluid density, $\mathbf{b} = -g\mathbf{j}$ is the specific body force, g is the acceleration due to the gravity and $\mathbf{j}$ is the unit vector along the y-direction.

The continuity equation is identically verified and Eq. $(6)_2$ implies

$$p = p(y) = \rho g(h - y) + p_h, \tag{7}$$

where $p_h = p(h)$ is the dimensional pressure at the upper plate. The pressure at the lower plate is $p_0 = p(0) = \rho gh + p_h$. Replacing $p(y)$ in Eqs. (2) and the results in Eq. $(6)_1$, we find the governing equations for the velocity field, namely

$$\mu\sqrt{\alpha\rho g(h-y)+1}\frac{\partial^2 v(y,t)}{\partial y^2} - \frac{\mu\alpha\rho g}{2\sqrt{\alpha\rho g(h-y)+1}}\frac{\partial v(y,t)}{\partial y} = \rho\frac{\partial v(y,t)}{\partial t}; \tag{8}$$
$$0 < y < h, t > 0,$$

respectively

$$\mu[\alpha\rho g(h-y)+1]^2\frac{\partial^2 v(y,t)}{\partial y^2} - 2\mu\alpha\rho g[\alpha\rho g(h-y)+1]\frac{\partial v(y,t)}{\partial y}$$
$$= \rho\frac{\partial v(y,t)}{\partial t}; 0 < y < h, t > 0, \tag{9}$$

corresponding to the two types of fluids with pressure-dependent viscosity.

The corresponding non-trivial shear-stresses, as it results from Eqs. (1), (2) and (7) are given by

$$\tau(y,t) = \mu\sqrt{\alpha\rho g(h-y)+1}\frac{\partial v(y,t)}{\partial y} \text{ or}$$
$$\tau(y,t) = \mu[\alpha\rho g(h-y)+1]^2\frac{\partial v(y,t)}{\partial y}, \tag{10}$$

while the appropriate initial and boundary conditions are

$$v(y,0) = 0; 0 \le y \le h, \tag{11}$$
$$v(0,t) = V\cos(\omega t), v(h,t) = 0 \text{ or}$$
$$v(0,t) = V\sin(\omega t), v(h,t) = 0; t > 0. \tag{12}$$

In Eqs. (10) $\tau = S_{yx}$ is the non-trivial component of the extra-stress tensor S.

In order to determine solutions that are independent of the flow geometry, we introduce the following non-dimensional variables, functions and parameter

$$y^* = \frac{y}{h}, t^* = \frac{t}{t_0}, v^* = \frac{v}{V}, \tau^* = \frac{t_0 \tau}{\rho h V}, \alpha^* = \alpha \rho g h, \tag{13}$$

where $t_0 = h^2/\nu$ is a characteristic time which is suitable chosen and $\nu = \mu/\rho$ is the kinematic viscosity. Using the entities (13) in Eqs. (8) and (9) and dropping out the star notation, we get the next dimensionless governing equations

$$\begin{aligned} &\sqrt{\alpha(1-y)+1}\frac{\partial^2 v(y,t)}{\partial y^2} - \frac{\alpha}{2\sqrt{\alpha(1-y)+1}}\frac{\partial v(y,t)}{\partial y} \\ &= \frac{\partial v(y,t)}{\partial t}; 0 < y < 1, t > 0, \end{aligned} \tag{14}$$

respectively

$$\begin{aligned} &[\alpha(1-y)+1]^2 \frac{\partial^2 v(y,t)}{\partial y^2} - 2\alpha[\alpha(1-y)+1]\frac{\partial v(y,t)}{\partial y} = \frac{\partial v(y,t)}{\partial t}; \\ &0 < y < 1, t > 0. \end{aligned} \tag{15}$$

The adequate dimensionless initial and boundary conditions are

$$v(y,0) = 0; 0 \le y \le 1, \tag{16}$$

$$v(0,t) = \cos(\omega t), v(1,t) = 0 \text{ or } v(0,t) = \sin(\omega t), v(1,t) = 0; t > 0, \tag{17}$$

while the dimensionless non-trivial shear stresses are given by

$$\tau(y,t) = \sqrt{\alpha(1-y)+1}\frac{\partial v(y,t)}{\partial y} \text{ or } \tau(y,t) = [\alpha(1-y)+1]^2 \frac{\partial v(y,t)}{\partial y}. \tag{18}$$

It is well known the fact that the starting solutions $v_c(y,t)$ and $v_s(y,t)$, corresponding to the two motion problems of fluids in discussion, can be presented as sums of steady-state (permanent or long time [14]) and transient components. Some time after the motion initiation, the fluid moves according to the starting solutions. After this time, when the transients $v_{ct}(y,t)$ and $v_{st}(y,t)$ disappear or can be neglected, the fluid flows according to the steady-state components $v_{cp}(y,t)$ or $v_{sp}(y,t)$. In practice this time is very important for the experimentalists who want to know the moment when the steady-state is acquired. They need to determine the required time to reach the steady-state. In order to determine this time, it is necessary to know exact solutions at least for the steady-state components of the starting solutions.

To determine the steady-state solutions in a simple way, we denote by

$$v_p(y,t) = v_{cp}(y,t) + i v_{sp}(y,t), \tag{19}$$

the dimensionless complex velocity where i is the imaginary unit. This complex velocity, as well as its dimensionless real and imaginary components $v_{cp}(y,t)$ and $v_{sp}(y,t)$, is independent of the initial condition but has to satisfy the governing equation (14) or (15) and the boundary conditions

$$v_p(0,t) = e^{i\omega t}, v_p(1,t) = 0; t \in R. \tag{20}$$

3 Exact expressions for non-dimensional steady-state solutions

In order to determine exact expressions for the dimensionless steady-state components $v_{cp}(y,t)$ and $v_{sp}(y,t)$ of the starting solutions $v_c(y,t)$, respectively $v_s(y,t)$ corresponding to the problems in discussion, we have to solve adequate boundary values problems for the dimensionless complex velocity $v_p(y,t)$.

3.1. Case $\eta(p) = \mu\sqrt{\alpha(p-p_h)+1}$

The dimensionless complex velocity field corresponding to this case has to satisfy the partial differential equation (see also Eq. (14))

$$\sqrt{\alpha(1-y)+1}\frac{\partial^2 v_p(y,t)}{\partial y^2} - \frac{\alpha}{2\sqrt{\alpha(1-y)+1}}\frac{\partial v_p(y,t)}{\partial y} = \frac{\partial v_p(y,t)}{\partial t};$$
$$0 < y < 1, t > 0 \tag{21}$$

and the boundary conditions (20). In order to solve the partial differential equation (21) with the conditions (20), we make the change of independent variable

$$y = \frac{\alpha + 1 - r^2}{\alpha} \text{ or equivalently } r = \sqrt{\alpha(1-y)+1} \tag{22}$$

and attain to the following boundary value problem

$$\frac{\alpha^2}{4r}\frac{\partial^2 v_p(r,t)}{\partial r^2} = \frac{\partial v_p(r,t)}{\partial t}; 1 < r < a, t \in R, \tag{23}$$
$$v_p(1,t) = 0, v_p(a,t) = e^{i\omega t}; t \in R \tag{24}$$

where $a = \sqrt{\alpha + 1}$.

For this boundary value problem we are looking for a separable solution

$$v_p(r,t) = V(r)T(t). \tag{25}$$

Introducing $v_p(r,t)$ from Eq. (25) in (23), we find that

$$\alpha^2 \frac{d^2V(r)}{dr^2} - 4\lambda r V(r) = 0, \frac{dT(t)}{dt} - \lambda T(t) = 0, \tag{26}$$

where λ is constant. According to the boundary conditions (24), it results that $\lambda = i\omega$ and the function $V(r)$ has to satisfy the conditions

$$V(1) = 0, V(a) = 1. \tag{27}$$

The equation $(26)_1$ is an ordinary differential equation of Airy type (see [19], the exercise 34 on the page 251) whose general solution is of the form

$$V(r) = \sqrt{r}[C_1 J_{\frac{1}{3}}(\frac{4r}{3\alpha}\sqrt{-i\omega r}) + C_2 Y_{\frac{1}{3}}(\frac{4r}{3\alpha}\sqrt{-i\omega r})], \tag{28}$$

where C_1 and C_2 are arbitrary constants. Using the boundary conditions (27) we find the two constants and as a result

$$V(r) = \frac{\sqrt{r}}{\sqrt{a}} \frac{Y_{\frac{1}{3}}(\frac{4\sqrt{-i\omega}}{3\alpha}) J_{\frac{1}{3}}(\frac{4r}{3\alpha}\sqrt{-i\omega r}) - J_{\frac{1}{3}}(\frac{4\sqrt{-i\omega}}{3\alpha}) Y_{\frac{1}{3}}(\frac{4r}{3\alpha}\sqrt{-i\omega r})}{Y_{\frac{1}{3}}(\frac{4\sqrt{-i\omega}}{3\alpha}) J_{\frac{1}{3}}(\frac{4a}{3\alpha}\sqrt{-i\omega a}) - J_{\frac{1}{3}}(\frac{4\sqrt{-i\omega}}{3\alpha}) Y_{\frac{1}{3}}(\frac{4a}{3\alpha}\sqrt{-i\omega a})}. \tag{29}$$

Consequently, based on the definition (19) of the complex velocity $v_p(y,t)$ and the previous calculi, we can say that the dimensionless steady-state components $v_{cp}(y,t)$ and $v_{sp}(y,t)$ of the problem in discussion are given by the equalities

$$\begin{aligned} v_{cp}(y,t) = {} & \frac{\sqrt{r}}{\sqrt{a}} \\ & \times Re\left\{ \frac{Y_{\frac{1}{3}}(\frac{4\sqrt{-i\omega}}{3\alpha}) J_{\frac{1}{3}}(\frac{4r}{3\alpha}\sqrt{-i\omega r}) - J_{\frac{1}{3}}(\frac{4\sqrt{-i\omega}}{3\alpha}) Y_{\frac{1}{3}}(\frac{4r}{3\alpha}\sqrt{-i\omega r})}{Y_{\frac{1}{3}}(\frac{4\sqrt{-i\omega}}{3\alpha}) J_{\frac{1}{3}}(\frac{4a}{3\alpha}\sqrt{-i\omega a}) - J_{\frac{1}{3}}(\frac{4\sqrt{-i\omega}}{3\alpha}) Y_{\frac{1}{3}}(\frac{4a}{3\alpha}\sqrt{-i\omega a})} e^{i\omega t} \right\}, \end{aligned} \tag{30}$$

$$\begin{aligned} v_{sp}(y,t) = {} & \frac{\sqrt{r}}{\sqrt{a}} \\ & \times Im\left\{ \frac{Y_{\frac{1}{3}}(\frac{4\sqrt{-i\omega}}{3\alpha}) J_{\frac{1}{3}}(\frac{4r}{3\alpha}\sqrt{-i\omega r}) - J_{\frac{1}{3}}(\frac{4\sqrt{-i\omega}}{3\alpha}) Y_{\frac{1}{3}}(\frac{4r}{3\alpha}\sqrt{-i\omega r})}{Y_{\frac{1}{3}}(\frac{4\sqrt{-i\omega}}{3\alpha}) J_{\frac{1}{3}}(\frac{4a}{3\alpha}\sqrt{-i\omega a}) - J_{\frac{1}{3}}(\frac{4\sqrt{-i\omega}}{3\alpha}) Y_{\frac{1}{3}}(\frac{4a}{3\alpha}\sqrt{-i\omega a})} e^{i\omega t} \right\}, \end{aligned} \tag{31}$$

where $r=\sqrt{\alpha(1-y)+1}$ while Re and Im denote the real, respectively the imaginary part of that which follows.

The solutions given by Eqs. (30) and (31), as we already mentioned before, are independent of the initial condition (16) but satisfy the boundary conditions $(17)_{1,2}$, respectively $(17)_{3,4}$ and the governing equation (14). In order to determine the dimensionless steady-state frictional forces per unit area exerted by the fluid on the plates, we firstly have to determine the corresponding shear stresses $\tau_{cp}(y,t)$ and $\tau_{sp}(y,t)$ by substituting the expressions of $v_{cp}(y,t)$ and $v_{sp}(y,t)$ from Eqs (30) and (31) in $(18)_1$. The dimensionless steady-state frictional forces per unit area exerted by the fluid on the upper plate, for instance, are given by

$$\tau_{cp}(1,t)=-\frac{2}{\alpha\sqrt{a}}\times \tag{32}$$
$$Re\left\{\frac{J_{\frac{1}{3}}(\frac{4\sqrt{-i\omega}}{3\alpha})Y_{\frac{4}{3}}(\frac{4\sqrt{-i\omega}}{3\alpha})-Y_{\frac{1}{3}}(\frac{4\sqrt{-i\omega}}{3\alpha})J_{\frac{4}{3}}(\frac{4\sqrt{-i\omega}}{3\alpha})}{Y_{\frac{1}{3}}(\frac{4\sqrt{-i\omega}}{3\alpha})J_{\frac{1}{3}}(\frac{4a}{3\alpha}\sqrt{-i\omega a})-J_{\frac{1}{3}}(\frac{4\sqrt{-i\omega}}{3\alpha})Y_{\frac{1}{3}}(\frac{4a}{3\alpha}\sqrt{-i\omega a})}\sqrt{-i\omega}e^{i\omega t}\right\},$$

$$\tau_{sp}(1,t)=-\frac{2}{\alpha\sqrt{a}}\times \tag{33}$$
$$Im\left\{\frac{J_{\frac{1}{3}}(\frac{4\sqrt{-i\omega}}{3\alpha})Y_{\frac{4}{3}}(\frac{4\sqrt{-i\omega}}{3\alpha})-Y_{\frac{1}{3}}(\frac{4\sqrt{-i\omega}}{3\alpha})J_{\frac{4}{3}}(\frac{4\sqrt{-i\omega}}{3\alpha})}{Y_{\frac{1}{3}}(\frac{4\sqrt{-i\omega}}{3\alpha})J_{\frac{1}{3}}(\frac{4a}{3\alpha}\sqrt{-i\omega a})-J_{\frac{1}{3}}(\frac{4\sqrt{-i\omega}}{3\alpha})Y_{\frac{1}{3}}(\frac{4a}{3\alpha}\sqrt{-i\omega a})}\sqrt{-i\omega}e^{i\omega t}\right\},$$

3.2. Case $\eta(p)=\mu[\alpha(p-p_h)+1]^2$

In this case we have to solve the following partial differential equation

$$[\alpha(1-y)+1]^2\frac{\partial^2 v_p(y,t)}{\partial y^2}-2\alpha[\alpha(1-y)+1]\frac{\partial v_p(y,t)}{\partial y}=\frac{\partial v_p(y,t)}{\partial t}, \tag{34}$$
$$0<y<1, t\in R$$

with the same boundary conditions (20). Making the change of variable

$$y=\frac{\alpha+1-e^r}{\alpha}\ \text{or equivalently}\ r=\ln[\alpha(1-y)+1], \tag{35}$$

in Eq. (34), we attain to the partial differential equation with constant coefficients

$$\frac{\partial^2 v_p(r,t)}{\partial r^2}+\frac{\partial v_p(r,t)}{\partial r}=\frac{1}{\alpha^2}\frac{\partial v_p(r,t)}{\partial t}, 0<r<b, t\in R \tag{36}$$

where $b = \ln(\alpha + 1)$. The corresponding boundary conditions are

$$v_p(0,t) = 0, v_p(b,t) = e^{i\omega t}, t \in R. \tag{37}$$

Looking again for a separable solution of the same form (25), we find that the new function $V(r)$ has to satisfy the ordinary differential equation

$$\frac{d^2V(r)}{dr^2} + \frac{dV(r)}{dr} - \frac{\lambda}{\alpha^2}V(r) = 0 \tag{38}$$

and the boundary conditions

$$V(0) = 0, V(b) = 1. \tag{39}$$

The solution of the ordinary differential equation (38) with the boundary conditions (39) is given by

$$V(r) = \frac{e^{r_1 r} - e^{r_2 r}}{e^{r_1 b} - e^{r_2 b}}, r_{1,2} = \frac{-\alpha \pm \sqrt{\alpha^2 + 4i\omega}}{2\alpha} \tag{40}$$

and the dimensionless steady-state velocities fields corresponding to this case are

$$v_{cp}(y,t) = Re\{\frac{[\alpha(1-y)+1]^{r_1} - [\alpha(1-y)+1]^{r_2}}{(\alpha+1)^{r_1} - (\alpha+1)^{r_2}} e^{i\omega t}\}, \tag{41}$$

$$v_{sp}(y,t) = Im\{\frac{[\alpha(1-y)+1]^{r_1} - [\alpha(1-y)+1]^{r_2}}{(\alpha+1)^{r_1} - (\alpha+1)^{r_2}} e^{i\omega t}\}, \tag{42}$$

The corresponding shear-stresses $\tau_{cp}(y,t)$ and $\tau_{sp}(y,t)$ are immediately obtained substituting $v_{cp}(y,t)$ and $v_{sp}(y,t)$ from Eqs. (41) and (42) in $(18)_2$. Using them, we can determine the dimensionless steady-state frictional forces per unit area exerted by the fluid to the plates. Dimensionless steady-state frictional forces per unit area exerted by the fluid to the upper plate, for example, are given by

$$\tau_{cp}(1,t) = \alpha Re\{\frac{r_2 - r_1}{(\alpha+1)^{r_1} - (\alpha+1)^{r_2}} e^{i\omega t}\},$$
$$\tau_{sp}(1,t) = \alpha Im\{\frac{r_2 - r_1}{(\alpha+1)^{r_1} - (\alpha+1)^{r_2}} e^{i\omega t}\}. \tag{43}$$

4 Limiting cases

In order to bring to light the accuracy of results that have been previously obtained, we now consider some limiting cases whose steady-state solutions can be directly determined or as asymptotical approximations of present solutions.

4.1 Case $\omega = 0$ (Simple Couette flow)

The flow between two parallel plates, one of them moving in its plane with a constant velocity V and the other one being stationary is called the simple Couette flow [20]. Dimensionless steady solutions corresponding to this motion, namely

$$v_{Cp}(y) = \frac{\sqrt{\alpha(1-y)+1}-1}{\sqrt{\alpha+1}-1} \text{ and } v_{Cp}(y) = \frac{(\alpha+1)(1-y)}{\alpha(1-y)+1}, \tag{44}$$

can be immediately obtained solving the ordinary differential equations

$$\frac{d}{dy}[\sqrt{\alpha(1-y)+1}\frac{dv(y)}{dy}] = 0, \text{ respectively } \frac{d}{dy}[[\alpha(1-y)+1]^2\frac{dv(y)}{dy}] = 0; \tag{45}$$

$$0 < y < 1,$$

with the boundary conditions (see Eqs. $(17)_{1,2}$)

$$v(0) = 1, v(1) = 0. \tag{46}$$

Simple computations show that the velocity field $v_{Cp}(y)$ given by Eq. $(44)_2$ can be immediately obtained as a limiting case of $v_{cp}(y,t)$ given by Eq. (41) when $\omega \to 0$. In addition Fig. 1, as it was to be expected, clearly show that for small enough values of the oscillations' frequency ω and suitable values of the time t (so that the product ωt to be small enough), the diagrams of the steady-state solution $v_{cp}(y,t)$ given by Eq. (30) are almost identical to those of the steady solution $v_{Cp}(y,t)$ given by Eqs. $(44)_1$.

It is worth pointing out the fact that, in both cases, the dimensionless steady components

$$\tau_{Cp} = -\frac{\alpha}{2(\sqrt{\alpha+1}-1)} \text{ and } \tau_{Cp} = -(\alpha+1), \tag{47}$$

of the shear stress τ_C corresponding to the simple Couette flow of Newtonian fluids with power-law dependence of viscosity on the pressure are constant on the whole flow domain, although the adequate velocities which are given by Eqs. (44) are functions of the spatial variable y.

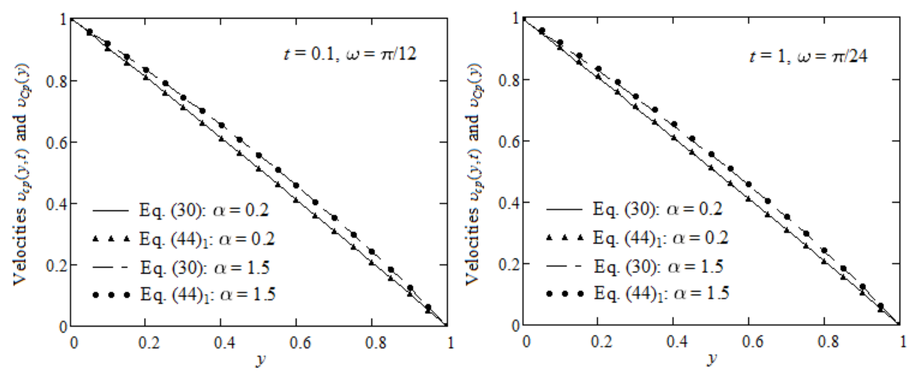

Figure 1: Profiles of the permanent solutions $v_{cp}(y,t)$ and $v_{Cp}(y)$ given by Eqs. (30), respectively $(44)_1$.

4.2 Case $\alpha = 0$ (Flows of incompressible Newtonian fluids)

The dimensionless steady-state solutions [21]

$$v_{Ncp}(y,t) = Re\{\frac{sh[(1-y)\sqrt{i\omega}]}{sh(\sqrt{i\omega})}e^{i\omega t}\}, v_{Nsp}(y,t) = Im\{\frac{sh[(1-y)\sqrt{i\omega}]}{sh(\sqrt{i\omega})}e^{i\omega t}\}, \tag{48}$$

corresponding to incompressible Newtonian fluids performing the same motions as in Section 3, can be easy determined using the separable variable method. Figs. 2, 3, 4 and 5, as expected, show that the diagrams of the dimensionless steady-state velocity fields $v_{cp}(y,t)$ and $v_{sp}(y,t)$ given by Eqs. (30) and (31), respectively (41) and (42) tend to superpose over those of $v_{Ncp}(y,t)$ and $v_{Nsp}(y,t)$ given by Eqs. (48).

Furthermore, using the asymptotic approximations

$$J_\nu(z) \approx \sqrt{\frac{2}{\pi z}}\cos[z - \frac{(2\nu+1)\pi}{4}], Y_\nu(z) \approx \sqrt{\frac{2}{\pi z}}\sin[z - \frac{(2\nu+1)\pi}{4}] \tag{49}$$

for $|z| \gg 1$, it is not difficult to show that the dimensionless steady-state components $v_{cp}(y,t)$ and $v_{sp}(y,t)$ given by Eqs. (30), respectively (31) can be well enough approximated by the equalities

$$v_{cp}(y,t) \approx \frac{\sqrt{\alpha(1-y)+1}}{\sqrt{\alpha+1}} Re\{\frac{\sin\{\frac{4\sqrt{-i\omega}}{3\alpha}[1-\sqrt[4]{[\alpha(1-y)+1]^3}]\}}{\sin\{\frac{4\sqrt{-i\omega}}{3\alpha}[1-\sqrt[4]{(\alpha+1)^3}]\}}e^{i\omega t}\}. \tag{50}$$

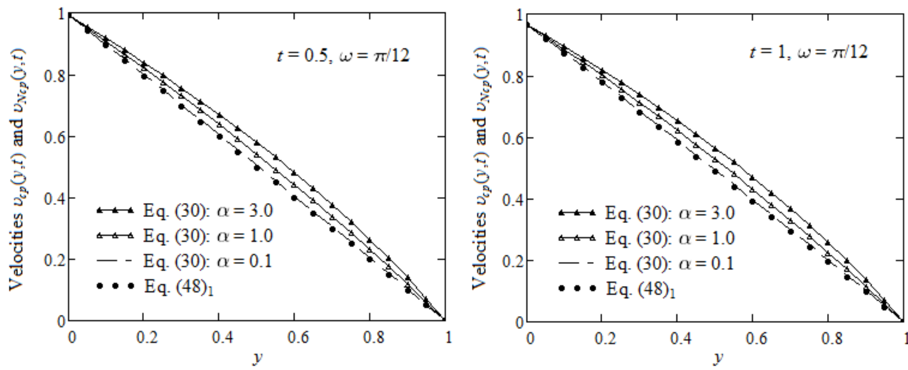

Figure 2: Profiles of the permanent solutions $v_{cp}(y,t)$ for three decreasing values of the parameter α and $v_{Ncp}(y,t)$ given by Eqs. (30), respectively $(48)_1$.

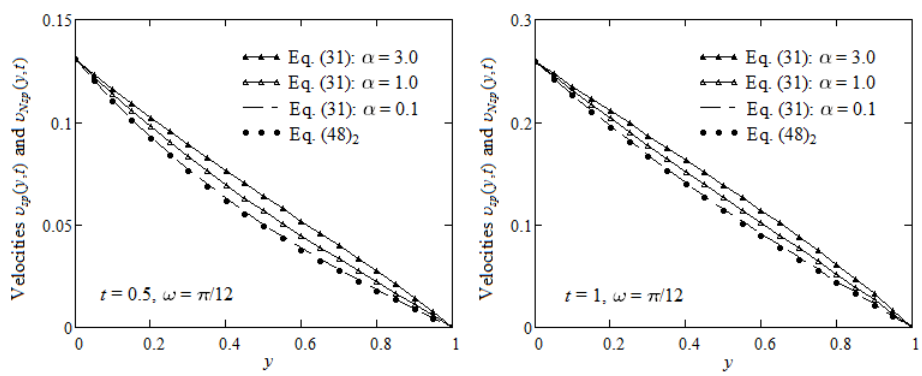

Figure 3: Profiles of the permanent solutions $v_{sp}(y,t)$ for three decreasing values of the parameter α and $v_{Nsp}(y,t)$ given by Eqs. (31), respectively $(48)_2$.

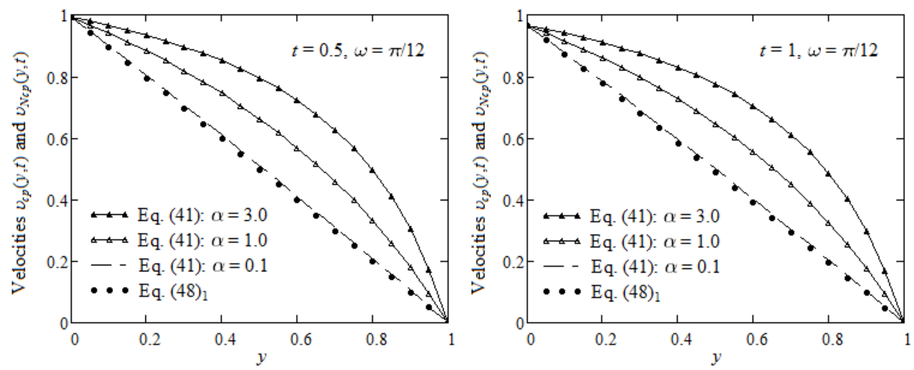

Figure 4: Profiles of the permanent solutions $v_{cp}(y,t)$ for three decreasing values of the parameter α and $v_{Ncp}(y,t)$ given by Eqs. (41), respectively $(48)_1$.

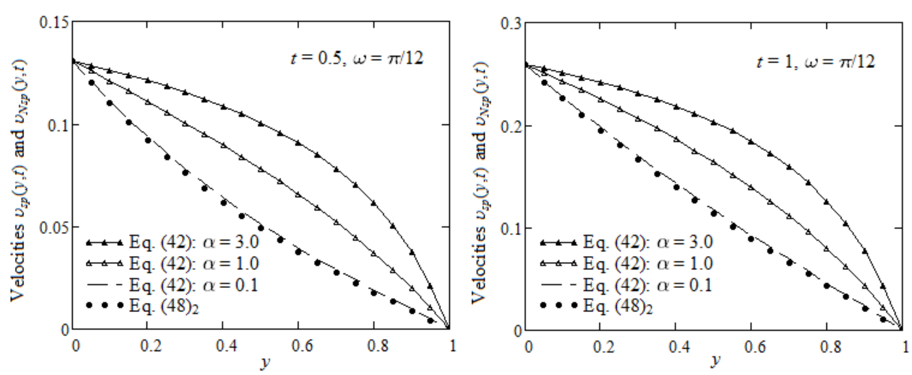

Figure 5: Profiles of the permanent solutions $v_{sp}(y,t)$ for three decreasing values of the parameter α and $v_{Nsp}(y,t)$ given by Eqs. (42), respectively $(48)_2$.

$$v_{sp}(y,t) \approx \frac{\sqrt{\alpha(1-y)+1}}{\sqrt{\alpha+1}} Im\{\frac{\sin\{\frac{4\sqrt{-i\omega}}{3\alpha}[1-\sqrt[4]{[\alpha(1-y)+1]^3}]\}}{\sin\{\frac{4\sqrt{-i\omega}}{3\alpha}[1-\sqrt[4]{(\alpha+1)^3}]\}} e^{i\omega t}\}. \tag{51}$$

for small enough values of α.

Now, using the Taylor series for $[1+\alpha(1-y)]^{3/4}$ and $(1+\alpha)^{3/4}$ around the zero value of the dimensionless pressure-viscosity coefficient α, the identity

$$\sin[(1-y)\sqrt{-i\omega}] = -ish[(1-y)\sqrt{i\omega}], \tag{52}$$

and taking the limit of Eqs. (50) and (51) when $\alpha \to 0$ we recover Eqs. (48).

4.3 Case $\alpha = \omega = 0$ (Simple Couette flow of incompressible Newtonian fluids)

Finally, taking the limit of either one of Eqs. (44) when $\alpha \to 0$ or the limit of Eq. $(48)_1$ if $\omega \to 0$, we recover the steady component

$$v_{NCp}(y) = 1 - y, \tag{53}$$

of the dimensionless velocity field $v_{NC}(y,t)$ corresponding to the simple Couette flow of incompressible Newtonian fluids. The corresponding shear stress, namely $\tau_{NCp} = -1$, is obtained taking the limit of Eq. $(47)_1$ or $(47)_2$ when $\alpha \to 0$.

5 Conclusions

Exact expressions for the steady-state components of the starting solutions corresponding to some oscillatory motions of two classes of incompressible Newtonian fluids with power-law dependence of viscosity on the pressure between two infinite horizontal parallel plates are established in simple forms using suitable changes of the independent spatial variable. The fluid motion is generated by the lower plate that oscillates in its plane. The obtained solutions are independent of the initial condition but they satisfy the boundary conditions and governing equations. They are important for the experimentalists who want to eliminate the transients from their experiments. For that they have to know the required time to reach the steady-state. This is the time after which the fluid moves according to the steady-state solutions. Graphically, it is the time after which the diagrams of starting solutions (which can be numerically obtained) are almost identical to those of the corresponding steady-state solutions.

To validate the results, some limiting cases are considered and different graphical representations are presented in Figs. 1-5. The dimensionless velocity fields $v_{Cp}(y)$ corresponding to the simple Couette flow of fluids in consideration are also determined. From Figs. 1, as it was to be expected, it clearly results that for small values of the oscillations' frequency ω the diagrams of the dimensionless steady-state solution $v_{cp}(yt)$ given by Eq. (30) are almost identically to those of the steady solution $v_{Cp}(y)$ given by Eq. $(44)_1$ which corresponds to the simple Couette flow of fluids with pressure-dependent viscosity given by Eq. $(2)_1$. In the second case when the dependence of viscosity on the pressure is given by Eq. $(2)_2$, it is easy to prove that the adequate velocity field $v_{Cp}(y)$ given by Eq. $(44)_2$ is just the limit of $v_{cp}(y,t)$ given by Eq. (41) when $\omega \to 0$. It is worth to point out the fact that, as it results from Eqs. (47), the steady-state components τ_{Cp} of the dimensionless non-trivial shear stresses τ_C corresponding to the simple Couette flow of both fluids in consideration is constant on the whole flow domain although the corresponding velocity fields given by Eqs. $(44)_1$ or $(44)_2$ are functions of y.

Figs. 2, 3, 4 and 5 are here included to show that, as expected, the diagrams of the solutions $v_{cp}(y,t)$ and $v_{sp}(y,t)$ corresponding to oscillatory motions of fluids with pressure-dependent viscosity tend to superpose over those of the Newtonian fluids performing the same motions when the dimensionless pressure-viscosity coefficient $\alpha \to 0$. The boundary conditions are clearly satisfied. In all cases the fluid velocity smoothly decreases from maximum values on the lower plate to the zero value on the stationary plate. It is a decreasing function with respect to the parameter α and this behaviour is in accordance with the dependence of the fluid viscosity on the pressure. If the parameter α increases, the fluid viscosity also brings up and the fluid flows slower.

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