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NON-EQUILIBRIUM THERMODYNAMICS FRAMEWORK FOR FLUID FLOW AND POROSITY DYNAMICS IN POROUS ISOTROPIC MEDIA*

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Dedicated to Dr. Vasile Drăgan on the occasion of his 70th anniversary

Abstract

In previous papers in the linear and anisotropic case, constitutive relations, rate equations, temperature and energy equations were derived by the authors to describe the mechanical, thermal and transport properties of fluid-saturated crystals with porous channels defects, using a model developed by one of us (L. R.) in the framework of non-equilibrium thermodynamics. A structural permeability tensor à la Kubik, r_{ij}, its gradient and its flux $\mathcal{V}_{ijk}$ were introduced as internal variables in the thermodynamic state vector. Here, we work out in the isotropic and perfect isotropic linear cases the constitutive functions for the stress tensor, the entropy density, the chemical potentials, and also the rate equations for r_{ij}, $\mathcal{V}_{ijk}$, the fluid-concentration and the heat fluxes, describing disturbances propagating with finite velocity

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and presenting a relaxation time. The porous defects modify the thermal conductivity and when they have a density higher than a suitable characteristic value the thermal conductivity decreases. Furthermore, the closure of the system of equations, describing the media under consideration and linearized around a thermodynamic equilibrium state is obtained. The derived results may have great relevance in biology, medical sciences and in several technological sectors, like seismic engineering and nanotechnology (where high-frequency waves propagation is present and the properties variation rate of the considered medium is faster than the relaxation times of the fluxes towards their equilibrium value).
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1 Introduction

The study of media with porous defects may have relevance in the description of phenomena accompanying flows of mass in porous structures and find applications in applied sciences. Here, we use a thermodynamic theory (see [1], [2] and also [3], [4]), developed in the framework of Extended Irreversible Thermodynamics, [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], with internal variables. More precisely, in [1] and [2] for the media under consideration the basic equations were established, the Liu's theorem [17] was applied and in a special case the constitutive theory and the rate equations for the fluxes and the porosity field were constructed as objective functions using Smith's theorem [18]. In [3] and [4] constitutive relations, rate equations and other resutls were derived for the same media in the anisotropic case. In this paper we investigate the behaviour of isotropic and perfect isotropic porous structures filled by a fluid flow, having a particular spatial symmetry properties, using a mathematical theory for isotropic cartesian tensors [19], [20]. The influence of porous channels on the other fields, occurring inside the considered medium, is described by a stuctural permeability tensor à la Kubik [21], giving a macroscopic characterization of the porous matrix. In [22], [23], [24], [25], [26], [27], [28], [29], [30], [31], [32] and [33] models, with some applications, for media with defects having the form of a network of very thin tubes, like porous channels and dislocations, were formulated, using the same methods of non-equilibrium thermodynamics in the case for instance of piezoelectric, elastic, semiconductor and superlattice structures. Also in [34], [35], [36], [37] and [38] non-equilibrium temperatures and heat equation were studied in media with internal variables and in the same thermodynamic framework of non-equilibrium thermodynamics. A relatively high temperature gradient could produce, for instance, a migration of defects inside the system. The results, obtained in this paper have great interest in several technological sectors (see [41]) like physics of soil, medical sciences, seismic engineering, in acoustic pollution, to build barrier against it, in pharmaceutics and in nanotechnology, where the rate of variation of the properties of the system is faster than the time scale characterizing the relaxation times of the fluxes towards their respective equilibrium value, there are situations of high-frequency waves propagation and the volume element size L of these nanostructures along some directions is so small that it becomes comparable (or smaller than) the free mean path l of the heat carriers ($L \ll l$, i.e. the Knudsen number Kn, $Kn = \frac{l}{L}$, is such that $\frac{l}{L} \gg 1$). The porous defects change the thermal conductivity, and from experiments and theoretic studies it was seen that the porous defects have a minor effect on the thermal conductivity when their density is smaller than a characteristic value, which depends on the material and the temperature, but when their density is higher than this characteristic value, the thermal conductivity decreases. The paper is organised as follows. In Section 2 the governing equations of the model, describing the mechanical, thermal and transport properties of solid structure with porous channels, saturated by a fluid flow, derived in the framework of extended thermodynamics with internal variable, are presented. The basic balances and the rate equations for porosity field, its flux, the heat flux and the fluid-concentration flux are introduced [1]. In the Section 3, the results obtained in the linear anisotropic case for the constitutive relations and the rate equations in [3] and for the temperature and energy equations [4] are presented. Sections 4 and 5 deal with the study of porous media filled by a fluid flow, when they are isotropic under orthogonal transformations and the Cartesian components of the equations describing them do not depend on all the Cartesian components of the independent variables of the chosen thermodynamic state vector. The case where the symmetry properties of the system under consideration are invariant with respect to all rotations of axes frame (isotropic case) and the case where these symmetry properties are invariant with respect to all rotations and inversions of axes frame (perfect isotropic case) are treated in detail and the constitutive, rate, temperature and energy equations are worked out in these two different cases. In particular, the generalized Maxwell-Cattaneo-Vernotte and Fick-Nonnenmacher rate equations for the heat and fluid-concentration fluxes and the rate equations for the porosity field and its flux are derived, showing the influence of porous defects on the transport properties of the considered media. Finally, the closure of the whole system of equations, describing the behaviour of the isotropic and perfect isotropic porous structures under consideration, is deduced. The Appendix is dedicated to the study of particular special forms for third, fourth, fifth and sixth order isotropic tensors, having symmetry properties, coming from the symmetry of the strain tensor ε_{ij} and the structural permeability tensor r_{ij}, and also from the used model. The expressions are cumbersome but are useful in computer programming for physical phenomena simulations. The obtained results can be applied to simpler real cases, where it is possible to neglect the influence of some fields occurring inside the examined media. Other different approaches for porous structures saturated by fluid flows are in [39], [40] and [41].

2 Governing Equations

The aim of this paper is to study structures with defects of porous channels saturated by a fluid flow. To describe as the defects field evolves, we introduce in the thermodynamic state vector the symmetric structural permeability field r_{ij} à la Kubik [21], its gradient $r_{ij,k}$ and its flux $\mathcal{V}_{ijk}$. Here, we present a model for fluid-satured porous nanocrystals, developed in [1], [2] and also in [3] and [4], in the framework of extended irreversible thermodynamics with internal variables. In [21] a representative elementary sphere volume Ω of a porous skeleton filled by a fluid flow is considered, large enough to give a representation of its statistical properties in order to use average statical procedures to define this porosity tensor from the macroscopic point of view. We assume that the mass of the fluid filling the porous channels inside the solid matrix and this same matrix constitute a two-components mixture. Indicating by ρ_1 the mass of the fluid transported through the elastic porous crystal of density ρ_2, we describe this fluid flow by two variables: the *concentration of the fluid* $c = \frac{\rho_1}{\rho}$ and the *flux* of this fluid j_i^c (see (3)), in such way that

$$\rho = \rho_1 + \rho_2. \tag{1}$$

The following continuity equations are valid for the mixture of continua as a whole and also for each its constituent

$$\dot{\rho} + \rho v_{i,i} = 0, \quad \frac{\partial \rho_1}{\partial t} + (\rho_1 v_{1i})_{,i} = 0, \quad \frac{\partial \rho_2}{\partial t} + (\rho_2 v_{2i})_{,i} = 0, \tag{2}$$

where a superimposed dot denotes the material derivative, $\frac{d}{dt} = \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla$, v_{1i} and v_{2i} are the velocities of the fluid particles and the particles of the elastic body, respectively and we suppose that there are not source terms. The barycentric velocity v_i of the mixture and the fluid flux j_i^c are defined as follows

$$\rho v_i = \rho_1 v_{1i} + \rho_2 v_{2i}, \quad j_i^c = \rho_1 (v_{1i} - v_i). \tag{3}$$

The thermal field is described by the *absolute temperature* T, its gradient $T_{,i}$ and the *heat flux* q_i. The mechanical field is described by the symmetric total stress tensor τ_{ij}, referred to the whole system considered as a mixture, and by the small strain tensor ε_{ij}, $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$, with $\boldsymbol{u}$ the displacement vector.

We choose the following thermodynamic state vector

$$C = \{\varepsilon_{ij}, c, T, r_{ij}, j_i^c, q_i, \mathcal{V}_{ijk}, c_{,i}, T_{,i}, r_{ij,k}\}, \tag{4}$$

where, we have taken into consideration the gradients $c_{,i}, T_{,i}$ and $r_{ij,k}$, and have ignored the viscous effects, that we have studied in [41].
We consider two groups of laws. The first group concerns the classical balance equations [9]:
the balance of mass having the form

$$\rho \dot{c} + j_{i,i}^c = 0; \tag{5}$$

the momentum balance

$$\rho \dot{v}_i - \tau_{ji,j} - f_i = 0, \tag{6}$$

where f_i denotes a *body force*, that in the following will be disregarded for the sake of simplicity;
the internal energy balance

$$\rho \dot{e} - \tau_{ji} v_{i,j} + q_{i,i} - \rho h = 0, \tag{7}$$

where e is the internal energy density, h is the *energy source density* (also neglected in the following) and $v_{i,j}$ is the gradient of the velocity of the body;

the second group of laws deals with the rate equations for the structural permeability field r_{ij}, its flux $\mathcal{V}_{ijk}$, the heat flux q_i and the fluid-concentration flux j_i^c, constructed in such a way that they obey the objectivity and frame-indifference principles (see [42], [43] and [44]).

We choose for these evolution equations the following form

$$\overset{*}{r}_{ij} + \mathcal{V}_{ijk,k} - \mathcal{R}_{ij}(C) = 0, \tag{8}$$

$$\overset{*}{\mathcal{V}}_{ijk} - V_{ijk}(C) = 0, \tag{9}$$

$$\overset{*}{q}_i - Q_i(C) = 0, \tag{10}$$

$$\overset{*}{j}{}^c_i - J^c_i(C) = 0, \tag{11}$$

where the symbol (*) denotes the *Zaremba-Jaumann* derivative, defined for the vectors, the second rank tensor and the third rank tensor present in (8)-(9) as follows

$$\overset{*}{j}{}^c_i = \dot{j}^c_i - w_{ik}j^c_k, \quad \overset{*}{q}_i = \dot{q}_i - w_{ik}q_k, \quad \overset{*}{r}_{ij} = \dot{r}_{ij} - w_{ik}r_{kj} - w_{jk}r_{ik}\ , \tag{12}$$

$$\overset{*}{\mathcal{V}}_{ijk} = \dot{\mathcal{V}}_{ijk} - w_{ip}\mathcal{V}_{pjk} - w_{jp}\mathcal{V}_{ipk} - w_{kp}\mathcal{V}_{ijp}, \tag{13}$$

with $w_{ij} = v_{i,j} - \frac{d\varepsilon_{ij}}{dt}$, being $w_{ij} = \frac{1}{2}(v_{i,j} - v_{j,i})$ and $\frac{d\varepsilon_{ij}}{dt} = \frac{1}{2}(v_{i,j} + v_{j,i})$ the antisymmetric and the symmetric part of $v_{i,j}$, respectively. In (8)-(11) $\mathcal{R}_{ij}(C)$ is the source term describing the creation or annihilation of porous channels, $V_{ijk}(C)$, $Q_i(C)$ and $J^c_i(C)$ are the sources of the porosity field flux, the heat flux and the fluid-concentration flux. They are constitutive functions of the independent variables of the thermodynamic state vector space (4). In the rate equations (8)-(11) we do not consider the flux terms of r_{ij}, $\mathcal{V}_{ijk}$, q_i and j^c_i, in such a way that we can close the system of equations governing the behaviour of the media taken into account. Furthermore, in (8)-(11) we use for w_{ij} the expression $w_{ij} = v_{i,j} - \frac{\partial\varepsilon_{ij}}{\partial t}$, to obtain relations in linear approximation.

3 Constitutive relations, rate equations, temperature and energy equations in the anisotropic case

In order that the physical processes occurring in the considered porous structure filled by a fluid flow are real, all the admissible solutions of the proposed governing equations should be restricted by the following entropy inequality

$$\rho\dot{S} + \phi_{i,i} - \frac{\rho h}{T} = \sigma^{(s)} \geq 0, \tag{14}$$

where S is the *entropy density*, $\dfrac{\rho h}{T}$ is the *external entropy production source* (neglected in the following), $\sigma^{(s)}$ is the *internal entropy production* and ϕ_i is the *entropy flux*. In [1] Liu's theorem [17], that considers all balance and evolution equations as mathematical constraints for the general validity of the inequality (14), was applied, assuming that the density mass ρ of the considered defective nanocrystals is constant, and the state laws, the generalized affinities, the entropy inequality and the following functional form of the free energy F

$$F = F(\varepsilon_{ij}, c, T, r_{ij}, j_i^c, q_i, \mathcal{V}_{ijk}) \tag{15}$$

were obtained.

In [2] the set of constitutive functions (dependent functions on the set (4) of independent variables)

$$W = \{\tau_{ij}, e, \mathcal{R}_{ij}, J_i^c, Q_i, V_{ijk}, S, \phi_i, \Pi^c, \Pi_{ij}^r\} \tag{16}$$

(with Π^c the *chemical potential of the fluid concentration field* and Π_{ij}^r a *potential related to the structural permeability field*), having the general form

$$W = \widetilde{W}(C), \tag{17}$$

were constructed in a special case as objective polynomial constitutive functions using Smith's theorem [18]. In (17) both C and W are evaluated at the same point and time.

Furthermore, in [3] the free energy (15) was expanded up the second-order approximation around a reference state, representing a thermodynamic equilibrium state, indicated by the subscript "$_0$", where it was supposed that

$$(\varepsilon_{ij})_0 = 0, \quad (\tau_{ij})_0 = 0, \quad (u_i)_0 = u_{0i}, \quad (v_i)_0 = v_{0i}, \tag{18}$$

and the deviations of some variables from this reference state were indicated by

$$\theta = T - T_0, \ \left|\frac{\theta}{T_0}\right| \ll 1, \quad \tilde{e} = e - e_0, \ \left|\frac{\tilde{e}}{e_0}\right| \ll 1, \quad \mathcal{C} = c - c_0, \ \left|\frac{\mathcal{C}}{c_0}\right| \ll 1, \tag{19}$$

$$\mathcal{S} = S - S_0, \ \left|\frac{\mathcal{S}}{S_0}\right| \ll 1, \quad R_{ij} = r_{ij} - r_{0ij}, \ \left|\frac{R_{ij}}{r_{0ij}}\right| \ll 1. \tag{20}$$

In this thermodynamic state we have

$$\begin{aligned}&(j_i^c)_0 = 0, \ (q_i)_0 = 0, \ (\mathcal{V}_{ijk})_0 = 0, \ \left(\Pi_{ij}^r\right)_0 = 0, \ (\Pi^c)_0 = 0,\\ &\left(\Pi_{ijk}^{\nu}\right)_0 = 0, \ (\Pi_i^q)_0 = 0, \ \left(\Pi_i^{j^c}\right)_0 = 0,\end{aligned} \tag{21}$$

being Π^{ν}_{ijk}, Π^{q}_{i} and $\Pi^{j^c}_{i}$ the *generalized affinities* conjugated to the respective fluxes $\mathcal{V}_{ijk}$, q_i and j^c_i. The potentials and the generalized affinities were obtained in [3] as partial derivative of F with respect to own conjugate variable. In particular the following relations were obtained in the anisotropic case in the linear approximation:
the constitutive relations

$$\tau_{ij} = c_{ijlm}\varepsilon_{lm} - \lambda^{\theta\varepsilon}_{ij}\theta + \lambda^{r\varepsilon}_{ijlm}R_{lm} - \lambda^{c\varepsilon}_{ij}\mathcal{C}, \tag{22}$$

$$S = S_0 + \frac{\lambda^{\theta\varepsilon}_{ij}}{\rho}\varepsilon_{ij} + \frac{c_v}{T_0}\theta - \frac{\lambda^{r\theta}_{ij}}{\rho}R_{ij} - \frac{\lambda^{\theta c}}{\rho}\mathcal{C}, \tag{23}$$

$$\Pi^{r}_{ij} = \lambda^{r\varepsilon}_{ijlm}\varepsilon_{lm} + \lambda^{r\theta}_{ij}\theta + \lambda^{rr}_{ijlm}R_{lm} + \lambda^{rc}_{ij}\mathcal{C}, \tag{24}$$

$$\Pi^{c} = -\frac{\lambda^{c\varepsilon}_{ij}}{\rho}\varepsilon_{ij} + \frac{\lambda^{\theta c}}{\rho}\theta + \frac{\lambda^{rc}_{ij}}{\rho}R_{ij} + \frac{\lambda^{c}}{\rho}\mathcal{C}, \tag{25}$$

the generalized affinities

$$\Pi^{\nu}_{ijk} = \lambda^{\nu\nu}_{ijklmn}\mathcal{V}_{lmn} + \lambda^{\nu q}_{ijkl}q_l + \lambda^{\nu j^c}_{ijkl}j^c_l, \tag{26}$$

$$\Pi^{q}_{i} = \lambda^{q\nu}_{ijkl}\mathcal{V}_{jkl} + \lambda^{qq}_{ij}q_j + \lambda^{qj^c}_{ij}j^c_j, \tag{27}$$

$$\Pi^{j^c}_{i} = \lambda^{j^c\nu}_{ijkl}\mathcal{V}_{jkl} + \lambda^{j^c q}_{ij}q_j + \lambda^{j^c j^c}_{ij}j^c_j. \tag{28}$$

The quantities like c_{ijlm}, $\lambda^{\theta\varepsilon}_{ij}$, $\lambda^{r\varepsilon}_{ijlm}$, which occur in (22)-(28), are called phenomenological tensors. For instance, c_{ijlm} is the elastic tensor of order four, $\lambda^{\theta\varepsilon}_{ij}$ is the thermoelastic tensor of order two, λ^{rc}_{ij} is a tensor of order two connected with the influence of the fluid-concentration flow on the porosity field, $\lambda^{r\theta}_{ij}$ is a tensor of order two connected with the influence of the porosity field on the temperature field. In (23) c_v is the specific heat. Furthermore, the constant phenomenological coefficients satisfy the following symmetric relations, see [3] (being defined in terms of second derivatives of F and for the symmetry of ε_{ij} and r_{ij})

$$\begin{gathered}
c_{ijlm} = c_{jilm} = c_{ijml} = c_{jiml} = c_{lmij} = c_{mlij} = c_{mlji} = c_{lmji},\\
\lambda^{r\varepsilon}_{ijlm} = \lambda^{r\varepsilon}_{jilm} = \lambda^{r\varepsilon}_{ijml} = \lambda^{r\varepsilon}_{jiml} = \lambda^{r\varepsilon}_{lmji} = \lambda^{r\varepsilon}_{lmij} = \lambda^{r\varepsilon}_{mlji} = \lambda^{r\epsilon}_{mlij},\\
\lambda^{rr}_{ijlm} = \lambda^{rr}_{ijml} = \lambda^{rr}_{jilm} = \lambda^{rr}_{jiml} = \lambda^{rr}_{lmij} = \lambda^{rr}_{lmji} = \lambda^{rr}_{mlij} = \lambda^{rr}_{mlji},\\
\lambda^{\theta\varepsilon}_{ij} = \lambda^{\theta\varepsilon}_{ji}, \quad \lambda^{qq}_{ij} = \lambda^{qq}_{ji}, \quad \lambda^{rc}_{ij} = \lambda^{rc}_{ji}, \quad \lambda^{c\epsilon}_{ij} = \lambda^{c\varepsilon}_{ji}, \quad \lambda^{r\theta}_{ij} = \lambda^{r\theta}_{ji},\\
\lambda^{\nu\nu}_{ijklmn} = \lambda^{\nu\nu}_{lmnijk}, \quad \lambda^{\nu q}_{ijkl} = \lambda^{\nu q}_{lijk}, \quad \lambda^{j^c j^c}_{ij} = \lambda^{j^c j^c}_{ji}, \lambda^{j^c q}_{ij} = \lambda^{j^c q}_{ji}, \quad \lambda^{\nu j^c}_{ij} = \lambda^{\nu j^c}_{ji}.
\end{gathered}$$

In [3], in the case where it is possible to replace Zaremba-Jaumann derivative by the material derivative, the following rate equations were obtained, supposing that the source terms $\mathcal{R}_{ij}$, V_{ijk}, J^c_i, Q_i can be expressed as linear objective polynomials having constant coefficients, in terms of the independent variables:

$$\dot{r}_{ij} + \mathcal{V}_{ijk,k} = \beta^1_{ijkl}\varepsilon_{kl} + \beta^2_{ijkl}r_{kl} + \beta^3_{ijk}j^c_k + \beta^4_{ijk}q_k + \beta^5_{ijklm}\mathcal{V}_{klm} + \beta^6_{ijk}c_{,k} + \beta^7_{ijk}T_{,k} + \beta^8_{ijklm}r_{kl,m}, \tag{29}$$

where

$$\beta^s_{ijkl} = \beta^s_{jikl} = \beta^s_{ijlk} = \beta^s_{jilk}, \quad (s = 1, 2), \quad \beta^8_{ijklm} = \beta^8_{jiklm} = \beta^8_{ijlkm} = \beta^8_{jilkm}, \tag{30}$$

$$\beta^r_{ijk} = \beta^r_{jik}, \quad (r = 3, 4, 6, 7), \quad \beta^5_{ijklm} = \beta^5_{jiklm}, \quad \mathcal{V}_{ijk,k} = \mathcal{V}_{jik,k}, \tag{31}$$

because the phenomenological tensors $\boldsymbol{\beta}^s$ $(s = 1, 2, \ldots, 8)$ are symmetric in the indexes $\{i, j\}$, by virtue of the symmetry of r_{ij}, $\boldsymbol{\beta}^s$ $(s = 1, 2, 8)$ are also symmetric in the indexes $\{k, l\}$, because they are dummy with the indexes of the symmetric tensors ε_{kl}, r_{kl} and $r_{kl,m}$, respectively, and $\mathcal{V}_{ijk,k} = \mathcal{V}_{jik,k}$ is symmetric in the indexes $\{i, j\}$, by virtue of the symmetry of r_{ij};

$$\dot{\mathcal{V}}_{ijk} = \gamma^1_{ijkl}j^c_l + \gamma^2_{ijkl}q_l + \gamma^3_{ijklmn}\mathcal{V}_{lmn} + \gamma^4_{ijkl}c_{,l} + \gamma^5_{ijkl}T_{,l} + \gamma^6_{ijklmn}r_{lm,n}, \tag{32}$$

where

$$\gamma^6_{ijklmn} = \gamma^6_{ijkmln}, \tag{33}$$

because of the symmetry $r_{lm,n}$;

$$\tau^q \dot{q}_i = \chi^1_{ij}j^c_j - q_i + \chi^3_{ijkl}\mathcal{V}_{jkl} + \chi^4_{ij}c_{,j} - \chi^5_{ij}T_{,j} + \chi^6_{ijkl}r_{jk,l}, \tag{34}$$

where χ^1_{ij} is the thermodiffusive kinetic tensor, χ^4_{ij} is the thermodiffusive tensor, χ^5_{ij} is the heat conductivity tensor and

$$\chi^6_{ijkl} = \chi^6_{ikjl}, \tag{35}$$

by virtue of the symmetry of the tensor $r_{jk,l}$;

$$\tau^{j^c} \dot{j}^c_i = -j^c_i + \xi^2_{ij}q_j + \xi^3_{ijkl}\mathcal{V}_{jkl} - \xi^4_{ij}c_{,j} + \xi^5_{ij}T_{,j} + \xi^6_{ijkl}r_{jk,l}, \tag{36}$$

where ξ^4_{ik} and ξ^5_{ik} are the diffusion tensor and the thermodiffusive tensor, respectively, and

$$\xi^6_{ijkl} = \xi^6_{ikjl}, \tag{37}$$

because of the symmetry of $r_{jk,l}$.

The rate equation (34) generalizes Maxwell-Vernotte-Cattaneo relation (obtained when the only influence of the temperature gradient field is taken into consideration)

$$\tau^q \dot{q}_i = -q_i - \chi^5_{ij} T_{,j}.$$

This equation presents a relaxation time for the heat flux and describes thermal disturbances having finite velocity of propagation (see [45] and [46]). When the relaxation time τ^q goes to zero it reduces to anisotropic Fourier equation

$$q_i = -\chi^5_{ij} T_{,j},$$

leading to infinite speeds propagation of thermal signals.
The rate equation (36) for the mass flux generalizes the Fick-Nonnenmacher law

$$\tau^{j^c} \dot{j}^c_i = -j^c_i - \xi^4_{ij} c_{,j},$$

describing propagation of disturbances with finite velocity and presenting a relaxation time for the fluid-concentration flux.

Furthermore, we obtained in [4] *the generalized telegraph temperature equation* linearized around the thermodynamic equilibrium state defined by (18)-(21)

$$\begin{aligned}\tau^q \ddot{T} + \dot{T} = &-\gamma_{ij}(\tau^q \ddot{\varepsilon}_{ij} + \dot{\varepsilon}_{ij}) + \varphi(\tau^q \ddot{c} + \dot{c}) + \eta_{ij}(\tau^q \ddot{r}_{ij} + \dot{r}_{ij}) \\ &+ k_{ij} T_{,ji} - \nu^1_{ij} j^c_{j,i} - \nu^3_{ijkl} \mathcal{V}_{jkl,i} - \nu^4_{ij} c_{,ji} - \nu^6_{ijkl} r_{jk,li}, \end{aligned} \tag{38}$$

where

$$\gamma_{ij} = \frac{T_0}{c_v \rho} \lambda^{\theta\varepsilon}_{ij}, \quad \varphi = \frac{T_0}{c_v} \lambda^{\theta c}, \quad \eta_{ij} = \frac{T_0}{c_v \rho} \lambda^{r\theta}_{ij}, \quad k_{ij} = \frac{\chi^5_{ij}}{c_v \rho}, \tag{39}$$

$$\nu^1_{ij} = \frac{\chi^1_{ij}}{c_v \rho}, \quad \nu^3_{ijkl} = \frac{\chi^3_{ijkl}}{c_v \rho}, \quad \nu^4_{ij} = \frac{\chi^4_{ij}}{c_v \rho}, \quad \nu^6_{ijkl} = \frac{\chi^6_{ijkl}}{c_v \rho}, \tag{40}$$

the superimposed dot " $\dot{}$ " indicates the linearized time derivative $\frac{d}{dt} = \frac{\partial}{\partial t} + \boldsymbol{v}_0 \cdot \mathrm{grad}$ and the deviations of the fields from the thermodynamic equilibrium state are indicated by the same symbols of the fields themselves. The evolution equations (29), (32), (34), (36) and (38) allow finite speeds for the field disturbances and describe fast phenomena whose relaxation times are comparable or higher than the relaxation times of the media under consideration. In (29), (32), (34) and (36) the last three terms (in equation (38) the last four terms) describe non-local effects, because they relate the rate equations to the inhomogeneities of the system.

Also, in [3], we have obtained a linear form for the first law of Thermodynamics

$$\rho \dot{e} = T_0 \lambda^{\theta\varepsilon}_{ij} \dot{\varepsilon}_{ij} + \rho c_v \dot{T} - T_0 \lambda^{r\theta}_{ij} \dot{r}_{ij} - T_0 \lambda^{\theta c} \dot{c}. \tag{41}$$

4 Isotropic porous media with respect to all rotations of axes frame

The existence of spatial symmetry properties in a porous structure may simplify the form of the constitutive equations, the generalized affinities and the rate equations, in such a way that the Cartesian components of these equations do not depend on all the Cartesian components of the independent variables of the thermodynamic state vector C (4). This statement is called Curie symmetry principle [9].

In the following two Subsections we will study the form of the balance equations, the constitutive relations, the rate equations, the temperature equation and the closure of equations system describing the behaviour of the media under consideration having symmetry properties *invariant with respect to all rotations of axes frame.*

First, we examine the form of isotropic tensors of rank up to six.

The tensors of rank up to three take the form [19]

$$L_i = 0, \quad L_{ij} = L\delta_{ij}, \quad L_{ijk} = L \in_{ijk}, \tag{42}$$

where $\in_{ijk}$ is the Levi-Civita tensor and L is a scalar.

Tensors of order four L_{ijkl} must have the form [19]

$$L_{ijkl} = L_1\delta_{ij}\delta_{kl} + L_2\delta_{ik}\delta_{jl} + L_3\delta_{il}\delta_{jk}, \tag{43}$$

with L_i $(i = 1, 2, 3)$ scalars.

Tensors of order five L_{ijklm} and of *order six* L_{ijklmn} must have the form, respectively, (see [20])

$$\begin{aligned} L_{ijklm} = {} & L_1 \in_{ijk} \delta_{lm} + L_2 \in_{ijl} \delta_{km} + L_3 \in_{ijm} \delta_{kl} + L_4 \in_{ikl} \delta_{jm} \\ & + L_5 \in_{ikm} \delta_{lj} + L_6 \in_{ilm} \delta_{jk}, \end{aligned} \tag{44}$$

$$\begin{aligned} L_{ijklmn} = {} & L_1\delta_{ij}\delta_{kl}\delta_{mn} + L_2\delta_{ij}\delta_{km}\delta_{ln} + L_3\delta_{ij}\delta_{kn}\delta_{lm} + L_4\delta_{ik}\delta_{jl}\delta_{mn} \\ & + L_5\delta_{ik}\delta_{jm}\delta_{ln} + L_6\delta_{ik}\delta_{jn}\delta_{lm} + L_7\delta_{il}\delta_{jk}\delta_{mn} + L_8\delta_{il}\delta_{jm}\delta_{kn} \\ & + L_9\delta_{il}\delta_{jn}\delta_{km} + L_{10}\delta_{im}\delta_{jk}\delta_{ln} + L_{11}\delta_{im}\delta_{jl}\delta_{kn} + L_{12}\delta_{im}\delta_{jn}\delta_{kl} \\ & + L_{13}\delta_{in}\delta_{jk}\delta_{lm} + L_{14}\delta_{in}\delta_{jl}\delta_{km} + L_{15}\delta_{in}\delta_{jm}\delta_{kl}, \end{aligned} \tag{45}$$

with L_i $(i = 1, 2, \ldots, 15)$ scalars.

4.1 Constitutive relations, generalized affinities and rate equations in the isotropic case

Taking into account (42)-(45), in the case of an isotropic medium with respect to all rotations of axes frame the *isotropic constitutive relations* for τ_{ij}, Π^r_{ij}, S and Π^c, derived from (22)-(25) with $\lambda^{\theta\varepsilon}_{ij}$, $\lambda^{c\varepsilon}_{ij}$, $\lambda^{r\theta}_{ij}$ and λ^{rc}_{ij} having the form $(42)_2$ and c_{ijlm}, $\lambda^{r\varepsilon}_{ijlm}$ and λ^{rr}_{ijlm} taking the form (83) of Appendix A, because of their particular symmetries, are:
for *the stress tensor*

$$\tau_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij} - \lambda^{\theta\varepsilon}\delta_{ij}\theta + \lambda^{r\varepsilon}_1\delta_{ij}R_{kk} + \lambda^{r\varepsilon}_2 R_{ij} - \lambda^{c\varepsilon}\delta_{ij}\mathcal{C}, \tag{46}$$

where λ and μ are the well known Lamé constants, that represent the two significant independent components of c_{ijlm}, and $\lambda^{r\varepsilon}_1$ and $\lambda^{r\varepsilon}_2$ are the two significant independent components of $\lambda^{r\varepsilon}_{ijlm}$;
for *the entropy density*

$$S = S_0 + \frac{\lambda^{\theta\varepsilon}}{\rho}\varepsilon_{ii} + \frac{c_v}{T_0}\theta - \frac{\lambda^{r\theta}}{\rho}R_{ii} - \frac{\lambda^{\theta c}}{\rho}\mathcal{C}; \tag{47}$$

for *the potential of porosity field*

$$\Pi^r_{ij} = \lambda^{r\varepsilon}_1\delta_{ij}\varepsilon_{kk} + \lambda^{r\varepsilon}_2\varepsilon_{ij} + \lambda^{r\theta}\delta_{ij}\theta + \lambda^{rr}_1\delta_{ij}R_{kk} + \lambda^{rr}_2 R_{ij} - \lambda^{rc}\delta_{ij}\mathcal{C}, \tag{48}$$

where λ^{rr}_1 and λ^{rr}_2 are the two significant independent components of λ^{rr}_{ijlm};
for *the chemical potential of the fluid-concentration*

$$\Pi^c = -\frac{\lambda^{c\varepsilon}}{\rho}\varepsilon_{ii} + \frac{\lambda^{\theta c}}{\rho}\theta + \frac{\lambda^{rc}}{\rho}R_{ii} + \frac{\lambda^{c}}{\rho}\mathcal{C}. \tag{49}$$

Also, by virtue of (42)-(45), from (26)-(28), we deduce the expressions for *the isotropic generalized affinities*, Π^ν_{ijk}, Π^q_i and $\Pi^{j^c}_i$, where the tensors λ^{qq}_{ij}, $\lambda^{qj^c}_{ij}$ and $\lambda^{j^cj^c}_{ij}$ have the form $(42)_2$, $\lambda^{\nu q}_{ijkl}$ and $\lambda^{\nu j^c}_{ijkl}$ keep the form (89) and the tensor $\lambda^{\nu\nu}_{ijklmn}$ of order six assumes the form (97) of Appendix A, because of its particular symmetry. In particular, we have:
the generalized affinity conjugated to the flux $\mathcal{V}_{ijk}$

$$\begin{aligned}\Pi^\nu_{ijk} = {} & [\lambda^{\nu\nu}_1(\delta_{ij}\delta_{kl}\delta_{mn} + \delta_{in}\delta_{jk}\delta_{lm}) + \lambda^{\nu\nu}_2(\delta_{ij}\delta_{km}\delta_{ln} + \delta_{ik}\delta_{jn}\delta_{lm}) \\ & + \lambda^{\nu\nu}_3\delta_{ij}\delta_{kn}\delta_{lm} + \lambda^{\nu\nu}_4(\delta_{ik}\delta_{jl}\delta_{mn} + \delta_{im}\delta_{jk}\delta_{nl}) + \lambda^{\nu\nu}_5\delta_{ik}\delta_{jm}\delta_{ln} \\ & + \lambda^{\nu\nu}_6\delta_{il}\delta_{jk}\delta_{mn} + \lambda^{\nu\nu}_7\delta_{il}\delta_{jm}\delta_{kn} + \lambda^{\nu\nu}_8\delta_{il}\delta_{jn}\delta_{km} \\ & + \lambda^{\nu\nu}_9\delta_{im}\delta_{jl}\delta_{kn} + \lambda^{\nu\nu}_{10}(\delta_{im}\delta_{jn}\delta_{kl} + \delta_{in}\delta_{jl}\delta_{km}) + \lambda^{\nu\nu}_{11}\delta_{in}\delta_{jm}\delta_{kl}]\mathcal{V}_{lmn} \\ & + [\lambda^{\nu q}_1\delta_{ik}\delta_{jl} + \lambda^{\nu q}_2(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk})]q_l \\ & + [\lambda^{\nu j^c}_1\delta_{ik}\delta_{jl} + \lambda^{\nu j^c}_2(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk})]j^c_l,\end{aligned} \tag{50}$$

where $\lambda_s^{\nu\nu}$ $(s = 1, \dots 11)$ are the 11 significant independent components of $\lambda_{ijklmn}^{\nu\nu}$, $\lambda_1^{\nu q}$, $\lambda_2^{\nu q}$ the two significant independent components of $\lambda_{ijlm}^{\nu q}$ and $\lambda_1^{\nu j^c}$, $\lambda_2^{\nu j^c}$ the two significant independent components of the tensor $\lambda_{ijlm}^{\nu j^c}$.
Equation (50) gives

$$\begin{aligned}\Pi_{ijk}^{\nu} = {} & \lambda_1^{\nu\nu}\left(\delta_{ij}\mathcal{V}_{kll} + \delta_{jk}\mathcal{V}_{lli}\right) + \lambda_2^{\nu\nu}\left(\delta_{ij}\mathcal{V}_{lkl} + \delta_{ik}\mathcal{V}_{llj}\right) + \lambda_3^{\nu\nu}\delta_{ij}\mathcal{V}_{llk} \\ & + \lambda_4^{\nu\nu}\left(\delta_{ik}\mathcal{V}_{jll} + \delta_{jk}\mathcal{V}_{lil}\right) + \lambda_5^{\nu\nu}\delta_{ik}\mathcal{V}_{ljl} + \lambda_6^{\nu\nu}\delta_{jk}\mathcal{V}_{ill} + \lambda_7^{\nu\nu}\mathcal{V}_{ijk} \\ & + \lambda_8^{\nu\nu}\mathcal{V}_{ikj} + \lambda_9^{\nu\nu}\mathcal{V}_{jik} + \lambda_{10}^{\nu\nu}\left(\mathcal{V}_{kij} + \mathcal{V}_{jki}\right) + \lambda_{11}^{\nu\nu}\mathcal{V}_{kji} \\ & + \lambda_1^{\nu q}\delta_{ik}q_j + \lambda_2^{\nu q}\left(\delta_{ij}q_k + \delta_{jk}q_i\right) + \lambda_1^{\nu j^c}\delta_{ik}j_j^c + \lambda_2^{\nu j^c}\left(\delta_{ij}j_k^c + \delta_{jk}j_i^c\right);\end{aligned} \tag{51}$$

the generalized affinity conjugated to the heat flux q_i

$$\Pi_i^q = \lambda_1^{\nu q}\mathcal{V}_{kik} + \lambda_2^{\nu q}\left(\mathcal{V}_{ikk} + \mathcal{V}_{kki}\right) + \lambda^{qq}q_i + \lambda^{qj^c}j_i^c; \tag{52}$$

the generalized affinity conjugated to the fluid-concentration flux j_i^c

$$\Pi_i^{j^c} = \lambda_1^{\nu j^c}\mathcal{V}_{kik} + \lambda_2^{\nu j^c}\left(\mathcal{V}_{ikk} + \mathcal{V}_{kki}\right) + \lambda^{j^c q}q_i + \lambda^{j^c j^c}j_i^c. \tag{53}$$

The isotropic rate equations for the fluxes and the internal variable are derived from (29), (32), (34) and (36).
In particular, *for the structural permeability tensor* r_{ij}, because of the tensors β_{ijk}^s $(s = 3, 4, 6, 7)$ of order three vanish (see Subsection A.1 of Appendix A), the fourth order tensor β_{ijkl}^1 and β_{ijkl}^2 have the form (83) of the Appendix A and the fifth order tensors β_{ijklm}^5 and β_{ijklm}^8 assume the form (77) and (80), respectively, of the Appendix A, we work out

$$\begin{aligned}\dot{r}_{ij} + \mathcal{V}_{ijk,k} = {} & [\beta_1^1\delta_{ij}\delta_{kl} + \beta_2^1(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})]\varepsilon_{kl} \\ & + [\beta_1^2\delta_{ij}\delta_{kl} + \beta_2^2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})]r_{kl} \\ & + [\beta_1^5(\in_{ikl}\delta_{jm} + \in_{jkl}\delta_{im}) + \beta_2^5(\in_{ikm}\delta_{lj} + \in_{jkm}\delta_{li}) \\ & + \beta_3^5(\in_{ilm}\delta_{jk} + \in_{jlm}\delta_{ik})]\mathcal{V}_{klm} \\ & + \beta^8(\in_{ikm}\delta_{lj} + \in_{jkm}\delta_{li} + \in_{ilm}\delta_{jk} + \in_{jlm}\delta_{ik})r_{kl,m},\end{aligned} \tag{54}$$

in which β_1^1, β_2^1 and β_1^2, β_2^2, are the two significant independent components of β_{ijkl}^1 and β_{ijkl}^2, respectively, β_s^5 $(s = 1, 2, 3)$ are the three significant independent components of β_{ijklm}^5 and β^8 is the only one significant independent component of β_{ijklm}^8, due to its particular symmetry.
Equation (54) gives

$$\begin{aligned}\dot{r}_{ij} + \mathcal{V}_{ijk,k} = {} & \beta_1^1\delta_{ij}\varepsilon_{kk} + \beta_2^1\varepsilon_{ij} + \beta_1^2\delta_{ij}r_{kk} + \beta_2^2 r_{ij} + \beta_1^5(\in_{ikl}\mathcal{V}_{klj} + \in_{jkl}\mathcal{V}_{kli}) \\ & + \beta_2^5(\in_{ikl}\mathcal{V}_{kjl} + \in_{jkl}\mathcal{V}_{kil}) + \beta_3^5(\in_{ilk}\mathcal{V}_{jlk} + \in_{jlk}\mathcal{V}_{ilk}) \\ & + \beta^8(\in_{ikm}r_{kj,m} + \in_{jkm}r_{ki,m} + \in_{jlm}r_{il,m});\end{aligned} \tag{55}$$

for *the flux* $\mathcal{V}_{ijk}$ of the structural permeability tensor r_{ij}, taking into account that the fourth order tensors γ^r_{ijkl} $(r = 1, 2, 4, 5)$ and the sixth order tensor γ^6_{ijklmn} have the form (43) and (100), respectively, of the Appendix A, we have:

$$
\begin{aligned}
\dot{\mathcal{V}}_{ijk} &= (\gamma^1_1\delta_{ij}\delta_{kl} + \gamma^1_2\delta_{ik}\delta_{jl} + \gamma^1_3\delta_{il}\delta_{jk})j^c_l + (\gamma^2_1\delta_{ij}\delta_{kl} + \gamma^2_2\delta_{ik}\delta_{jl} + \gamma^2_3\delta_{il}\delta_{jk})q_l \\
&+ (\gamma^3_1\delta_{ij}\delta_{kl}\delta_{mn} + \gamma^3_2\delta_{ij}\delta_{km}\delta_{ln} + \gamma^3_3\delta_{ij}\delta_{kn}\delta_{lm} + \gamma^3_4\delta_{ik}\delta_{jl}\delta_{mn} \\
&+ \gamma^3_5\delta_{ik}\delta_{jm}\delta_{ln} + \gamma^3_6\delta_{ik}\delta_{jn}\delta_{lm} + \gamma^3_7\delta_{il}\delta_{jk}\delta_{mn} + \gamma^3_8\delta_{il}\delta_{jm}\delta_{kn} + \gamma^3_9\delta_{il}\delta_{jn}\delta_{km} \\
&+ \gamma^3_{10}\delta_{im}\delta_{jk}\delta_{ln} + \gamma^3_{11}\delta_{im}\delta_{jl}\delta_{kn} + \gamma^3_{12}\delta_{im}\delta_{jn}\delta_{kl} \\
&+ \gamma^3_{13}\delta_{in}\delta_{jk}\delta_{lm} + \gamma^3_{14}\delta_{in}\delta_{jl}\delta_{km} + \gamma^3_{15}\delta_{in}\delta_{jm}\delta_{kl})\mathcal{V}_{lmn} \\
&+ (\gamma^4_1\delta_{ij}\delta_{kl} + \gamma^4_2\delta_{ik}\delta_{jl} + \gamma^4_3\delta_{il}\delta_{jk})c_{,l} + (\gamma^5_1\delta_{ij}\delta_{kl} + \gamma^5_2\delta_{ik}\delta_{jl} + \gamma^5_3\delta_{il}\delta_{jk})T_{,l} \\
&+ [\gamma^6_1(\delta_{kl}\delta_{mn} + \delta_{km}\delta_{ln})\delta_{ij} + \gamma^6_2\delta_{ij}\delta_{kn}\delta_{lm} + \gamma^6_3(\delta_{jl}\delta_{mn} + \delta_{jm}\delta_{ln})\delta_{ik} \\
&+ \gamma^6_4\delta_{ik}\delta_{jn}\delta_{lm} + \gamma^6_5(\delta_{il}\delta_{mn} + \delta_{im}\delta_{ln})\delta_{jk} + \gamma^6_6(\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl})\delta_{kn} \qquad (56) \\
&+ \gamma^6_7(\delta_{il}\delta_{km} + \delta_{im}\delta_{kl})\delta_{jn} + \gamma^6_8\delta_{in}\delta_{jk}\delta_{lm} + \gamma^6_9(\delta_{jl}\delta_{km} + \delta_{jm}\delta_{kl})\delta_{in}]r_{lm,n},
\end{aligned}
$$

in which γ^6_s $(s = 1, \ldots 9)$ are the 9 significant independent components of γ^6_{ijklmn}.
Equation (56) can be written as follows

$$
\begin{aligned}
\dot{\mathcal{V}}_{ijk} &= \gamma^1_1\delta_{ij}j^c_k + \gamma^1_2\delta_{ik}j^c_j + \gamma^1_3\delta_{jk}j^c_i + \gamma^2_1\delta_{ij}q_k + \gamma^2_2\delta_{ik}q_j + \gamma^2_3\delta_{jk}q_i + \gamma^3_1\delta_{ij}\mathcal{V}_{kll} \\
&+ \gamma^3_2\delta_{ij}\mathcal{V}_{lkl} + \gamma^3_3\delta_{ij}\mathcal{V}_{llk} + \gamma^3_4\delta_{ik}\mathcal{V}_{jll} + \gamma^3_5\delta_{ik}\mathcal{V}_{ljl} + \gamma^3_6\delta_{ik}\mathcal{V}_{llj} + \gamma^3_7\delta_{jk}\mathcal{V}_{ill} \\
&+ \gamma^3_8\mathcal{V}_{ijk} + \gamma^3_9\mathcal{V}_{ikj} + \gamma^3_{10}\delta_{jk}\mathcal{V}_{lil} + \gamma^3_{11}\mathcal{V}_{jik} + \gamma^3_{12}\mathcal{V}_{kij} + \gamma^3_{13}\delta_{jk}\mathcal{V}_{lli} \\
&+ \gamma^3_{14}\mathcal{V}_{jki} + \gamma^3_{15}\mathcal{V}_{kji} + \gamma^4_1\delta_{ij}c_{,k} + \gamma^4_2\delta_{ik}c_{,j} + \gamma^4_3\delta_{jk}c_{,i} + \gamma^5_1\delta_{ij}T_{,k} \\
&+ \gamma^5_2\delta_{ik}T_{,j} + \gamma^5_3\delta_{jk}T_{,i} + \gamma^6_1\delta_{ij}r_{kl,l} + \gamma^6_2\delta_{ij}r_{ll,k} + \gamma^6_3\delta_{ik}r_{jl,l} \\
&+ \gamma^6_4\delta_{ik}r_{ll,j} + \gamma^6_5\delta_{jk}r_{il,l} + \gamma^6_6 r_{ij,k} + \gamma^6_7 r_{ik,j} + \gamma^6_8\delta_{jk}r_{ll,i} + \gamma^6_9 r_{jk,i}; \qquad (57)
\end{aligned}
$$

for *the heat flux* q_i, because the fourth order tensors χ^3_{ijkl} and χ^6_{ijkl} have the form (43) and (92) of the Appendix A, we obtain the following expression

$$
\tau^q\dot{q}_i = \chi^1 j^c_i - q_i + \chi^3_1\mathcal{V}_{ikk} + \chi^3_2\mathcal{V}_{kik} + \chi^3_3\mathcal{V}_{kki} + \chi^4 c_{,i} - \chi^5 T_{,i} + \chi^6_1 r_{ik,k} + \chi^6_2 r_{kk,i}, \qquad (58)
$$

with χ^6_1 and χ^6_2 the two significant independent components of χ^6_{ijkl}.

In the case where the coefficients χ^1, χ^3_s $(s = 1, 2, 3)$, χ^4, χ^6_1, and χ^6_2 are negligible, equation (58) becomes the well-known Maxwell-Cattaneo-Vernotte equation $\tau^q\dot{q}_i + q_i = -\chi^5 T_{,i}$, allowing finite speeds of thermal propagation and giving Fourier equation $q_i = -\chi^5 T_{,i}$, describing thermal disturbances with infinite velocity of propagation, when the relaxation time τ^q goes to zero;

for *the fluid-concentration flux* j_i^c, taking into consideration that the fourth order tensors ξ^3_{ijkl} and ξ^6_{ijkl} have the form (43) and (92), respectively, of the Appendix A, we obtain

$$\tau^{j^c}\dot{j}_i^c = -j_i^c + \xi^2 q_i + \xi_1^3 \mathcal{V}_{ikk} + \xi_2^3 \mathcal{V}_{kik} + \xi_3^3 \mathcal{V}_{kki} + \xi^4 c_{,i} + \xi^5 T_{,i} + \xi_1^6 r_{ik,k} + \xi_2^6 r_{kk,i}, \tag{59}$$

in which ξ_1^6 and ξ_2^6 are the two significant independent components of ξ^6_{ijkl}.

The isotropic generalized telegraph temperature equation is deduced from (38), when the second order tensors k_{ij}, γ_{ij}, η_{ij}, ν^1_{ij} and ν^2_{ij} have the form $(42)_2$ and the fourth order tensors ν^3_{ijkl} and ν^6_{ijkl} assume, respectively, the form (43) and (92) of Appendix A:

$$\begin{aligned}\tau^q\ddot{T} + \dot{T} = kT_{,ii} - \gamma(\tau^q\ddot{\varepsilon}_{ii} + \dot{\varepsilon}_{ii}) + \varphi\,(\tau^q\ddot{c} + \dot{c}) + \eta\,(\tau^q\ddot{r}_{ii} + \dot{r}_{ii}) - \nu^1 j^c_{i,i} \\ - \nu^4 c_{,ii} - \left(\nu_1^3\mathcal{V}_{ijj,i} + \nu_2^3\mathcal{V}_{jij,i} + \nu_3^3\mathcal{V}_{jji,i}\right) - \left(\nu_1^6 r_{ij,ji} + \nu_2^6 r_{jj,ii}\right),\end{aligned} \tag{60}$$

in which ν_1^6 and ν_2^6 are the two significant independent components of ν^6_{ijkl}. The evolution equations (55), (57), (58), (59) and (60) describe disturbances with finite velocity and fast phenomena having relaxation times comparable or higher than the relaxation times of the materials taken into account. Also, in these equations there are terms taking into consideration non-local effects and relating these rate equations to the inhomogeneities present in the system.

The isotropic linearized internal energy balance is worked out from (41), when the second order tensors $\lambda^{\theta\varepsilon}_{ij}$ and $\lambda^{r\theta}_{ij}$ have the form $(42)_2$:

$$\rho\dot{e} = T_0\lambda^{\theta\varepsilon}\dot{u}_{i,i} + \rho c_v \dot{T} - T_0\lambda^{r\theta}\dot{r}_{ii} - T_0\lambda^{\theta c}\dot{c}. \tag{61}$$

4.2 Closure of the governing system of equations in the isotropic case

In this Subsection, to close the system of equations describing linear isotropic porous media filled by a fluid flow, we linearize the balance equations (5), (6) and the rate equations (55), (57), (58) and (59) around the equilibrium state (18)-(21). Taking into account the constitutive relations (46) and (47), the linearized temperature equation (60) and internal energy balance equation (61), the definitions $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ and $v_i = \dot{u}_i$, indicating the deviations of the fields from the thermodynamic equilibrium state by the same symbols of the fields themselves, and considering the case where we may replace the material derivative by the partial time derivative, we obtain the following closed system of 45 equations for 45 unknowns: 1 for c, 3 for u_i, 6 for r_{ij}, 27 for $\mathcal{V}_{ijk}$, 3 for q_i, 3 for j_i^c, 1 for T and 1 for e

$$\rho\frac{\partial c}{\partial t} = -j^c_{i,i}, \tag{62}$$

$$\rho\frac{\partial^2 u_i}{\partial t^2} = (\lambda+\mu)u_{k,ki} + \mu u_{i,kk} - \lambda^{\theta\varepsilon}T_{,i} + \lambda_1^{r\varepsilon}r_{kk,i} + \lambda_2^{r\varepsilon}r_{ik,k} - \lambda^{c\varepsilon}c_{,i}, \tag{63}$$

$$\begin{aligned}\frac{\partial r_{ij}}{\partial t} &= -\mathcal{V}_{ijk,k} + \beta_1^1\delta_{ij}u_{k,k} + \frac{1}{2}\beta_2^1(u_{i,j}+u_{j,i}) + \beta_1^2\delta_{ij}r_{kk} + \beta_2^2 r_{ij} \\ &+ \beta_1^5(\in_{ikl}\mathcal{V}_{klj} + \in_{jkl}\mathcal{V}_{kli}) + \beta_2^5(\in_{ikl}\mathcal{V}_{kjl} + \in_{jkl}\mathcal{V}_{kil}) \\ &+ \beta_3^5(\in_{ilk}\mathcal{V}_{jlk} + \in_{jlk}\mathcal{V}_{ilk}) + \beta^8(\in_{ikm}r_{kj,m} + \in_{jkm}r_{ki,m} + \in_{jlm}r_{il,m}),\end{aligned} \tag{64}$$

$$\begin{aligned}\frac{\partial \mathcal{V}_{ijk}}{\partial t} &= \gamma_1^1\delta_{ij}j_k^c + \gamma_2^1\delta_{ik}j_j^c + \gamma_3^1\delta_{jk}j_i^c + \gamma_1^2\delta_{ij}q_k + \gamma_2^2\delta_{ik}q_j + \gamma_3^2\delta_{jk}q_i \\ &+ \gamma_1^3\delta_{ij}\mathcal{V}_{kll} + \gamma_2^3\delta_{ij}\mathcal{V}_{lkl} + \gamma_3^3\delta_{ij}\mathcal{V}_{llk} + \gamma_4^3\delta_{ik}\mathcal{V}_{jll} + \gamma_5^3\delta_{ik}\mathcal{V}_{ljl} + \gamma_6^3\delta_{ik}\mathcal{V}_{llj} \\ &+ \gamma_7^3\delta_{jk}\mathcal{V}_{ill} + \gamma_8^3\mathcal{V}_{ijk} + \gamma_9^3\mathcal{V}_{ikj} + \gamma_{10}^3\delta_{jk}\mathcal{V}_{lil} + \gamma_{11}^3\mathcal{V}_{jik} + \gamma_{12}^3\mathcal{V}_{kij} \\ &+ \gamma_{13}^3\delta_{jk}\mathcal{V}_{lli} + \gamma_{14}^3\mathcal{V}_{jki} + \gamma_{15}^3\mathcal{V}_{kji} + \gamma_1^4\delta_{ij}c_{,k} + \gamma_2^4\delta_{ik}c_{,j} + \gamma_3^4\delta_{jk}c_{,i} \\ &+ \gamma_1^5\delta_{ij}T_{,k} + \gamma_2^5\delta_{ik}T_{,j} + \gamma_3^5\delta_{jk}T_{,i} + \gamma_1^6\delta_{ij}r_{kl,l} + \gamma_2^6\delta_{ij}r_{ll,k} + \gamma_3^6\delta_{ik}r_{jl,l} \\ &+ \gamma_4^6\delta_{ik}r_{ll,j} + \gamma_5^6\delta_{jk}r_{il,l} + \gamma_6^6 r_{ij,k} + \gamma_7^6 r_{ik,j} + \gamma_8^6\delta_{jk}r_{ll,i} + \gamma_9^6 r_{jk,i},\end{aligned} \tag{65}$$

$$\begin{aligned}\tau^q\frac{\partial q_i}{\partial t} &= \chi^1 j_i^c - q_i + \chi_1^3\mathcal{V}_{ikk} + \chi_2^3\mathcal{V}_{kik} + \chi_3^3\mathcal{V}_{kki} + \chi^4 c_{,i} \\ &- \chi^5 T_{,i} + \chi_1^6 r_{ik,k} + \chi_2^6 r_{kk,i},\end{aligned} \tag{66}$$

$$\begin{aligned}\tau^{j^c}\frac{\partial j_i^c}{\partial t} &= -j_i^c + \xi^2 q_i + \xi_1^3\mathcal{V}_{ikk} + \xi_2^3\mathcal{V}_{kik} + \xi_3^3\mathcal{V}_{kki} + \xi^4 c_{,i} \\ &+ \xi^5 T_{,i} + \xi_1^6 r_{ik,k} + \xi_2^6 r_{kk,i},\end{aligned} \tag{67}$$

$$\begin{aligned}\tau^q\frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} &= kT_{,ii} - \gamma\left(\tau^q\frac{\partial^2 u_{i,i}}{\partial t^2} + \frac{\partial u_{i,i}}{\partial t}\right) + \varphi\left(\tau^q\frac{\partial^2 c}{\partial t^2} + \frac{\partial c}{\partial t}\right) \\ &+ \eta\left(\tau^q\frac{\partial^2 r_{ii}}{\partial t^2} + \frac{\partial r_{ii}}{\partial t}\right) - \nu^1 j^c_{i,i} - \nu^4 c_{,ii} \\ &- \left(\nu_1^3\mathcal{V}_{ijj,i} + \nu_2^3\mathcal{V}_{jij,i} + \nu_3^3\mathcal{V}_{jji,i}\right) - \left(\nu_1^6 r_{ij,ji} + \nu_2^6 r_{jj,ii}\right),\end{aligned} \tag{68}$$

$$\rho\frac{\partial e}{\partial t} = T_0\lambda^{\theta\varepsilon}\frac{\partial u_{i,i}}{\partial t} + \rho c_v\frac{\partial T}{\partial t} - T_0\lambda^{r\theta}\frac{\partial r_{ii}}{\partial t} - T_0\lambda^{\theta c}\frac{\partial c}{\partial t}. \tag{69}$$

Notice that also in the case where we do not take into consideration equation (69) for the internal energy, the system of equations (62)-(68) is still closed.

5 Isotropic porous media with respect to all rotations and inversions of axes frame (perfect isotropic case)

In this Section we consider perfect isotropic porous media having *symmetry properties invariant with respect to all rotations and to inversions of the frame of axes.*
In this case *the tensors of odd order vanish* [19], i.e.

$$L_i = 0, \quad L_{ijk} = 0, \quad L_{ijklm} = 0, \tag{70}$$

the tensors of even order are given by $(42)_2$, (43) and (45) and *take equal forms to those valid in the isotropic case*, coming from special symmetry properties (see Subsections A.3 and A.4 of the Appendix A).

5.1 Constitutive relations, generalized affinities, rate, temperature and energy equations in perfect isotropic case

Notice that all the tensors that appear in equations (22)-(28) and (32), (34), (36) and (38) are of even order, so that the constitutive relations, the generalized affinities, the rate equations for the porosity field flux, the heat flux and the fluid-concentration flux, the temperature and energy equations remain unchanged, with respect the isotropic case, and assume the form (46)-(49), (51)-(53), (57)-(59) and (60), (61). Taking into account relations (70), the only different equation in this case is the rate equation (29) for the internal variable r_{ij}, that takes the form

$$\dot{r}_{ij} = -\mathcal{V}_{ijk,k} + \beta_1^1 \delta_{ij} \varepsilon_{kk} + \beta_2^1 \varepsilon_{ij} + \beta_1^2 \delta_{ij} r_{kk} + \beta_2^2 r_{ij}. \tag{71}$$

5.2 Closure of the governing system of equations in the perfect isotropic case

Linearizing the balance equations (5), (6) and the rate equations (71) and (57)-(59) around the equilibrium state (18)-(21), taking into account the constitutive relations (46) and (47), the linearized temperature and energy equations (60) and (61), equations (62), (63), (65)-(69) remain unchanged and relation (71) takes the form

$$\frac{\partial r_{ij}}{\partial t} = -\mathcal{V}_{ijk,k} + \beta_1^1 \delta_{ij} \varepsilon_{kk} + \beta_2^1 \varepsilon_{ij} + \beta_1^2 \delta_{ij} r_{kk} + \beta_2^2 r_{ij}, \tag{72}$$

where we have considered the case in which the material derivative may be replaced by the partial time derivative and the deviations of the fields from the thermodynamic equilibrium state have been indicated by the fields themselves. Thus, in total we have a closed set of 45 equations for the 45 unknowns c, u_i, r_{ij}, $\mathcal{V}_{ijk}$, q_i, j_i^c, T and e. The obtained results can be applied to real situations. The derived system of equations is very complex but in simpler cases it is possible to find analytical or numerical solutions. In particular, in [22] we have studied coupled porosity and fluid-concentration waves, calculating the dispersion relation and the propagation modes of these complex waves.

Conclusions

In this paper we have obtained a description of isotropic and perfect isotropic porous media filled by a fluid flow, in the framework of rational extended irreversible thermodynamics with internal variables, where the structural permeability tensor r_{ij} (with its gradient $r_{ij,k}$) and its flux $\mathcal{V}_{ijk}$ are introduced as internal variables, in the thermodynamic state vector. Here, the results obtained in previous papers for anisotropic porous media, are specialized when the considered media have symmetry properties invariant under orthogonal transformations of the axes frame. It was assumed that the mass density is constant, the body force and heat source are negligible and the constitutive equations, the generalized affinities, the rate equations for dissipative fluxes, presenting a relaxation time, and the closure of system of equations describing the behaviour of the considered media were worked out in the isotropic and perfect isotropic cases. It was seen that porous channels influence mechanical, thermal and transport properties of these media. In particular, when the density of porous defects is higher than its characteristic value the thermal conductivity decreases. The generalized Maxwell-Vernotte-Cattaneo, Fick-Nonnenmacher and telegraph temperature equations were obtained as particular cases. The study of fluid-saturated porous media has a great interest in applied sciences, like geology, hydrology, pharmaceutics and nanotechnology, where there are situations of propagation of high-frequency waves.

A Particular cases of isotropic and perfect isotropic tensors with special properties

In the following Subsections we will consider isotropic tensors of odd order (third and fifth), and isotropic and perfect isotropic tensors of even order (fourth and sixth), having special symmetry properties. We emphasize that the perfect isotropic tensors of odd order (first, third and fifth) are null (see (70)). Also the isotropic tensors of first order are null (see $(42)_1$). The results related to the tensors of odd order are valid only in the isotropic case, when these tensors are invariant in form with respect to all rotations of axes frame (see Section 4), while the results related to the tensors of even order are valid both in the isotropic case and in the perfect isotropic case, when these tensors are invariant in form with respect to all rotations and inversions of axes frame, (see Sections 4 and 5).

A.1 Special form for isotropic tensors of order three

In the case where a third order isotropic tensor L_{ijk} has the symmetry

$$L_{ijk} = L_{jik}, \tag{73}$$

(valid for the third order tensors β^s_{ijk} ($s = 3, 4, 6, 7$) in the rate equation (29)), we have $L_{ijk} = 0$.
In fact, from relation $(42)_3$ we can write

$$L_{jik} = L \in_{jik} = -L \in_{ijk}, \tag{74}$$

and equating this last relation with $(42)_3$ we immediately deduce $L = 0$.

A.2 Special form for isotropic tensors of order five

In the following we study the form of isotropic tensors of order five having special symmetries.

A.2.1 Case where a fifth order isotropic tensor L_{ijklm} has one particular symmetry

In the case when

$$L_{ijklm} = L_{jiklm}, \tag{75}$$

(valid for the tensor β^5_{ijklm} in equation (29)) we show that *the number of the significant independent components of this tensor reduces from* 6 *to* 3.
In fact, from (44) we have

$$\begin{aligned} L_{jiklm} = &- L_1 \in_{ijk} \delta_{lm} - L_2 \in_{ijl} \delta_{km} - L_3 \in_{ijm} \delta_{kl} + L_4 \in_{jkl} \delta_{im} \\ &+ L_5 \in_{jkm} \delta_{li} + L_6 \in_{jlm} \delta_{ik}. \end{aligned} \tag{76}$$

Equating (44) and (76) we obtain

$$L_{ijklm} = A_1(\in_{ikl}\delta_{jm} + \in_{jkl}\delta_{im}) + A_2(\in_{ikm}\delta_{lj} + \in_{jkm}\delta_{li}) + A_3(\in_{ilm}\delta_{jk} + \in_{jlm}\delta_{ik}), \quad (77)$$

where $A_1 = L_4$, $A_2 = L_5$ and $A_3 = L_6$.

A.2.2 Case where a fifth order isotropic tensor L_{ijklm} presents two symmetries

In the case when

$$L_{ijklm} = L_{jiklm}, \quad L_{ijklm} = L_{ijlkm}, \quad (78)$$

(valid for the tensor β^8_{ijklm} in equation (29)) we show that *the significant independent component of this tensor is only one.*
In fact, from (77) we have

$$L_{ijlkm} = -A_1(\in_{ikl}\delta_{jm} + \in_{jkl}\delta_{im}) + A_2(\in_{ilm}\delta_{kj} + \in_{jlm}\delta_{ki}) + A_3(\in_{ikm}\delta_{jl} + \in_{jkm}\delta_{il}). \quad (79)$$

Equating (77) and (79) we finally work out

$$L_{ijlkm} = L(\in_{ikm}\delta_{lj} + \in_{jkm}\delta_{li} + \in_{ilm}\delta_{jk} + \in_{jlm}\delta_{ik}), \quad (80)$$

where $L \equiv A_2 = A_3$.

A.3 Special form for fourth order isotropic and perfect isotropic tensors

In this Subsection we will treat special symmetry properties of a fourth order tensor L_{ijkl} and we will demonstrate that L_{ijkl} can be expressed only by *two significant independent components* that will be called A_1 and A_2.

A.3.1 Case where a fourth order isotropic tensor L_{ijkl} has one particular type of symmetry

In the case when

$$L_{ijkl} = L_{jikl}, \quad (81)$$

(valid for tensors β^1_{ijkl} and β^2_{ijkl} in equation (29)), from relation (43) we have

$$L_{jikl} = L_1\delta_{ji}\delta_{kl} + L_2\delta_{jk}\delta_{il} + L_3\delta_{jl}\delta_{ik}. \quad (82)$$

Adding equations (43) and (82), using $L_{ijkl} = L_{jikl}$ and multiplying by 1/2, we obtain

$$L_{ijkl} = A_1\delta_{ij}\delta_{kl} + A_2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{83}$$

where $A_1 = L_1$ and $A_2 = (L_2 + L_3)/2$.

A.3.2 Case where a fourth order isotropic tensor L_{ijkl} has three symmetries

In the case when

$$L_{ijkl} = L_{jikl}, \quad L_{ijkl} = L_{ijlk}, \quad L_{ijkl} = L_{klij}, \tag{84}$$

equivalent to the following chain of equalities

$$L_{ijlm} = L_{jilm} = L_{ijml} = L_{jiml} = L_{lmij} = L_{mlij} = L_{mlji} = L_{lmji}, \tag{85}$$

(valid for the tensors c_{ijkl}, $\lambda^{r\varepsilon}_{ijkl}$ and λ^{rr}_{ijkl} present in equations (22) and (24)), from (83) (that includes the symmetry $(84)_1$) we can see that also the symmetry $(84)_2$ is true, as well as $(84)_3$, because

$$L_{klij} = A_1\delta_{kl}\delta_{ij} + A_2(\delta_{ki}\delta_{lj} + \delta_{kj}\delta_{il}) = L_{ijkl}. \tag{86}$$

The other symmetries in (85) are also satisfied. Thus, we use for the tensors c_{ijkl}, $\lambda^{r\varepsilon}_{ijkl}$ and λ^{rr}_{ijkl} expression (83) again.

A.3.3 Case where a fourth order isotropic tensor L_{ijkl} has one particular symmetry of another type

In the case when

$$L_{ijkl} = L_{lijk}, \tag{87}$$

(valid for the coefficients $\lambda^{\nu q}_{ijkl}$ and $\lambda^{\nu j^c}_{ijkl}$ in equations (26)-(28)) from relation (43) we deduce

$$L_{lijk} = L_1\delta_{li}\delta_{jk} + L_2\delta_{lj}\delta_{ik} + L_3\delta_{lk}\delta_{ij}. \tag{88}$$

Using the same procedure seen in Subsection A.3.2, we obtain

$$L_{ijkl} = A_1\delta_{ik}\delta_{jl} + A_2(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}), \tag{89}$$

where $A_1 = L_2$ and $A_2 = (L_1 + L_3)/2$.

It is useful to emphasize that the same result (89) is obtained if the symmetries $L_{ijkl} = L_{ilkj}$ and/or $L_{ijkl} = L_{kjil}$ are valid. These results are not used in this paper.

A.3.4 Case where a fourth order isotropic tensor L_{ijkl} has the symmetry $L_{ijkl} = L_{ikjl}$

In the case when

$$L_{ijkl} = L_{ikjl}, \tag{90}$$

(valid for the tensors χ^6_{ijkl} in equation (34) and ξ^6_{ijkl} in equation (36)), from relation (43) we have

$$L_{ikjl} = L_1\delta_{ik}\delta_{jl} + L_2\delta_{ij}\delta_{kl} + L_3\delta_{il}\delta_{kj}. \tag{91}$$

Using the same procedure seen in Subsection A.3.2, we obtain

$$L_{ijkl} = A_1\delta_{il}\delta_{jk} + A_2(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}), \tag{92}$$

with $A_1 = L_3$ and $A_2 = (L_1 + L_2)/2$.

A.3.5 Case where a fourth order isotropic tensor L_{ijkl} has two symmetries

In the case when

$$L_{ijkl} = L_{ikjl}, \quad L_{ijkl} = L_{ljki}, \tag{93}$$

equivalent to the following chain of equalities

$$L_{ijkl} = L_{ikjl} = L_{ljki} = L_{lkji}, \tag{94}$$

(valid for the tensor ν^6_{ijkl} in the temperature equation (38)), from (92) (that includes the symmetry $(93)_1$) we can see that also the symmetry $(93)_2$ is satisfied, so that we use for the tensor ν^6_{ijkl} expression (92) again.

A.4 Special form for isotropic and perfect isotropic tensors of order six

In this Subsection we will treat special symmetry properties of a sixth order tensor L_{ijklmn} and we will demonstrate that the number of its *significant independent components* is reduced.

A.4.1 Case where a sixth order isotropic tensor L_{ijklmn} has one particular symmetry

In the case when

$$L_{ijklmn} = L_{lmnijk}, \tag{95}$$

(valid for the tensor $\lambda^{\nu\nu}_{ijklmn}$ in equation (26)) we show that the number of *the significant independent components of this tensor reduce from* 15 *to* 11. In fact, writing relation (45) in the case of L_{lmnijk} (i.e. by exchanging indexes $\{i,j,k\}$ with indexes $\{l,m,n\}$), we obtain

$$\begin{aligned} L_{lmnijk} =& L_1\delta_{lm}\delta_{ni}\delta_{jk} + L_2\delta_{lm}\delta_{nj}\delta_{ik} + L_3\delta_{lm}\delta_{nk}\delta_{ij} + L_4\delta_{ln}\delta_{mi}\delta_{jk} \\ &+ L_5\delta_{ln}\delta_{mj}\delta_{ik} + L_6\delta_{ln}\delta_{mk}\delta_{ij} + L_7\delta_{li}\delta_{mn}\delta_{jk} + L_8\delta_{li}\delta_{mj}\delta_{nk} \\ &+ L_9\delta_{li}\delta_{mk}\delta_{nj} + L_{10}\delta_{lj}\delta_{mn}\delta_{ik} + L_{11}\delta_{lj}\delta_{mi}\delta_{nk} + L_{12}\delta_{lj}\delta_{mk}\delta_{ni} \\ &+ L_{13}\delta_{lk}\delta_{mn}\delta_{ij} + L_{14}\delta_{lk}\delta_{mi}\delta_{nj} + L_{15}\delta_{lk}\delta_{mj}\delta_{ni}. \end{aligned} \tag{96}$$

Adding relation (96) to (45) and multiplying by 1/2, we work out

$$\begin{aligned} L_{ijklmn} =& A_1(\delta_{ij}\delta_{kl}\delta_{mn} + \delta_{in}\delta_{jk}\delta_{lm}) + A_2(\delta_{ij}\delta_{km}\delta_{ln} + \delta_{ik}\delta_{jn}\delta_{lm}) \\ &+ A_3\delta_{ij}\delta_{kn}\delta_{lm} + A_4(\delta_{ik}\delta_{jl}\delta_{mn} + \delta_{im}\delta_{jk}\delta_{nl}) + A_5\delta_{ik}\delta_{jm}\delta_{ln} \\ &+ A_6\delta_{il}\delta_{jk}\delta_{mn} + A_7\delta_{il}\delta_{jm}\delta_{kn} + A_8\delta_{il}\delta_{jn}\delta_{km} + A_9\delta_{im}\delta_{jl}\delta_{kn} \\ &+ A_{10}(\delta_{im}\delta_{jn}\delta_{kl} + \delta_{in}\delta_{jl}\delta_{km}) + A_{11}\delta_{in}\delta_{jm}\delta_{kl}, \end{aligned} \tag{97}$$

with $A_1 = (L_1 + L_{13})/2$, $A_2 = (L_2 + L_6)/2$, $A_3 = L_3$, $A_4 = (L_4 + L_{10})/2$, $A_5 = L_5$, $A_6 = L_7$, $A_7 = L_8$, $A_8 = L_9$, $A_9 = L_{11}$, $A_{10} = (L_{12} + L_{14})/2$, $A_{11} = L_{15}$.

A.4.2 Case where a sixth order isotropic tensor L_{ijklmn} has one particular symmetry of another type

In the case when

$$L_{ijklmn} = L_{ijkmln}, \tag{98}$$

(valid for the tensor γ^6_{ijklmn} in equation (32)) we show that the number of *the significant independent components of this tensor reduce from* 15 *to* 9. In fact, writing relation (45) in the case of L_{ijkmln} (i.e. by exchanging index l with index m), we have

$$\begin{aligned} L_{ijkmln} =& L_1\delta_{ij}\delta_{km}\delta_{ln} + L_2\delta_{ij}\delta_{kl}\delta_{mn} + L_3\delta_{ij}\delta_{kn}\delta_{ml} + L_4\delta_{ik}\delta_{jm}\delta_{ln} \\ &+ L_5\delta_{ik}\delta_{jl}\delta_{mn} + L_6\delta_{ik}\delta_{jn}\delta_{ml} + L_7\delta_{im}\delta_{jk}\delta_{ln} + L_8\delta_{im}\delta_{jl}\delta_{kn} \\ &+ L_9\delta_{im}\delta_{jn}\delta_{kl} + L_{10}\delta_{il}\delta_{jk}\delta_{mn} + L_{11}\delta_{il}\delta_{jm}\delta_{kn} + L_{12}\delta_{il}\delta_{jn}\delta_{km} \\ &+ L_{13}\delta_{in}\delta_{jk}\delta_{ml} + L_{14}\delta_{in}\delta_{jm}\delta_{kl} + L_{15}\delta_{in}\delta_{jl}\delta_{km}. \end{aligned} \tag{99}$$

Adding this relation to (45), using (98) and multiplying by 1/2, we have

$$\begin{aligned} L_{ijklmn} =& A_1(\delta_{kl}\delta_{mn} + \delta_{km}\delta_{ln})\delta_{ij} + A_2\delta_{ij}\delta_{kn}\delta_{lm} + A_3(\delta_{jl}\delta_{mn} + \delta_{jm}\delta_{ln})\delta_{ik} \\ &+ A_4\delta_{ik}\delta_{jn}\delta_{lm} + A_5(\delta_{il}\delta_{mn} + \delta_{im}\delta_{ln})\delta_{jk} + A_6(\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl})\delta_{kn} \\ &+ A_7(\delta_{il}\delta_{km} + \delta_{im}\delta_{kl})\delta_{jn} + A_8\delta_{in}\delta_{jk}\delta_{lm} + A_9(\delta_{jl}\delta_{km} + \delta_{jm}\delta_{kl})\delta_{in}, \end{aligned} \tag{100}$$

where $A_1 = (L_1 + L_2)/2,\ A_2 = L_3,\ A_3 = (L_4 + L_5)/2,\ A_4 = L_6,\ A_5 = (L_7 + L_{10})/2,\ A_6 = (L_8 + L_{11})/2,\ A_7 = (L_9 + L_{12})/2,\ A_8 = L_{13},\ A_9 = (L_{14} + L_{15})/2$.

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