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COEFFICIENT ESTIMATES FOR A SUBCLASS OF BI-UNIVALENT FUNCTIONS ASSOCIATED WITH THE AGHALARY-EBADIAN-WANG OPERATOR*

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Abstract

In this paper, we introduce and investigate a new subclass of analytic and bi-univalent functions defined in the open unit disk $\mathbb{U}$, which is associated with the Aghalary-Ebadian-Wang operator. Furthermore, we find estimates on the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ of functions in the new class and improve some recent works.

MSC: 30C45; 30C50.

keywords: Univalent functions, Bi-univalent functions, Coefficient estimates, Aghalary-Ebadian-Wang operator.

1 Introduction

Let $\mathcal{A}$ be a class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

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which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Also we let $\mathcal{S}$ to denote the class of functions $f \in \mathcal{A}$ which are univalent in $\mathbb{U}$.

The Koebe one-quarter theorem [5] ensures that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{S}$ contains a disk of radius $\frac{1}{4}$. So every function $f \in \mathcal{S}$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z \qquad\qquad (z \in \mathbb{U}),$$

and

$$f(f^{-1}(w)) = w \qquad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 + \cdots. \tag{2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1).

Lewin [9] investigated the class Σ of bi-univalent functions and showed that $|a_2| < 1.51$ for the functions belonging to Σ. Subsequently, Brannan and Clunie [4] conjectured that $|a_2| \leq \sqrt{2}$. Kedzierawski [8] proved this conjecture for a special case when the function f and f^{-1} are starlike functions. Tan [12] obtained the bound for $|a_2|$ namely $|a_2| \leq 1.485$ which is the best known estimate for functions in the class Σ. Recently there interest to study the bi-univalent functions class Σ (see [6, 7, 13, 14]) and obtain non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. The coefficient estimate problem i.e. bound of $|a_n|$ $(n \in \mathbb{N} - \{1, 2\})$ for each $f \in \Sigma$ given by [2] is still an open problem.

The Hadamard product of two analytic functions $f(z) = z + \sum\limits_{n=2}^{\infty} a_nz^n$ and $h(z) = z + \sum\limits_{n=2}^{\infty} b_nz^n$, is defined as

$$(f * h)(z) = z + \sum_{n=2}^{\infty} a_nb_nz^n. \tag{3}$$

The Pochhammer symbol (or the shifted factorial) defined as follows:

$$(\alpha)_n = \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)} = \begin{cases} 1 & n = 0, \\ \alpha(\alpha + 1)(\alpha + 2) \cdots (\alpha + n - 1) & n = 1, 2, 3, \cdots \end{cases}$$

Then define the function $\phi_a^\lambda(b,c;z)$, by

$$\phi_a^\lambda(b,c;z) := 1 + \sum_{n=1}^{\infty} A_n \psi_n z^n, \quad (z \in \mathbb{U}) \tag{4}$$

where

$$A_n := \left(\frac{a}{a+n}\right)^\lambda, \qquad \psi_n := \frac{(b)_n}{(c)_n}$$

and $b \in \mathbb{R}; c \in \mathbb{R} \setminus \mathbb{Z}_0^-; a \in \mathbb{C} \setminus \mathbb{Z}_0^-$ $(\mathbb{Z}_0^- := \{0,-1,-2,-3,\cdots\}); \lambda \geq 0$.
By using the function $\phi_a^\lambda(b,c;z)$ given by (4), Aghalary-Ebadian-Wang [1], introduced the following convolution operator,

$$L_a^\lambda(b,c;\beta)f(z) := \phi_a^\lambda(b,c;z) * \left(\frac{f(z)}{z}\right)^\beta,$$

where $f \in \mathcal{A}; \beta \in \mathbb{C} \setminus \{0\}; z \in \mathbb{U}$.

In the present paper by using the function $L_a^\lambda(b,c;\beta)$, we introduce a new subclass of the bi-univalent functions and we find estimates on the coefficients $|a_2|$ and $|a_3|$ by employing the techniques used earlier by Srivastava et al.[11] (see also [7]). Our results generalize and improve those in related work of H. Shojaei [10].

2 The subclass $L_\Sigma^{h,p}(\alpha)$

In this section, we introduce and investigate the general subclass $L_\Sigma^{h,p}(\alpha)$.

Definition 1. *Let the analytic functions $h, p : \mathbb{U} \to \mathbb{C}$ be so constrained that*

$$\min\{\mathfrak{Re}(h(z)), \mathfrak{Re}(p(z))\} > 0 \tag{5}$$

and $h(0) = p(0) = 1$.
We say that a function $f \in L_\Sigma^{h,p}(\alpha)$, $(0 \leq \alpha \leq 1)$, if $f \in \Sigma$ and f satisfies the following conditions:

$$\frac{\left(zL_a^\lambda(b,c;\beta)f(z)\right)'}{(1-\alpha) + \alpha L_a^\lambda(b,c;\beta)f(z)} \in h(\mathbb{U}) \qquad (z \in \mathbb{U}) \tag{6}$$

and

$$\frac{\left(wL_a^\lambda(b,c;\beta)g(w)\right)'}{(1-\alpha) + \alpha L_a^\lambda(b,c;\beta)g(w)} \in p(\mathbb{U}) \qquad (w \in \mathbb{U}), \tag{7}$$

where the function $g(w) = f^{-1}(w)$ is given by (2).

Remark 1. *There are many choices of the functions $h(z)$ and $p(z)$ which would provide interesting subclasses of the analytic function class $\mathcal{A}$.*

1. *If we take $h(z) = p(z) = \left(\frac{1+z}{1-z}\right)^{\mu}$ $(z \in \mathbb{U}, 0 < \mu \leq 1)$, then the functions $h(z)$ and $p(z)$ satisfy the hypotheses of Definition 1. Clearly, if $f \in L_{\Sigma}^{h,p}(\alpha)$, then we have*

$$\left|\frac{\left(zL_a^{\lambda}(b,c;\beta)f(z)\right)'}{(1-\alpha)+\alpha L_a^{\lambda}(b,c;\beta)f(z)}\right| < \frac{\mu\pi}{2} \qquad (0 \leq \alpha \leq 1) \tag{8}$$

and

$$\left|\frac{\left(wL_a^{\lambda}(b,c;\beta)g(w)\right)'}{(1-\alpha)+\alpha L_a^{\lambda}(b,c;\beta)g(w)}\right| < \frac{\mu\pi}{2} \qquad (0 \leq \alpha \leq 1). \tag{9}$$

2. *If we take $h(z) = p(z) = \frac{1+(1-2\gamma)z}{1-z}$ $(z \in \mathbb{U}, 0 \leq \gamma < 1)$, then the functions $h(z)$ and $p(z)$ satisfy the hypotheses of Definition 1. Clearly, if $f \in L_{\Sigma}^{h,p}(\alpha)$, then we have*

$$\mathfrak{Re}\left(\frac{\left(zL_a^{\lambda}(b,c;\beta)f(z)\right)'}{(1-\alpha)+\alpha L_a^{\lambda}(b,c;\beta)f(z)}\right) > \gamma \qquad (0 \leq \gamma < 1)$$

and

$$\mathfrak{Re}\left(\frac{\left(wL_a^{\lambda}(b,c;\beta)g(w)\right)'}{(1-\alpha)+\alpha L_a^{\lambda}(b,c;\beta)g(w)}\right) > \gamma \qquad (0 \leq \gamma < 1).$$

3. *For $h(z) = p(z) = \left(\frac{1+z}{1-z}\right)^{\mu}$, we have $L_{\Sigma}^{h,p}(\alpha) = \mathcal{M}_{\Sigma}^{L}(\mu,\alpha)$, where the class $\mathcal{M}_{\Sigma}^{L}(\mu,\alpha)$ was studied by H. Shojaei [10].*

4. *For $h(z) = p(z) = \frac{1+(1-2\gamma)z}{1-z}$, we have $L_{\Sigma}^{h,p}(\alpha) = \mathcal{N}_{\Sigma}^{L}(\mu,\alpha)$, where the class $\mathcal{N}_{\Sigma}^{L}(\gamma,\alpha)$ was studied by H. Shojaei [10].*

It is easy to see that the class $L_{\Sigma}^{h,p}(\alpha)$ is not empty. As an example, let $f(z) = \frac{z}{1-z}$, then $g(w) = f^{-1}(w) = \frac{w}{1+w} \in \mathcal{S}$, so $f \in \Sigma$.
Now, by appropriate choice of the parameters $L_a^{\lambda}(b,c;\beta)$, we show that $f(z)$ and $g(w)$ satisfy in relations (6) and (7). Set $\lambda = 0, b = c$ and $\beta = 1$, we have

$$\phi_a^0(b,b;z) = 1 + z + z^2 + \cdots = \frac{1}{1-z},$$

and

$$L_a^0(b,b;1)f(z) := \phi_a^0(b,b;z) * \frac{f(z)}{z} = \frac{1}{1-z} * \frac{1}{1-z} = \frac{1}{1-z}.$$

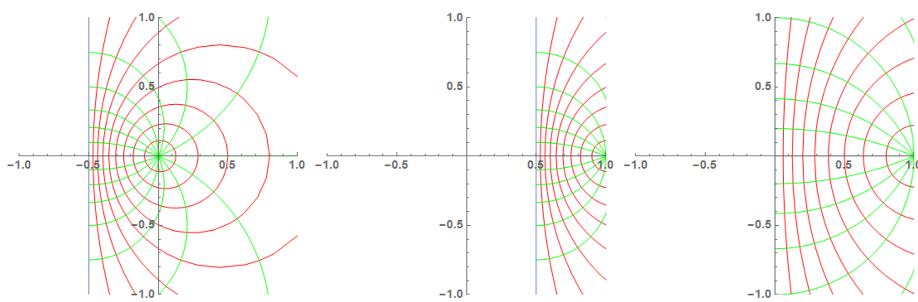
Suppose $h(z) = p(z) = \dfrac{1+z}{1-z}$ and $\mu = 1$, we have

$$k(z) := \frac{\left(z \cdot \dfrac{1}{1-z}\right)'}{(1-\alpha) + \alpha \cdot \dfrac{1}{1-z}} = \frac{1}{1-z} \in h(\mathbb{U}).$$

The same manner holds for $g(w)$ as follows:

$$\frac{\left(w \cdot \dfrac{1}{1+w}\right)'}{(1-\alpha) + \alpha \cdot \dfrac{1}{1+w}} = \frac{1}{1+w} \in p(\mathbb{U}).$$

The images of $\mathbb{U}$ under $f(z), k(z)$ and $h(z) = p(z) = \dfrac{1+z}{1-z}$ are shown in Fig.1. Thus $f(z) \in L_{\Sigma}^{h,p}(\alpha)$ and therefore $L_{\Sigma}^{h,p}(\alpha)$ is not empty.



(a) $f(z)$ (b) $k(z)$ (c) $h(z)$

Figure 1: Images of $\mathbb{D}$ under $f(z), k(z)$ and $h(z)$.

3 Coefficient Estimates

For proof of the theorem, we need the following lemma.

Lemma 1. *(see [5]). If $p \in P$, then $|c_k| \leqslant 2$ for each k, where P is the family of all functions $p(z)$ analytic in $\mathbb{U}$ for which $\mathfrak{Re}(p(z)) > 0$, $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ for $z \in \mathbb{U}$.*

Theorem 1. *Let $f(z)$ given by the Taylor Maclaurin series expansion* (1) *be in the class $L_{\Sigma}^{h,p}(\alpha)$. Then,*

$$|a_2| \leq \min\left\{\sqrt{\frac{|h'(0)|^2+|p'(0)|^2}{|2(2-\alpha)^2\beta^2A_1^2\psi_1^2|}},\right. \tag{10}$$
$$\left.\sqrt{\frac{|h''(0)|+|p''(0)|}{|(3-\alpha)\beta(\beta+1)A_2\psi_2-2\alpha(2-\alpha)\beta^2A_1^2\psi_1^2|}}\right\}$$

and

$$|a_3| \leq \min\left\{\frac{|h'(0)|^2+|p'(0)|^2}{|(2-\alpha)^2\beta A_1^2\psi_1^2|}+\frac{|h''(0)|+|p''(0)|}{|2(3-\alpha)\beta A_2\psi_2|},\right. \tag{11}$$
$$\left.\frac{|h''(0)|+|p''(0)|}{|2(3-\alpha)\beta A_2\psi_2|}+\frac{|h''(0)|+|p''(0)|}{|(3-\alpha)\beta(\beta+1)A_2\psi_2-2\alpha(2-\alpha)^2\beta^2A_1^2\psi_1^2|}\right\}.$$

Proof. First of all, it follows from the conditions (6) and (7) that,

$$\frac{\left(zL_a^{\lambda}(b,c;\beta)f(z)\right)'}{(1-\alpha)+\alpha L_a^{\lambda}(b,c;\beta)f(z)} = h(z) \qquad (z\in\mathbb{U}) \tag{12}$$

and

$$\frac{\left(wL_a^{\lambda}(b,c;\beta)g(w)\right)'}{(1-\alpha)+\alpha L_a^{\lambda}(b,c;\beta)g(w)} = p(w) \qquad (w\in\mathbb{U}), \tag{13}$$

where the function $g(w)=f^{-1}(w)$ is given by (2), respectively, $h(z)$ and $p(w)$ satisfy the conditions of Definition1. Furthermore, the functions $h(z)$ and $p(w)$ have the following Taylor-Maclaurin series expansions:

$$h(z) = 1+h_1z+h_2z^2+\cdots, \tag{14}$$

and

$$p(w) = 1+p_1w+p_2w^2+\cdots. \tag{15}$$

Now, by comparing the series expansions (14) and (15) by the coefficients (12) and (13), we get

$$2\beta A_1\psi_1a_2 = h_1+\alpha\beta A_1\psi_1a_2, \tag{16}$$

$$3A_2\psi_2[\frac{\beta(\beta-1)}{2}a_2^2+\beta a_3] = h_2+\alpha\beta A_1\psi_1a_2h_1+\alpha A_2\psi_2[\frac{\beta(\beta-1)}{2}a_2^2+\beta a_3], \tag{17}$$

$$-2\beta A_1\psi_1 a_2 = p_1 - \alpha\beta A_1\psi_1 a_2, \tag{18}$$

$$3A_2\psi_2[\frac{\beta(\beta+3)}{2}a_2^2 - \beta a_3] = p_2 - \alpha\beta A_1\psi_1 a_2 p_1 + \alpha A_2\psi_2[\frac{\beta(\beta+3)}{2}a_2^2 - \beta a_3]. \tag{19}$$

From (16) and (18), we obtain

$$h_1 = -p_1, \tag{20}$$

and

$$h_1^2 + p_1^2 = 2(2-\alpha)^2\beta^2 A_1^2\psi_1^2 a_2^2. \tag{21}$$

Therefore,

$$|a_2|^2 \leq \frac{|h'(0)|^2 + |p'(0)|^2}{2(2-\alpha)^2\beta^2 A_1^2\psi_1^2}.$$

Also, From (17) and (19), we find that

$$h_2 + p_2 = (3-\alpha)\beta(\beta+1)A_2\psi_2 a_2^2 - \frac{2\alpha h_1^2}{2-\alpha}. \tag{22}$$

On the other hand, From (16), we have

$$h_1^2 = (2-\alpha)^2\beta^2 A_1^2\psi_1^2 a_2^2, \tag{23}$$

by substituting the value of h_1^2 from (23) into (22), it follows that

$$a_2^2 = \frac{h_2 + p_2}{(3-\alpha)\beta(\beta+1)A_2\psi_2 - 2\alpha(2-\alpha)^2\beta^2 A_1^2\psi_1^2}, \tag{24}$$

thus,

$$|a_2|^2 \leq \frac{|h''(0)| + |p''(0)|}{(3-\alpha)\beta(\beta+1)A_2\psi_2 - 2\alpha(2-\alpha)\beta^2 A_1^2\psi_1^2}.$$

So we get the desired estimate on the coefficient $|a_2|$ as asserted in (10). Next, in order to find the bound on the coefficient $|a_3|$, we subtract (19) from (17). We thus get

$$2(3-\alpha)\beta A_2\psi_2(a_3 - a_2^2) = h_2 - p_2. \tag{25}$$

Upon substituting the value of a_2^2 from (21) into (25), it follows that

$$a_3 = \frac{h_1^2 + p_1^2}{(2-\alpha)^2 \beta A_1^2 \psi_1^2} + \frac{h_2 - p_2}{2(3-\alpha)\beta A_2 \psi_2}.$$

We thus find that

$$|a_3| \leq \frac{|h'(0)|^2 + |p'(0)|^2}{|(2-\alpha)^2 \beta A_1^2 \psi_1^2|} + \frac{|h''(0)| + |p''(0)|}{|2(3-\alpha)\beta A_2 \psi_2|}.$$

Upon substituting the value of a_2^2 from (24) into (25), it follows that

$$a_3 = \frac{h_2 - p_2}{2(3-\alpha)\beta A_2 \psi_2} + \frac{h_2 + p_2}{(3-\alpha)\beta(\beta+1) A_2 \psi_2 - 2\alpha(2-\alpha)^2 \beta^2 A_1^2 \psi_1^2},$$

consequently, we have

$$|a_3| \leq \frac{|h''(0)| + |p''(0)|}{|2(3-\alpha)\beta A_2 \psi_2|} + \frac{|h''(0)| + |p''(0)|}{|(3-\alpha)\beta(\beta+1) A_2 \psi_2 - 2\alpha(2-\alpha)^2 \beta^2 A_1^2 \psi_1^2|}.$$

This evidently completes the proof of Theorem (1).

$\square$

4 Corollaries and Consequences

By setting $h(z) = p(z) = \left(\frac{1+z}{1-z}\right)^{\mu}$ and $L_a^{\lambda}(b, c; \beta) = L_1^1(b, b; 1)$ in Theorem 1. we get following result.

Corollary 1. *Let the function $f(z)$ given by the Taylor-Maclaurin series expansion* (1) *be in the bi-univalent function class $\mathcal{M}_{\Sigma}^{L}(\mu, 1)$. Then*

$$|a_2| \leq \sqrt{\frac{24}{5}\mu}$$

and

$$|a_3| \leq \frac{39}{5}\mu^2.$$

Remark 2. *Corollary 1 is an improvement of the following estimates obtained by H. Shojaei [10].*

Corollary 2. *(see [10]) Let the function $f(z)$ given by the Taylor-Maclaurin series expansion* (1) *be in the bi-univalent function class $\mathcal{M}_{\Sigma}^{L}(\mu, 1)$. Then*

$$|a_2| \leq \sqrt{30\mu}$$

and

$$|a_3| \leq 11\mu.$$

By setting $h(z) = p(z) = \dfrac{1 + (1 - 2\gamma)z}{1 - z}$ and $L_a^{\lambda}(b, c; \beta) = L_1^1(b, b; 1)$ in Theorem 1, we get the following consequence.

Corollary 3. *Let the function $f(z)$ given by the Taylor-Maclaurin series expansion* (1) *be in the bi-univalent function class $\mathcal{N}_{\Sigma}^{L}(\mu, 1)$. Then*

$$|a_2| \leq \sqrt{\frac{48}{5}(1 - \gamma)}.$$

Remark 3. *Corollary 3 is an improvement of the following estimates obtained by H. Shojaei [10].*

Corollary 4. *(see[10]) Let the function $f(z)$ given by the Taylor-Maclaurin series expansion* (1) *be in the bi-univalent function class $\mathcal{N}_{\Sigma}^{L}(\mu, 1)$. Then*

$$|a_2| \leq 3\sqrt{2(1 - \gamma)}.$$

References

[1] R. Aghalary, A. Ebadian, and Z.G. Wang, Subordination and superordination result involving certain convolution operators, *Bull. Iranian Math. Soc.* 36(1), 137-147, 2010.

[2] R. M. Ali, S. K. Lee, V. Ravichandran and S. Subramaniam, Coefficient estimates for bi-univalent Ma-Minda starlike and convex functions, *Appl. Math. Lett.* 25: 344-351, 2012.

[3] M. K. Aouf, R. M. El-Ashwah, and A. Abd-Eltawab, New Subclasses of Biunivalent Functions Involving Dziok-Srivastava Operator, *ISRN Mathematical Analysis*, Article ID 387178, 5 pages, 2013.

[4] D. A. Brannan and J. G. Clunie (Eds.), Aspects of Contemporary Complex Analysis, *Proceedings of the NATO Advanced Study Institute held at the University of Durham*, July 1-20, 1979, Academic Press, New York and London, 1980.

[5] P. L. Duren, *Univalent Functions*, Springer-Verlag, New York, Berlin, 1983.

[6] T. Hayami and S. Owa, Coefficient bounds for bi-univalent functions, *Pan-American Mathematical Journal*, 22:15-26, 2012.

[7] B. A. Frasin and M. K. Aouf, New subclasses of bi-univalent functions, *Appl. Math. Lett.* 24:1569-1573, 2011.

[8] A. W. Kedzierawski, Some remarks on bi-univalent functions, *Ann. Univ. Mariae Curie-Sklodowska Sect. A.* 39:77-81, 1985.

[9] M. Lewin, On a coefficient problem for bi-univalent functions, *Proc. Amer. Math. Soc*, 18:63-68, 1967.

[10] H. Shojaei, Certain subclasses of bi-univalent functions associated with the Aghalary-Ebadian-Wang operator, *Sahand Communications in Mathematical Analysis (SCMA)*, 7(1):63-73, 2017.

[11] H. M. Srivastava, A. K. Mishra and P. Gochhayat, Certain subclasses of analytic and biunivalent functions, *Appl. Math. Lett.*, 23:1188-1192, 2010.

[12] D. L. Tan, Coefficient estimates for bi-univalent functions, *Chinese Ann. Math. Ser. A.* 5: 559-568, 1984.

[13] Q. H. Xu, Y. C. Gui, and H. M. Srivastava, Coefficient estimates for a certain subclass of analytic and bi-univalent functions, *Appl. Math. Lett.* 25:990-994, 2012.

[14] Q. H. Xu, H.G. Xiao, and H. M. Srivastava, A certain general subclass of analytic and bi-univalent functions and associated cofficient estimate problems, *Appl. Math. Comput.* 218:11461-11465, 2012.