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GENERALIZED EXPONENTIALLY STABLE LINEAR TIME-VARYING DISCRETE BEHAVIORS*

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Dedicated to Dr. Vasile Drăgan on the occasion of his 70^{th} anniversary

Abstract

This paper presents a new approach to formulating exponential behaviors like stability/instability for the linear time-varying systems and for the adjoint one. The classical concept of uniform exponential stability is generalized. Using this generalized concepts, some results extending existing uniform exponential stability conditions for linear time-varying systems are derived. As special cases for these results, some conditions are derived for the adjoint system. A characterization of the generalized concepts in terms of Lyapunov sequences is also given. Also, an example is included to further illustrate the connection with the classical concept of uniform exponential stability.
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1 Introduction

The notion of (uniform) exponential stability is well-known for linear time-varying (LTV) discrete-time systems (see, e.g., [11]) and its applications are widely developed. We refer here to the monographs of Agarwal [1], Elaydi [7], Lakshmikantham, Trigiante [8] and Gil [10] for details and further references.

In the area of LTV discrete-time systems, the attempt to extend the classical notion of exponential stability has been pursued along two approaches. One that provides various nonuniform concepts, where some exponential loss of hyperbolicity along the trajectories is allowed, and second generalized concepts that represent a kind of uniform hyperbolicity. Regarding the second direction, where this paper is situated, we can point out the first attempt done by James Muldowney [16] for the case of ODE's. More recent results in continuous case can be found in [3], [12], [13] and [22]. On the other hand, from the point of view of the discrete-time systems we may refer the reader to [2], [12], [20] and [21] for details. All these papers deal only with the generalized exponential dichotomy, respectively generalized exponential trichotomy concepts. It is important to point out that all above earlier results (both for continuous and discrete ones) consider only systems defined on the entire axis. Thus, it arrives naturally to consider the case when the system is defined only on the semi-axes, the so-called one-side systems.

Having this in mind, in this paper, we will make an attempt to investigate the notion of exponential stability in a more general setting, a generalized one for LTV discrete-time systems defined only on the semi-axes. The idea to consider this notion is motivated by the research reached in this direction in the above articles. But, considering the generalized concept from the above references mutatis mutandis for one-side systems in fact we recover the classical concept of uniform exponential stability (see, e.g., [18]), and from this point of view we required some adjustments. To the best of our knowledge, this is the first time that this concept is reported in the literature. Also, it is natural to view what's happening with the dual system. Thus, starting with this new concept of stability, we obtain that the dual system became in fact instable.

Using this extended concept, in this paper we give a simple and concrete example illustrating the relationship between the considered concept and the classical one of uniform exponential stability. Also, generalized exponential stability analysis of a LTV system is carried out in terms of Lyapunov sequences. The application of such a sequence to the dual system is also established. This paper is a companion of our earlier work [17] where some preliminary results have been presented.

We proceed as follows: In Section 2 we present some preliminaries regarding the uniform exponential stability for the LTV discrete-time systems and the dual one. Then, in Section 3 we first formulate the generalized behaviors approaches, we build connections between the original system and its dual, and we give an example to illustrate the considered concept. Subsection 3.1 is devoted to necessary and sufficient conditions, carry out an asymptotic analysis for the LTV discrete-time system, and establish its link to the dual one. In Subsection 3.2, we introduce the notion of Lyapunov sequence and we show how generalized behaviors can be characterized in terms of Lyapunov sequences. Finally, Section 4 draw conclusions and discuss several future directions.

Notation: The notations used in this paper are generally standard. We recall some of them for the readers' convenience. X denotes a real or complex Banach space, X^* topological dual space of X, θ^* the null element of X^*, $\mathcal{B}(X)$ denotes the Banach algebra of all bounded linear operators from X into itself. The norms of both these spaces will be denoted by $\|\cdot\|$. I is the identity operator on X. $\mathbb{N}$ indicates the set of all nonnegative integers, Δ the set of all pairs (m, n) of integers satisfying the inequality $m \geq n$.

2 Preliminaries

In this work we deal with LTV discrete-time systems (also called one-side system), described by

$$x_{n+1} = A(n)x_n, \tag{$\mathfrak{A}$}$$

where $A : \mathbb{N} \longrightarrow \mathcal{B}(X)$ is a sequence of invertible operators in $\mathcal{B}(X)$. Every solution $x = (x_n)_{n\in\mathbb{N}}$ of the LTV system ($\mathfrak{A}$) is given by $x_m = \mathcal{A}(m, n)x_n$ for all $(m, n) \in \Delta$, where $\mathcal{A} : \Delta \longrightarrow \mathcal{B}(X)$, the so-called Cauchy evolution map is given by

$$\mathcal{A}(m, n) = \begin{cases} A(m-1)\cdots A(n), & m > n, \\ I, & m = n. \end{cases}$$

As it is well known, this verifies the propagator property

$$\mathcal{A}(m,n)\mathcal{A}(n,p) = \mathcal{A}(m,p)$$

for all (m,n) and $(n,p) \in \Delta$. Let $y = (y_n^*)_{n\in\mathbb{N}}$ be a sequence in X^*. The LTV dual discrete-time system associated to $(\mathfrak{A})$ is given by

$$y_{n+1} = B(n)y_n \tag{$\mathfrak{B}$}$$

where

$$B(n) = (A^{-1})^*(n) = (A^*)^{-1}(n),$$

for all $n \in \mathbb{N}$. This shows that

$$\mathcal{B}(m,n) = (\mathcal{A}^{-1})^*(m,n) = (\mathcal{A}^*)^{-1}(m,n),$$

for all $(m,n) \in \Delta$. Further, one can see that

$$\mathcal{B}(m,n)\mathcal{B}(n,p) = \mathcal{B}(m,p),$$

for all $(m,n),(n,p) \in \Delta$. In order to be self contain we first recall the notions of uniform exponential stability respectively instability.

Definition 1. *The LTV system $(\mathfrak{A})$ is said to be:*

(a) (see, e.g., [18]) uniformly exponentially stable on $\mathbb{N}$ if there are some constants $N \geq 1$ and $\alpha > 0$ such that

$$\|\mathcal{A}(m,n)x\| \leq Ne^{-\alpha(m-n)}\|x\|, \tag{1}$$

(b) (see, e.g., [19]) uniformly exponentially instable if there are some constants $N \geq 1$ and $\alpha > 0$ such that

$$\|x\|e^{\alpha(m-n)} \leq N\|\mathcal{A}(m,n)x\|, \tag{2}$$

for all $(m,n) \in \Delta$ and $x \in X$.

3 Proposed approach

Further on, we shall consider a strictly positive sequence $(a_n)_{n\in\mathbb{N}}$ satisfying the property

$$\sum_{j=p}^{q} a_j \to +\infty \text{ as } q \to +\infty \text{ for fixed } p \in \mathbb{N}. \tag{3}$$

We set $s_n = a_0 + a_1 + \cdots + a_n$ for all $n \in \mathbb{N}$.

Definition 2. *The LTV system* ($\mathfrak{A}$) *is said to be:*

(a) generalized exponentially stable on $\mathbb{N}$ *if there exists a constant* $K \geq 1$ *and a strictly positive sequence* $(a_n)_{n\in\mathbb{N}}$ *satisfying (3) such that*

$$\|\mathcal{A}(m,n)x\| \leq Ke^{-(s_m-s_n)}\|x\|, \tag{4}$$

for all $(m,n) \in \Delta$ *and* $x \in X$.

(b) generalized exponentially instable on $\mathbb{N}$ *if there exists a constant* $K \geq 1$ *and a strictly positive sequence* $(a_n)_{n\in\mathbb{N}}$ *satisfying (3) such that*

$$\|x\|e^{s_m-s_n} \leq K\|\mathcal{A}(m,n)x\|, \tag{5}$$

for all $(m,n) \in \Delta$ *and* $x \in X$.

Note that: For $a_j = \alpha > 0$, in Definition 2 (a), for any $j \in \mathbb{N}$, we obtain the classical notion of uniform exponential stability. Further, we will show that for a LTV system ($\mathfrak{A}$) the concepts of generalized exponential stability and uniform exponential stability are distinct. This phenomenon is illustrated by the following example.

Example 1. *Let* $X = \mathbb{R}^2$ *with the norm* $\|x\| = \|(x_1,x_2)\| = |x_1| + |x_2|$ *and let the sequences* $(b_n)_{n\in\mathbb{N}}$ *be defined by* $b_n = \dfrac{n+1}{n+2}$, *for all* $n \in \mathbb{N}$. *Further, we consider the linear equation* $A_n : \mathbb{R}^2 \to \mathbb{R}^2$ *with the diagonal matrix of coefficients*

$$A_n = \begin{bmatrix} b_n & 0 \\ 0 & b_n \end{bmatrix}$$

defined for $n \in \mathbb{N}$. *One may show that* $(A_n)_{n\in\mathbb{N}}$ *does not admit a uniform exponential stability, but the generalized exponential stability is satisfied. Clearly, for every* $(m,n) \in \Delta$ *we have*

$$\begin{aligned} \mathcal{A}(m,n) &= \begin{bmatrix} b_{m-1}b_{m-2}\cdots b_n & 0 \\ 0 & b_{m-1}b_{m-2}\cdots b_n \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{n+1}{m+1} & 0 \\ 0 & \dfrac{n+1}{m+1} \end{bmatrix}. \end{aligned}$$

Then

$$\|\mathcal{A}(m,n)x\| = \frac{n+1}{m+1}\|x\|.$$

Now let $(a_n)_{n\in\mathbb{N}}$ *be defined by*

$$a_n = \ln\frac{n+2}{n+1} = \ln\left(1+\frac{1}{n+1}\right).$$

We deduce that

$$\begin{aligned} s_n = a_0 + a_1 + \cdots + a_n &= \ln 2 + \ln\frac{3}{2} + \cdots + \ln\frac{n+2}{n+1} \\ &= \ln\left(2\cdot\frac{3}{2}\cdots\frac{n+1}{n}\cdot\frac{n+2}{n+1}\right) = \ln(n+2) \end{aligned}$$

and

$$s_m - s_n = \ln\frac{m+2}{n+2},$$

for all $(m,n)\in\Delta$. *Indeed,*

$$a_p + a_{p+1} + \cdots + a_q = \ln\frac{q+2}{p+1} \longrightarrow +\infty$$

for $q\longrightarrow+\infty$ *and for all* $p\in\mathbb{N}$. *It follows that*

$$\|\mathcal{A}(m,n)x\| \le Ke^{-(s_m-s_n)}\|x\|$$

or, equivalently

$$\frac{n+1}{m+1} \le K\frac{n+2}{m+2}.$$

This is true for $K=1$ *because*

$$(n+1)(m+2) \le (n+2)(m+1) \Longleftrightarrow n \le m,$$

is always satisfied for all $(m,n)\in\Delta$. *This shows that* $(A_n)_{n\in\mathbb{N}}$ *admits a generalized exponential stability. Moreover,* $(A_n)_{n\in\mathbb{N}}$ *does not satisfy the inequality*

$$\frac{n+1}{m+1} \le Ke^{-\alpha(m-n)}$$

for all $(m,n)\in\Delta$. *Indeed, for* $n\in\mathbb{N}^*$ *and* $m=2n$ *we have*

$$\frac{n+1}{2n+1} \le Ke^{-\alpha n}$$

which for $n\longrightarrow\infty$ *gives a contradiction. Hence,* $(A_n)_{n\in\mathbb{N}}$ *does not admit a uniform exponential stability.*

In view of Definition 2 for the LTV system ($\mathfrak{A}$), we introduce now the concepts of generalized exponential stability/instability for the dual system ($\mathfrak{B}$), concluding this section with a result showing the relation between these concepts. Also, we refer the reader Chapter 5 from [1] for relations between the dual systems in the context of finite dimensional settings.

Definition 3. *The LTV system ($\mathfrak{B}$) is said to be*

(a) generalized exponentially stable on $\mathbb{N}$ if there exists a constant $K \geq 1$ and a strictly positive sequence $(a_n)_{n\in\mathbb{N}}$ satisfying (3) such that

$$\|\mathcal{B}(m,n)y^*\| \leq Ke^{-(s_m-s_n)}\|y^*\|, \tag{6}$$

for all $(m,n) \in \Delta$ and $y^* \in X^*$.

(b) generalized exponentially instable on $\mathbb{N}$ if there exists a constant $K \geq 1$ and a strictly positive sequence $(a_n)_{n\in\mathbb{N}}$ satisfying (3) such that

$$\|y^*\|e^{s_m-s_n} \leq K\|\mathcal{B}(m,n)y^*\|, \tag{7}$$

for all $(m,n) \in \Delta$ and $y^* \in X^*$.

Theorem 1. *(a) If the LTV discrete-time system ($\mathfrak{A}$) is generalized exponentially stable, then its dual ($\mathfrak{B}$) is generalized exponentially instable.*

(b) If the LTV discrete-time system ($\mathfrak{A}$) is generalized exponentially instable, then its dual ($\mathfrak{B}$) is generalized exponentially stable.

Proof. (a) Let $y^* \in X^*$ and $z \in X$. Then, there exists a unique $x \in X$ such that $z = \mathcal{A}^{-1}(m,n)x$. Hence we obtain

$$\begin{aligned}|y^*(z)| &= |y^*(\mathcal{A}^{-1}(m,n)x)| = |(\mathcal{A}^{-1})^*(m,n)y^*(x)| = |\mathcal{B}(m,n)y^*(x)| \\ &\leq \|\mathcal{B}(m,n)y^*\| \cdot \|x\| = \|\mathcal{B}(m,n)y^*\| \cdot \|\mathcal{A}(m,n)z\| \\ &\leq \|\mathcal{B}(m,n)y^*\|Ne^{-(s_m-s_n)}\|z\|.\end{aligned}$$

Taking the supremum with $\|z\| = 1$ we get

$$\|y^*\|e^{(s_m-s_n)} \leq N\|\mathcal{B}(m,n)y^*\|,$$

hence the conclusion.

(b) Let $y^* \in X^*$ and $z \in X$. Then

$$\begin{aligned}|\mathcal{B}(m,n)y^*(z)| &= |(A^{-1})^*(m,n)y^*(z)| = |y^*(\mathcal{A}^{-1}(m,n))z| \\ &\leq \|y^*\| \cdot \|\mathcal{A}^{-1}(m,n)z\| \\ &\leq \|y^*\|Ke^{-(s_m-s_n)}\|\mathcal{A}(m,n)\mathcal{A}^{-1}(m,n)z\| \\ &= \|y^*\|Ke^{-(s_m-s_n)}\|z\|.\end{aligned}$$

Taking the supremum with $\|z\| = 1$ we obtain

$$\|\mathcal{B}(m,n)y^*\| \leq Ke^{-(s_m - s_n)}\|y^*\|.$$

Hence the system $(\mathfrak{B})$ is generalized exponentially stable. □

3.1 Convergence and generalized behavior results.

Further, we present characterizations of LTV systems for the generalized exponential stability case.

Proposition 1. *For every LTV system* $(\mathfrak{A})$ *the following assertions are equivalent:*

(a) the system $(\mathfrak{A})$ *admits a generalized exponential stability;*

(b) there exists a constant $K \geq 1$ *and a sequence* $(a_n)_{n\in\mathbb{N}}$ *satisfying (3) such that*

$$\|\mathcal{A}(m,p)x\| \leq Ke^{-(s_m - s_n)}\|\mathcal{A}(n,p)x\|,$$

for all $(m,n),(n,p) \in \Delta$ *and* $x \in X$;

(c) there exist some constants $K \geq 1$, $r \in (0,1)$ *and a sequence* $(a_n)_{n\in\mathbb{N}}$ *satisfying (3) such that*

$$\|\mathcal{A}(m,p)x\| \leq Kr^{(s_m - s_n)}\|\mathcal{A}(n,p)x\|,$$

for all $(m,n),(n,p) \in \Delta$ *and* $x \in X$;

(d) there exist some constants $K \geq 1$, $r \in (0,1)$ *and a sequence* $(a_n)_{n\in\mathbb{N}}$ *satisfying (3) such that*

$$\|\mathcal{A}(m,n)x\| \leq Kr^{(s_m - s_n)}\|x\|,$$

for all $(m,n) \in \Delta$ *and* $x \in X$;

(e) there exists a constant $K \geq 1$ *and a strictly decreasing sequence* $(h_n)_{n\in\mathbb{N}}$ *from* $(0,1)$ *with* $\lim_{n\to\infty} h_n = 0$ *such that*

$$\|\mathcal{A}(m,p)x\| \leq K\frac{h_m}{h_n}\|\mathcal{A}(n,p)x\|,$$

for all $(m,n),(n,p) \in \Delta$ *and* $x \in X$;

(f) there exists a constant $K \geq 1$ and a strictly decreasing sequence $(h_n)_{n\in\mathbb{N}}$ from $(0,1)$ with $\lim_{n\to\infty} h_n = 0$ such that

$$\|\mathcal{A}(m,n)x\| \leq K\frac{h_m}{h_n}\|x\|,$$

for all $(m,n) \in \Delta$ and $x \in X$.

Proof. The equivalences between $(a) \Leftrightarrow (b)$, $(c) \Leftrightarrow (d)$ and $(e) \Leftrightarrow (f)$ are obvious. Further, for $r = 1/e$ we have that $(b) \Rightarrow (c)$ and $(a) \Rightarrow (d)$.

Clearly, $(s_n)_{n\in\mathbb{N}}$ is a strictly increasing sequence with $\lim_{n\to\infty} s_n = \infty$. Since $r \in (0,1)$ we conclude that $(h_n)_{n\in\mathbb{N}}$ defined for all $n \in \mathbb{N}$ by $h_n = r^{s_n}$ satisfies the hypothesis from (e) and (f). Thus $(c) \Rightarrow (e)$ and $(d) \Rightarrow (f)$.

Using the properties of $(h_n)_{n\in\mathbb{N}}$ we define $(a_n)_{n\in\mathbb{N}}$ by

$$a_n = \begin{cases} \ln\dfrac{1}{h_0}, & n = 0, \\ \ln\dfrac{h_{n-1}}{h_n}, & n > 0. \end{cases}$$

For each $(m,n) \in \Delta$ we obtain

$$\begin{aligned} s_m - s_n &= a_{n+1} + a_{n+2} + \cdots + a_m \\ &= \ln\frac{h_n}{h_{n+1}} + \ln\frac{h_{n+1}}{h_{n+2}} + \cdots + \ln\frac{h_{m-1}}{h_m} = \ln\frac{h_n}{h_m} \end{aligned}$$

and thus

$$\frac{h_n}{h_m} = e^{s_m - s_n}$$

or, equivalently

$$\frac{h_m}{h_n} = e^{-(s_m - s_n)}.$$

Therefore, $(e) \Rightarrow (b)$ and $(f) \Rightarrow (a)$. This completes the proof. □

We note the following immediate remark.

Remark 1. *For the case of uniform exponential stability, i.e., $a_j = \alpha > 0$ for all $j \in \mathbb{N}$ we have that $s_n = (n+1)\alpha$, for all $n \in \mathbb{N}$. Obviously, $s_m - s_n = \alpha(m-n)$, for all $(m,n) \in \Delta$. In this case the sequence $(h_n)_{n\in\mathbb{N}}$ defined by*

$$h_n = r^{s_n} = r^{\alpha(n+1)} = t^{n+1},$$

where $t = r^{\alpha}$, *satisfies the property*

$$\frac{h_m}{h_n} = \frac{t^{m+1}}{t^{n+1}} = t^{m-n} = h_{m-n-1},$$

for all $(m,n) \in \Delta$.

According to the previous proposition we can obtain necessary and sufficient criteria for the dual system $(\mathfrak{B})$. The proof follows the same line as the previous proposition and therefore is omitted.

Proposition 2. *For the dual* $(\mathfrak{B})$ *of the LTV* $(\mathfrak{A})$*, the following assertions are equivalent:*

(a) the system $(\mathfrak{B})$ *admits a generalized exponential instability;*

(b) there exists a constant $K \geq 1$ *and a sequence* $(a_n)_{n\in\mathbb{N}}$ *satisfying (3) such that*

$$\|\mathcal{B}(n,p)y^*\| e^{s_m - s_n} \leq K \|\mathcal{B}(m,p)y^*\|,$$

for all $(m,n),(n,p) \in \Delta$ *and* $y^* \in X^*$;

(c) there exist some constants $K \geq 1$, $r \in (1,\infty)$ *and a sequence* $(a_n)_{n\in\mathbb{N}}$ *satisfying (3) such that*

$$\|y^*\| r^{(s_m - s_n)} \leq K\| \leq K \|\mathcal{B}(m,n)y^*\|,$$

for all $(m,n) \in \Delta$ *and* $y^* \in X^*$;

(d) there exist some constants $K \geq 1$, $r \in (1,\infty)$ *and a sequence* $(a_n)_{n\in\mathbb{N}}$ *satisfying (3) such that*

$$\|\mathcal{B}(n,p)y^*\| r^{s_m - s_n} \leq K \|\mathcal{B}(m,p)y^*\|,$$

for all $(m,n),(n,p) \in \Delta$ *and* $y^* \in X^*$;

(e) there exists a constant $K \geq 1$ *and a strictly increasing sequence* $(h_n)_{n\in\mathbb{N}}$ *from* $(1,\infty)$ *with* $\lim_{n\to\infty} h_n = \infty$ *such that*

$$\|\mathcal{B}(n,p)y^*\| \frac{h_m}{h_n} \leq K \|\mathcal{B}(m,p)y^*\|,$$

for all $(m,n),(n,p) \in \Delta$ *and* $y^* \in X^*$;

(f) *there exists a constant $K \geq 1$ and a strictly increasing sequence $(h_n)_{n\in\mathbb{N}}$ from $(1,\infty)$ with $\lim\limits_{n\to\infty} h_n = \infty$ such that*

$$\|y^*\|\frac{h_m}{h_n} \leq K\|\mathcal{B}(m,n)y^*\|,$$

for all $(m,n) \in \Delta$ and $y^* \in X^*$.

Proposition 3. *If there exists a sequence $(b_j)_{j\in\mathbb{N}}$ satisfying (3) such that*

$$\sum_{m=n+1}^{+\infty} e^{\sum\limits_{j=n+1}^{m} b_j}\|\mathcal{A}(m,n)x\| \leq D\|x\|, \tag{8}$$

for all $n \geq 0$, $x \in X$ then LTV system ($\mathfrak{A}$) admits a generalized exponential stability.

Proof. One can easily see that

$$t_m - t_n = b_{n+1} + \cdots + b_m$$

implies that

$$\|\mathcal{A}(m,n)x\| \leq e^{-(t_m-t_n)}D\|x\|,$$

for all $(m,n) \in \Delta$, $x \in X$, with $m > n$. □

We conclude this section with some remarks.

Remark 2. *For the case of generalized exponential stability of LTV system ($\mathfrak{A}$), if we consider the sequence $(a_n)_{n\in\mathbb{N}}$ satisfying (3) with $\lim\limits_{n\to\infty} a_n = 0$, then there exists an $n \in \mathbb{N}$ such that*

$$\sum_{m=n+1}^{+\infty} e^{-(s_m-s_n)} = +\infty.$$

Indeed, let $\varepsilon > 0$. Then there exists a $k \in \mathbb{N}^*$ such that $a_j \in (0,\varepsilon)$ for all $j \in \mathbb{N}$ with $j \geq k$. Let $n \in \mathbb{N}$ with $n \geq k$. Then, for all $m \in \mathbb{N}$ with $m > n$ we deduce that

$$s_m - s_n = a_{n+1} + a_{n+2} + \ldots + a_m < (m-n)\varepsilon$$

which implies that

$$-(s_m - s_n) \geq -\varepsilon(m-n)$$

or, equivalently

$$e^{-(s_m - s_n)} \geq e^{-\varepsilon(m-n)}.$$

Thus, we obtain

$$\sum_{m=n+1}^{+\infty} e^{-(s_m - s_n)} \geq \sum_{m=n+1}^{+\infty} e^{-\varepsilon(m-n)} = \frac{e^{-\varepsilon}}{1 - e^{-\varepsilon}} = \frac{1}{e^{\varepsilon} - 1}.$$

Now, taking the limit for $\varepsilon \searrow 0$ yields

$$\sum_{m=n+1}^{\infty} e^{-(s_m - s_n)} \geq +\infty.$$

Remark 3. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence satisfying (3) and ($\mathfrak{A}$) admitting a generalized exponential stability.. If $\inf\limits_{n\geq 0} a_n = t > 0$, then*

$$\sum_{m=n+1}^{+\infty} \|\mathcal{A}(m,n)x\| < +\infty.$$

Indeed,

$$\begin{aligned}
\sum_{m=n+1}^{+\infty} \|\mathcal{A}(m,n)x\| &\leq K\|x\| \sum_{m=n+1}^{+\infty} e^{-(s_m - s_n)} \\
&= K\|x\| \sum_{m=n+1}^{+\infty} e^{-(a_{n+1} + a_{n+2} + \ldots + a_m)} \\
&\leq K\|x\| \sum_{m=n+1}^{+\infty} e^{-t(m-n)} \\
&= K\|x\| \left(e^{-t} + e^{-2t} + \ldots + e^{-jt} + \ldots\right) \\
&= \frac{K\|x\|}{e^t - 1}.
\end{aligned}$$

In this case, based on Theorem 2, point (iv) from [18] we conclude that LTV system ($\mathfrak{A}$) is uniformly exponentially stable.

Remark 4. *One can see that for the case of the generalized exponential stability the previous condition is not valid anymore. More precisely, in Example 1 we have that*

$$\inf_{n\geq 0} a_n = \inf_{n\geq 0} \ln\left(1 + \frac{1}{n+1}\right) = \ln 1 = 0$$

respectively

$$\sum_{m=n+1}^{+\infty} \|\mathcal{A}(m,n)x\| = \|x\| \sum_{m=n+1}^{+\infty} \frac{n+1}{m+1} = +\infty.$$

Remark 5. *If we consider the sequence $(a_n)_{n\in\mathbb{N}}$ satisfying (3) with $\lim_{n\to\infty} a_n = 0$, then left open the question to find the sequence $(b_j)_{j\in\mathbb{N}}$ such that Proposition 3 to be a necessary and sufficient condition.*

3.2 Lyapunov sequences

A powerful method that allows us to assure the stability properties for the LTV system $(\mathfrak{A})$ consists in the use of some auxiliary sequences. We now introduce such a class of sequences.

Definition 4. *We say that $L : \mathbb{N} \times X \to \mathbb{R}_+$ is a Lyapunov sequence for the system $(\mathfrak{A})$ if there exists a constant $K > 1$ such that*

$$\|x\| \le L(n,x) \le K\|x\|,$$

for all $n \in \mathbb{N}$ and $x \in X$.

Such Lyapunov sequences always exist. The trivial example of such Lyapunov sequences, also called Lyapunov norm, is $L(n,x) = \|x\|$. See, for example [15] for the case of ODE's, or Example 1.7.5 from [10] for the LTV system case. Among further developments in this area let us note [14]. The following theorem provides concrete information as to how the generalized exponential stability can be characterized in terms of Lyapunov sequences. Results for the classical concept of uniform exponential stability in terms of Lyapunov sequences can be found in [9], see Proposition 3.2, respectively Proposition 3.3 for the dual system.

Theorem 2. *The LTV system $(\mathfrak{A})$ is generalized exponentially stable if and only if there exists a Lyapunov sequence $L : \mathbb{N} \times X \to \mathbb{R}_+$ for $(\mathfrak{A})$ such that*

$$L(m, \mathcal{A}(m,n)x) \le e^{-(s_m - s_n)} L(n,x), \tag{9}$$

for all $n \in \mathbb{N}$ and $x \in X$.

Proof. Necessity. Suppose that LTV system $(\mathfrak{A})$ is generalized exponentially stable. We define $L : \mathbb{N} \times X \to \mathbb{R}_+$ by

$$L(n,x) = \sup_{m\ge n} e^{s_m - s_n} \|\mathcal{A}(m,n)x\|,$$

for all $n \in \mathbb{N}$ and $x \in X$. First, we observe that

$$L(n,x) \geq e^{s_n - s_n} \|\mathcal{A}(n,n)x\| = \|x\|.$$

On the other hand, we obtain

$$\begin{aligned} L(n,x) &= \sup_{m \geq n} e^{s_m - s_n} \|\mathcal{A}(m,n)x\| \\ &\leq \sup_{m \geq n} e^{s_m - s_n} K e^{-(s_m - s_n)} \|x\| = K\|x\|. \end{aligned}$$

Moreover,

$$\begin{aligned} L(m, \mathcal{A}(m,n)x) &= \sup_{k \geq m} e^{s_k - s_m} \|\mathcal{A}(k,m)\mathcal{A}(m,n)x\| \\ &= \sup_{k \geq m} e^{s_k - s_m} \|\mathcal{A}(k,n)x\| \leq \sup_{k \geq n} e^{-(s_m - s_n)} e^{s_k - s_n} \|\mathcal{A}(k,n)x\| \\ &= e^{-(s_m - s_n)} \sup_{k \geq n} e^{s_k - s_n} \|\mathcal{A}(k,n)x\| = e^{-(s_m - s_n)} L(n,x) \end{aligned}$$

for all $(m,n) \in \Delta$ and $x \in X$. This will conclude that (9) is verified.

Sufficiency. Let $L : \mathbb{N} \times X \to \mathbb{R}_+$ be a Lyapunov sequence that verifies (9). Hence,

$$\begin{aligned} \|\mathcal{A}(m,n)x\| &\leq L(m, \mathcal{A}(m,n)x\| \\ &\leq e^{-(s_m - s_n)} L(n,x) \leq K e^{-(s_m - s_n)} \|x\|. \end{aligned}$$

This shows generalized exponential stability of the LTV system ($\mathfrak{A}$) and completes our proof. □

Finally, we apply the derived results in terms of Lyapunov sequence for the dual system.

Definition 5. *We say that $L^* : \mathbb{N} \times X^* \to \mathbb{R}_+$ is a Lyapunov sequence for the system ($\mathfrak{B}$) if there exists a constant $K > 1$ such that*

$$\frac{1}{\|y^*\|} \leq L^*(n, y^*) \leq \frac{K}{\|y^*\|},$$

for all $n \in \mathbb{N}$ and $y^* \in X^*$, with $y^* \neq \theta^*$.

Theorem 3. *The LTV system ($\mathfrak{B}$) is generalized exponentially instable if and only if there exists a Lyapunov sequence $L^* : \mathbb{N} \times X^* \to \mathbb{R}_+$ for ($\mathfrak{B}$) such that*

$$L^*(m, \mathcal{B}(m,n)y^*) \leq e^{s_n - s_m} L^*(n, y^*), \tag{10}$$

for all $(m,n) \in \Delta$ and $y^* \in X^*$.

Proof. Suppose that LTV system $(\mathfrak{B})$ is generalized exponentially instable. We define $L^* : \mathbb{N} \times X^* \to \mathbb{R}_+$ by

$$L^*(n, y^*) = \begin{cases} \sup\limits_{m \geq n} \dfrac{e^{s_m - s_n}}{\|\mathcal{B}(m,n)y^*\|}, & \text{if} \quad y^* \neq \theta^*, \\ 0, & \text{if} \quad y^* = \theta^*. \end{cases}$$

for all $n \in \mathbb{N}$ and $y^* \in X^*$. Without loss of generality we assume that $y^* \neq \theta^*$. Thus

$$L^*(n, y^*) \geq \frac{e^{s_n - s_n}}{\|\mathcal{B}(n,n)y^*\|} = \frac{1}{\|y^*\|}.$$

On the other hand from (7) it follows that

$$L^*(n, y^*) \leq \frac{K}{\|y^*\|}.$$

We also have

$$\begin{aligned} L^*(m, \mathcal{B}(m,n)y^*) &= \sup_{k \geq m} \frac{e^{s_k - s_m}}{\|\mathcal{B}(k,m)\mathcal{B}(m,n)y^*\|} = \sup_{k \geq m} \frac{e^{s_k - s_m}}{\|\mathcal{B}(k,n)y^*\|} \\ &\leq \sup_{k \geq n} \frac{e^{s_k - s_n} e^{s_n - s_m}}{\|\mathcal{B}(k,n)y^*\|} = e^{s_n - s_m} L^*(n, y^*). \end{aligned}$$

This will conclude that (10) is verified.

Conversely, consider a Lyapunov sequences $L^* : \mathbb{N} \times X^* \to \mathbb{R}_+$ that verifies (10). Let $y^* \in X^*$ with $y^* \neq \theta^*$. Then we have

$$\begin{aligned} \frac{1}{\|\mathcal{B}(m,n)y^*\|} &\leq L^*(m, \mathcal{B}(m,n)y^*)\| \\ &\leq e^{s_n - s_m} L^*(n, y^*) \leq e^{s_n - s_m} \frac{K}{\|y^*\|}. \end{aligned}$$

This will conclude that system $(\mathfrak{B})$ is generalized exponentially instable. □

4 Conclusions

Uniform exponential behaviors like stability/instability, which have been proved so useful in characterizing LTV systems, have been extended here to the so-called concepts of generalized exponential stability/instability. This new concepts provide a better view regarding the grow rates from the classical ones. Furthermore, we have derived a set of generalized exponential stability conditions, also Lyapunov sequences for the LTV discrete-time system, and we have applied them to the adjoint one.

Several issues remain open for future research. First, one has to quantitatively improve the result in Proposition 3.4 so as the open problems results in [18] for the generalized exponential stability concept. Second, generalized exponential dichotomy/trichotomy should be defined for one-side systems. Third, one could consider the proposed approach to study the robustness for LTV one-side systems. Finally, should be of interest the consideration of the dual system in the absence of the invertibility property. In this case, the anticausal exponential stability is of interest. Such systems occur naturally in connection with the problem of the exponential stability in mean square of linear systems perturbed by Markov processes with an infinite number of states, see e.g. [4] and [5] for the discrete-time case and [6] for the continuous time case.

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