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ON THE NUMBER OF PARTITIONS INTO PARTS WITH THE MINIMAL PART k AND THE MINIMAL DIFFERENCE d^*

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Dedicated to Dr. Vasile Drăgan on the occasion of his 70$^{\text{th}}$ anniversary

Abstract

In this paper, the author considered two specializations of the identity q-Chu Vandermonde and derived two recurrence relations for the number of partitions of n into m parts with the smallest part greater than or equal to k and the minimal difference d.

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1 Introduction

For $|q| < 1$, the Rogers-Ramanujan functions are defined by

$$G(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} \tag{1}$$

and

$$H(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n}, \tag{2}$$

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where

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$$

is the q-shifted factorial with $(a;q)_0 = 1$.

Because the infinite product

$$(a;q)_\infty = \lim_{n\to\infty} (a;q)_n$$

diverges when $a \neq 0$ and $|q| \geqslant 1$, whenever $(a;q)_\infty$ appears in a formula, we shall assume that $|q| < 1$. For $|q| < 1$, the functions $G(q)$ and $H(q)$ satisfy the famous Rogers-Ramanujan identities [6, 7]:

1. $G(q) = \dfrac{1}{(q,q^4;q^5)_\infty}$,
2. $H(q) = \dfrac{1}{(q^2,q^3;q^5)_\infty}$,

where

$$(a_1, a_2 \ldots, a_n; q)_\infty = (a_1;q)_\infty (a_2;q)_\infty \cdots (a_n;q)_\infty.$$

The Rogers-Ramanujan identities are two of the most remarkable and important results in the theory of q-series, having a remarkable applicability in areas as enumerative combinatorics, number theory, representation theory, group theory, statistical physics, probability and complex analysis [1]. They were first discovered in 1894 by Rogers [7] and then rediscovered by Ramanujan in 1913. It is a well-known fact that there is a list of forty identities involving $G(q)$ and $H(q)$ that Ramanujan compiled. More details about these identities can be found in the classical texts by Andrews and Berndt [3].

Due to MacMahon [5], we have the following combinatorial version of the Rogers-Ramanujan identities:

1. The number of partitions of n into parts congruent to $\{1,4\} \bmod 5$ equals the number of partitions of n into parts with the minimal difference 2.

2. The number of partitions of n into parts congruent to $\{2,3\} \bmod 5$ equals the number of partitions of n with minimal part 2 and minimal difference 2.

In this paper, we consider $Q_m^{(d,k)}(n)$ the number of partitions of n into m parts where each part differs from the next by at least d and the smallest part is greater than or equal to k. According to [2, Theorem 11.4.2], we have

$$\sum_{n=0}^{\infty} Q_m^{(d,k)}(n)q^n = \frac{q^{km+d\binom{m}{2}}}{(q;q)_m}.$$

In general, k is considered a positive integer. Assuming that k is a nonnegative integer, we remark few special cases of $Q_m^{(d,k)}(n)$:

1. When k is a positive integer, $Q_m^{(1,k)}(n)$ denotes the number of partitions of n into distinct m parts, each part greater than or equal to k.

2. When k is a positive integer, $Q_m^{(0,k)}(n)$ denotes the number of partitions of n into m parts, each part greater than or equal to k.

3. $Q_m^{(1,0)}(n)$ denotes the number of partitions of n into distinct m parts or distinct $m-1$ parts, i.e.,

$$Q_m^{(1,0)}(n) = Q_m^{(1,1)}(n) + Q_{m-1}^{(1,1)}(n).$$

4. $Q_m^{(0,0)}(n)$ denotes the number of partitions of n into at most m parts, i.e.,

$$Q_m^{(0,0)}(n) = Q_0^{(0,1)}(n) + Q_1^{(0,1)}(n) + Q_2^{(0,1)}(n) + \cdots + Q_m^{(0,1)}(n).$$

   Instead of $Q_m^{(0,0)}(n)$, we will use the notation $p_m(n)$.

It is clear that the famous Rogers-Ramanujan identities can be rewritten in terms of $Q_m^{(d,k)}(n)$ as follows:

1. $\displaystyle\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} Q_m^{(2,1)}(n)q^n = \frac{1}{(q,q^4;q^5)_\infty}$,

2. $\displaystyle\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} Q_m^{(2,2)}(n)q^n = \frac{1}{(q^2,q^3;q^5)_\infty}$.

This approach allows us to derive combinatorial interpretations of the Rogers-Ramanujan identities in terms of $p_m(n)$:

1. The number of partitions of n into parts congruent to $\{1,4\} \bmod 5$ equals

$$\sum_{m=0}^{\infty} p_m(n-m^2).$$

2. The number of partitions of n into parts congruent to $\{2,3\}$ mod 5 equals
$$\sum_{m=0}^{\infty} p_m(n-m-m^2).$$

In this paper, motivated by these results, we shall provide some recurrence relations for $Q_m^{(d,k)}(n)$.

Theorem 1.1. *For $k>0$ and $d,m,n \geqslant 0$,*
$$Q_m^{(d,k)}(n) = \sum_{j=0}^{m} \sum_{r=0}^{n-(d-1)\binom{m}{2}} (-1)^j Q_{m-j}^{(d,k)}\left(r+k(m-j)+(d-1)\binom{m-j}{2}\right) \times P\left(k-1,j,n-r-(d-1)\binom{m}{2}\right),$$
where $P(k,m,n)$ denotes the number of partitions of n into at most m parts, each part less than or equal to k.

Theorem 1.2. *For $d,k,m,n \geqslant 0$,*
$$Q_m^{(d,k)}(n) = \sum_{j=0}^{\min(k,m)} \sum_{r=0}^{n+j-d\binom{m}{2}} (-1)^j Q_{m-j}^{(d,k)}\left(r+k(m-j)+d\binom{m-j}{2}\right) \times Q\left(k,j,n+j-r-d\binom{m}{2}\right),$$
where $Q(k,m,n)$ denotes the number of partitions of n into exactly m distinct parts, each part less than or equal to k.

Some special cases of Theorem 1.1 can be easily derived considering that
$$P(0,m,n)=\delta_{0,n},$$
where $\delta_{i,j}$ is the usual Kronecker delta function.

Corollary 1.3. *For $m,n \geqslant 0$,*

1. $Q_m^{(0,1)}(n) = \sum_{j=0}^{m} (-1)^{m-j} Q_j^{(0,1)}\left(n+j+\binom{m}{2}-\binom{j}{2}\right)$;

2. $Q_m^{(1,1)}(n) = \sum_{j=0}^{m} (-1)^{m-j} Q_j^{(1,1)}(n+j)$;

3. $Q_m^{(2,1)}(n) = \sum_{j=0}^{m} (-1)^{m-j} Q_j^{(2,1)} \left(n + j - \binom{m}{2} + \binom{j}{2} \right).$

On the other hand, taking into account that

$$P(1,m,n) = \begin{cases} 1, & \text{for } n \leqslant m, \\ 0, & \text{for } n > m, \end{cases}$$

by Theorem 1.1, we obtain the following relations.

Corollary 1.4. *For $m, n \geqslant 0$,*

1. $Q_m^{(0,2)}(n) = \sum_{j=0}^{m} \sum_{r=0}^{m-j} (-1)^{m-j} Q_j^{(0,2)} \left(n + 2j - r + \binom{m}{2} - \binom{j}{2} \right);$

2. $Q_m^{(1,2)}(n) = \sum_{j=0}^{m} \sum_{r=0}^{m-j} (-1)^{m-j} Q_j^{(1,2)} (n + 2j - r);$

3. $Q_m^{(2,2)}(n) = \sum_{j=0}^{m} \sum_{r=0}^{m-j} (-1)^{m-j} Q_j^{(2,2)} \left(n + 2j - r - \binom{m}{2} + \binom{j}{2} \right).$

The following recurrence relation can be obtained from Theorem 1.2, replacing k by 1 and considering that

$$Q(1,0,n) = \delta_{0,n} \qquad \text{and} \qquad Q(1,1,n) = \delta_{1,n}.$$

Corollary 1.5. *For $d, m, n \geqslant 0$,*

$$Q_{m+1}^{(d,1)}(n+1) = Q_{m+1}^{(d,1)}(n-m) + Q_m^{(d,1)}(n-dm).$$

Moreover, taking into account that

$$Q(2,0,n) = \delta_{0,n}, \qquad Q(2,1,n) = \delta_{1,n} + \delta_{2,n}, \qquad \text{and} \qquad Q(2,2,n) = \delta_{3,n},$$

the case $k = 2$ of Theorem 1.2 reads as follows.

Corollary 1.6. *For $m > 0$, $d, n \geqslant 0$,*

$$\begin{aligned} Q_m^{(d,2)}(n) = Q_m^{(d,2)}&(n-2m) + Q_{m-1}^{(d,2)}(n-dm+d-2) \\ &+ Q_{m-1}^{(d,2)}(n-dm+d-3) - Q_{m-2}^{(d,2)}(n-2dm+3d-5). \end{aligned}$$

2 Proof of Theorem 1.1

For any positive integers n, m and k, Andrews [1] examined the partitions of n into at most m parts, each part less than or equal to k and remarked few results for the partition function $P(k,m,n)$ which denotes the number of these restricted partitions (see for example [1, Eq. (3.2.6), Theorems 3.1 and 3.10]). The generating function of $P(k,m,n)$ is given by

$$\sum_{n=0}^{km} P(k,m,n)q^n = \begin{bmatrix} k+m \\ k \end{bmatrix}.$$

For $k>0$ and $n \geqslant 0$, we have the following specialization of the identity q-Chu Vandermonde I [4]:

$$\frac{q^{nk+\binom{n}{2}}}{(q;q)_n} = \sum_{j=0}^{n} (-1)^j \frac{q^{\binom{n-j}{2}}}{(q;q)_{n-j}} \begin{bmatrix} k-1+j \\ j \end{bmatrix}.$$

Considering this identity and the generating functions for $Q_m^{(d,k)}(n)$ and $P(k,m,n)$, we can write:

$$\begin{aligned} &\sum_{n=0}^{\infty} Q_m^{(1,k)}(n)q^n \\ &\qquad = \sum_{j=0}^{m} (-1)^j \left(\sum_{n=0}^{\infty} Q_{m-j}^{(1,0)}(n)q^n \right) \left(\sum_{n=0}^{\infty} P(k-1,j,n)q^n \right) \\ &\qquad = \sum_{n=0}^{\infty} \sum_{j=0}^{m} \sum_{r=0}^{n} (-1)^j Q_{m-j}^{(1,0)}(r) P(k-1,j,n-r) q^n. \end{aligned}$$

Extracting the coefficients of q^n in the last identity, we obtain

$$Q_m^{(1,k)}(n) = \sum_{j=0}^{m} \sum_{r=0}^{n} (-1)^j Q_{m-j}^{(1,0)}(r) P(k-1,j,n-r). \tag{3}$$

On the other hand, we have the relation

$$\begin{aligned} \frac{q^{km+d\binom{m}{2}}}{(q;q)_m} &= \sum_{n=km+(d-1)\binom{m}{2}}^{\infty} Q_m^{(d,k)}(n)q^n \\ &= \sum_{n=0}^{\infty} Q_m^{(d,k)} \left(n+km+(d-1)\binom{m}{2} \right) q^{n+km+(d-1)\binom{m}{2}}, \end{aligned}$$

that can be written as

$$\frac{q^{\binom{m}{2}}}{(q;q)_m} = \sum_{n=0}^{\infty} Q_m^{(d,k)}\left(n + km + (d-1)\binom{m}{2}\right) q^n.$$

Taking into account that

$$\frac{q^{\binom{m}{2}}}{(q;q)_m} = \sum_{n=0}^{\infty} Q_m^{(1,0)}(n)\, q^n,$$

we deduce

$$Q_m^{(d,k)}\left(n + km + (d-1)\binom{m}{2}\right) = Q_m^{(1,0)}(n). \tag{4}$$

In a similar way, we obtain

$$Q_m^{(d,k)}\left(n + (d-1)\binom{m}{2}\right) = Q_m^{(1,k)}(n). \tag{5}$$

The proof follows easily from (3)-(5).

3 Proof of Theorem 1.2

The proof of this theorem is quite similar to the proof of Theorem 1.1. Following the notation in Andrews's book [1], we denote by $Q(k,m,n)$ the number of ways in which the integer n can be expressed as a sum of exactly m distinct positive integers less than or equal to n, without regard to order. By [1, Theorem 3.3], we have

$$\sum_{n=0}^{\infty} Q(k,m,n) q^n = q^{\binom{m+1}{2}} \begin{bmatrix} k \\ m \end{bmatrix}.$$

For $n, k \geqslant 0$, we have the following specialization of the identity q-Chu Vandermonde I [4]:

$$\frac{q^{nk}}{(q;q)_n} = \sum_{j=0}^{\min(n,k)} (-1)^j \frac{q^{\binom{j}{2}}}{(q;q)_{n-j}} \begin{bmatrix} k \\ j \end{bmatrix}.$$

Considering this identity and the generating functions for $Q_m^{(d,k)}(n)$ and $Q(k,m,n)$, we can write

$$\begin{aligned}
&\sum_{n=0}^{\infty} Q_m^{(0,k)}(n) q^n \\
&\quad = \sum_{j=0}^{\min(k,m)} \frac{(-1)^j}{q^j} \left(\sum_{n=0}^{\infty} Q_{m-j}^{(0,0)}(n) q^n \right) \left(\sum_{n=0}^{\infty} Q(k,j,n) q^n \right) \\
&\quad = \sum_{n=0}^{\infty} \sum_{j=0}^{\min(k,m)} \sum_{r=0}^{n} (-1)^j Q_{m-j}^{(0,0)}(r) Q(k,j,n-r) q^{n-j} \\
&\quad = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{\min(k,m)} \sum_{r=0}^{n+j} (-1)^j Q_{m-j}^{(0,0)}(r) Q(k,j,n+j-r) \right) q^n.
\end{aligned}$$

Extracting the coefficients of q^n in the last identity, we obtain the identity

$$Q_m^{(0,k)}(n) = \sum_{j=0}^{\min(k,m)} \sum_{r=0}^{n+j} (-1)^j Q_{m-j}^{(0,0)}(r) Q(k,j,n+j-r). \tag{6}$$

Since

$$\begin{aligned}
\frac{q^{km+d\binom{m}{2}}}{(q;q)_m} &= \sum_{n=d\binom{m}{2}}^{\infty} Q_m^{(d,k)}(n) q^n \\
&= \sum_{n=0}^{\infty} Q_m^{(d,k)} \left(n + d\binom{m}{2} \right) q^{n+d\binom{m}{2}},
\end{aligned}$$

we deduce that

$$\frac{q^{km}}{(q;q)_m} = \sum_{n=0}^{\infty} Q_m^{(d,k)} \left(n + d\binom{m}{2} \right) q^n.$$

On the other hand, we have

$$\frac{q^{km}}{(q;q)_m} = \sum_{n=0}^{\infty} Q_m^{(0,k)}(n) q^n.$$

Now it is clear that

$$Q_m^{(d,k)} \left(n + d\binom{m}{2} \right) = Q_m^{(0,k)}(n). \tag{7}$$

The identity

$$Q_m^{(d,k)}\left(n+km+d\binom{m}{2}\right)=Q_m^{(0,0)}(n) \tag{8}$$

follows in a similar way. By (6)-(8), we arrive at our conclusion.

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