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ON A CLASS OF WEIGHTED COMPOSITION OPERATORS ON THE BERGMAN SPACE OF THE UPPER HALF PLANE*

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Abstract

In this paper we consider a class of weighted composition operators $R_a, a \in \mathbb{D}$ defined on the Bergman space $L^2_a(\mathbb{U}_+)$ of the upper half plane. We showed that these classes of operators are unitary, self-adjoint and have numerical radius 1. We calculated the fixed points of these unitary operators and characterized the reducing subspace of $T \in \mathcal{L}(L^2_a(\mathbb{U}_+))$ that commutes with R_a. We also derived various algebraic properties of bounded linear operators defined on $L^2_a(\mathbb{U}_+)$, in terms of certain distance estimates involving the weighted composition operators R_a. Our main focus is on Toeplitz operators defined on $L^2_a(\mathbb{U}_+)$.

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1 Introduction

Let $\mathbb{U}_+ = \{s = x + iy \in \mathbb{C} : y > 0\}$ be the upper half plane, and let $d\widetilde{A} = dxdy$ be the area measure on $\mathbb{U}_+$. Let $L^2(\mathbb{U}_+, d\widetilde{A})$ be the space of complex-valued, absolutely square integrable, measurable functions on $\mathbb{U}_+$

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with respect to the area measure $d\widetilde{A}$. The space $L^2(\mathbb{U}_+, d\widetilde{A})$ is a Hilbert space with the inner product defined by

$$\langle f, g\rangle = \int_{\mathbb{U}_+} f(s)\overline{g(s)}d\widetilde{A}(s).$$

Let $L_a^2(\mathbb{U}_+)$ be the subspace of $L^2(\mathbb{U}_+, d\widetilde{A})$ consisting of those functions of $L^2(\mathbb{U}_+, d\widetilde{A})$ that are analytic on $\mathbb{U}_+$. The space $L_a^2(\mathbb{U}_+)$ is a closed subspace of $L^2(\mathbb{U}_+, d\widetilde{A})$ and it is called the Bergman space of the upper half plane. It is well known that $L_a^2(\mathbb{U}_+)$ is a reproducing kernel Hilbert space [4] and the reproducing kernel is given by

$$K_w(z) = -\frac{1}{\pi(\overline{w}-z)^2},\ w,\ z \in \mathbb{U}_+.$$

The orthogonal(Bergman) projection from $L^2(\mathbb{U}_+, d\widetilde{A})$ onto $L_a^2(\mathbb{U}_+)$ is given by

$$(P_+f)(w) = \langle f, K_w\rangle,\ w \in \mathbb{U}_+.$$

Let $L^\infty(\mathbb{U}_+)$ be the space of complex valued, essentially bounded, Lebesgue measurable functions on $\mathbb{U}_+$. For $\varphi \in L^\infty(\mathbb{U}_+)$, define

$$||\varphi||_\infty = ess \sup_{s\in\mathbb{U}_+} |\varphi(s)|.$$

The space $L^\infty(\mathbb{U}_+)$ is a Banach space with respect to the essential supremum norm. For $\varphi \in L^\infty(\mathbb{U}_+)$, we define the Toeplitz operator T_φ from $L_a^2(\mathbb{U}_+)$ into $L_a^2(\mathbb{U}_+)$ with generating symbol φ by $T_\varphi f = P_+(\varphi f)$, where P_+ denote the orthogonal projection from $L^2(\mathbb{U}_+, d\widetilde{A})$ onto $L_a^2(\mathbb{U}_+)$. The Toeplitz operator T_φ is bounded and $||T_\varphi|| \leq ||\varphi||_\infty$. For more details see [4]. The big Hankel operator H_φ from $L_a^2(\mathbb{U}_+)$ into $(L_a^2(\mathbb{U}_+))^\perp$ is defined by $H_\varphi f = (I - P_+)(\varphi f), f \in L_a^2(\mathbb{U}_+)$. The little Hankel operator h_φ from $L_a^2(\mathbb{U}_+)$ into $\overline{L_a^2(\mathbb{U}_+)}$ is defined by $h_\varphi f = \overline{P}_+(\varphi f)$, where $\overline{P}_+$ is the projection operator from $L^2(\mathbb{U}_+, d\widetilde{A})$ onto $\overline{L_a^2(\mathbb{U}_+)} = \{\overline{f} : f \in L_a^2(\mathbb{U}_+)\}$. Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk and $dA(z)$ be the Lebesgue area measure on the open unit disk $\mathbb{D}$ normalized so that the measure of the disk $\mathbb{D}$ is 1. In rectangular and polar coordinates, we have $dA(z) = \frac{1}{\pi}dxdy = \frac{1}{\pi}rdrd\theta$. The Bergman space of open unit disk, $L_a^2(\mathbb{D})$ is defined to be the subspace of $L^2(\mathbb{D}, dA)$ consisting of analytic functions. The sequence of functions $e_n(z) = \sqrt{n+1}\, z^n, n = 0, 1, 2, \cdots, z \in \mathbb{D}$ form an orthonormal basis for $L_a^2(\mathbb{D})$. The Bergman kernel or the reproducing kernel of $\mathbb{D}$ of $L^2_a(\mathbb{D})$ is given by $K(z,w) = \frac{1}{(1-z\overline{w})^2}$. Let $L^\infty(\mathbb{D})$ be the space of all complex-valued, essentially bounded, Lebesgue measurable functions on $\mathbb{D}$ and $P : L^2(\mathbb{D}, dA) \to L^2_a(\mathbb{D})$ be the Bergman projection given by

$$Pf(z) = \int_{\mathbb{D}} K(z,w)f(w)dA(w) = \int_{\mathbb{D}} \frac{f(w)}{(1-z\overline{w})^2} dA(w).$$

For $\varphi \in L^\infty(\mathbb{D})$, we define the Toeplitz operator $\mathcal{T}_\varphi$ from $L^2_a(\mathbb{D})$ into itself by $\mathcal{T}_\varphi f = P(\varphi f), f \in L^2_a(\mathbb{D})$.

The layout of this paper is as follows: In section 2, we establish an isomorphism between $L^2_a(\mathbb{U}_+)$ and $L^2_a(\mathbb{D})$. We also introduce a class of weighted composition operators R_a, $a \in \mathbb{D}$ defined on $L^2_a(\mathbb{U}_+)$ which are also self-adjoint and unitary. We showed that these operators satisfy certain intertwining properties with Toeplitz, Hankel and little Hankel operators. In section 3, we established that the numerical radius of the operators R_a, $a \in \mathbb{D}$ are equal to 1 and calculated the fixed points of these unitary operators that are also involutions. We characterized the reducing subspaces of the operators $T \in \mathcal{L}(L^2_a(\mathbb{U}_+))$ that commutes with R_a for some $a \in \mathbb{D}$. In section 4, we showed that the Toeplitz operator T_G defined on $L^2_a(\mathbb{U}_+)$ with symbol $G \in h^\infty(\mathbb{U}_+)$ is positive if and only if the symbol $G \geq 0$. We showed that if $\varphi \geq 0$ and $\varphi \in h^\infty(\mathbb{U}_+)$ and $||R_a - T_\varphi|| < 1$, for some $a \in \mathbb{D}$, then T_G is invertible. Further we showed that if $\varphi \geq 0, \varphi \in h^\infty(\mathbb{U}_+)$ and $||R_a - T_\varphi|| \leq 1$, for some $a \in \mathbb{D}$ then T_φ is not invertible if and only if $||I - T_\varphi|| = ||I - \frac{T_\varphi}{2}|| = 1$. In section 5, we deal with Schatten class operators defined on $L^2_a(\mathbb{U}_+)$. We showed that if $\varphi \in L^\infty(\mathbb{U}_+)$ and T_φ is invertible with the polar decomposition $T_\varphi = U|T_\varphi|$, then for all $a \in \mathbb{D}$ and for every $A \in \mathcal{L}(L^2_a(\mathbb{U}_+))$ with $UA = AU$, the inequality $||(U - T_\varphi)A||_2 \leq ||(R_a - T_\varphi)A||_2 \leq ||(U + T_\varphi)A||_2$ holds. Further if $T \in \mathcal{L}(L^2_a(\mathbb{U}_+))$ is positive and $R_a - T$ is compact for some $a \in \mathbb{D}$, then $I - T$ is compact and if $R_a - T \in S_p (0 < p \leq \infty)$ for some $a \in \mathbb{D}$, then $I - T \in S_p$. We also established that if for some $a \in \mathbb{D}, R_a$ is a local maximum or a local minimum of $O_p = ||U - T||_p^p, p > 1$ where U is unitary, $T > 0$, then

$$M_a = \{(L^{'} \circ \tau_{\varsigma_a}) t_{\varsigma_a} (g \circ L \circ \tau_{\varsigma_a}) : g \in L^2_a(\mathbb{D}), \text{ g is even}\}$$

is a reducing subspace of T and $||I - T||_p < ||R_a - T||_p$.

2 On a class of weighted composition operators

In this section we establish an isomorphism between $L^2_a(\mathbb{U}_+)$ and $L^2_a(\mathbb{D})$. We also introduce a class of weighted composition operators R_a, $a \in \mathbb{D}$ defined

on $L_a^2(\mathbb{U}_+)$ which are also self-adjoint and unitary. We showed that these operators satisfy certain intertwining properties with Toeplitz, Hankel and little Hankel operators.
Let $L : \mathbb{U}_+ \to \mathbb{D}$ be defined by $L(s) = \dfrac{i-s}{i+s} = z$. Then L is one one and onto and $L^{-1} : \mathbb{D} \to \mathbb{U}_+$ is given by

$$L^{-1}(z) = i\frac{1-z}{1+z} = s.$$

Further $L^{'}(s) = \dfrac{-2i}{(i+s)^2}$ and $(L^{-1})^{'}(z) = \dfrac{-2i}{(1+z)^2}$. Let $\mathcal{W} : L_a^2(\mathbb{D}) \to L_a^2(\mathbb{U}_+)$ be defined by

$$(\mathcal{W}g)(s) = g(Ls)\frac{2i}{\sqrt{\pi}(i+s)^2}.$$

Then $\mathcal{W}^{-1} : L_a^2(\mathbb{U}_+) \to L_a^2(\mathbb{D})$ is given by

$$(\mathcal{W}^{-1}G)(z) = (2i)\sqrt{\pi}G\left(L^{-1}(z)\right)\frac{1}{(1+z)^2}.$$

Notice that $\mathcal{W}^{-1}\mathcal{W}g = g$ for all $g \in L_a^2(\mathbb{D})$ and $\mathcal{W}\mathcal{W}^{-1}G = G$ for all $G \in L_a^2(\mathbb{U}_+)$. This can be verified as follows:

$$\begin{aligned}
((\mathcal{W}^{-1}\mathcal{W})g)(z) &= \mathcal{W}^{-1}\left(g(Ls)\frac{(2i)}{\sqrt{\pi}(i+s)^2}\right) \\
&= \frac{(2i)}{\sqrt{\pi}}\mathcal{W}^{-1}\left(g(Ls)\frac{1}{(i+s)^2}\right) \\
&= (2i)\sqrt{\pi}\frac{(2i)}{\sqrt{\pi}}g(L(L^{-1}z))\frac{1}{(i+L^{-1}z)^2}\frac{1}{(1+z)^2} \\
&= (-4)g(z)\left(\frac{1}{i+\frac{i-iz}{1+z}}\right)^2\frac{1}{(1+z)^2} \\
&= (-4)g(z)\left(\frac{1+z}{i+iz+i-iz}\right)^2\frac{1}{(1+z)^2} \\
&= (-4)g(z)\frac{1}{(2i)^2} \\
&= g(z),\ z \in \mathbb{D},\ g \in L_a^2(\mathbb{D})
\end{aligned}$$

and

$$\begin{aligned}
(\mathcal{W}\mathcal{W}^{-1}G)(s) &= \mathcal{W}\left((2i)\sqrt{\pi}G(L^{-1}(z))\frac{1}{(1+z)^2}\right) \\
&= (2i)\sqrt{\pi}\mathcal{W}\left(G(L^{-1}(z))\frac{1}{(1+z)^2}\right) \\
&= (2i)\sqrt{\pi}\frac{(2i)}{\sqrt{\pi}}G(L^{-1}(Ls))\frac{1}{(i+Ls)^2}\frac{1}{(i+s)^2} \\
&= (-4)G(s)\left(\frac{1}{1+\frac{i-s}{i+s}}\right)^2\frac{1}{(i+s)^2} \\
&= (-4)G(s)\left(\frac{i+s}{i+s+i-s}\right)^2\frac{1}{(i+s)^2} \\
&= (-4)G(s)\frac{1}{(2i)^2} \\
&= G(s), s \in \mathbb{U}_+,\ G \in L^2_a(\mathbb{U}_+).
\end{aligned}$$

The functions $\tau_a(s)$ given by $\tau_a(s) = \dfrac{c+sd-1}{s-d+sc} = \dfrac{(c-1)+sd}{(1+c)s-d}$ are automorphisms of $\mathbb{U}_+$ where $a = c+id \in \mathbb{D}$ and $s \in \mathbb{U}_+$ and $\tau_a^{'}(s) = \dfrac{1-|a|^2}{[(1+c)s-d]^2}$. Let $t_a(s) = \dfrac{|a|^2-1}{[(1+c)s-d]^2}$. Thus $\tau_a^{'}(s) = -t_a(s)$. It is not difficult to see that $(\tau_a \circ \tau_a)(s) = s$ and $(t_a \circ \tau_a)(s)t_a(s) = 1$, for all $a \in \mathbb{D}, s \in \mathbb{U}_+$. For $a \in \mathbb{D}$ consider the map $R_a : L^2_a(\mathbb{U}_+) \to L^2_a(\mathbb{U}_+)$ defined by $(R_a f)(s) = (f \circ \tau_a)(s)t_a(s)$. For $s \in \mathbb{U}_+$,

$$\begin{aligned}
(R_a^2 f)(s) &= R_a[(f \circ \tau_a)(s)t_a(s)] \\
&= (f \circ \tau_a \circ \tau_a)(s)(t_a \circ \tau_a)(s)t_a(s) \\
&= f(s) \text{ since } (t_a \circ \tau_a)(s)t_a(s) = 1.
\end{aligned}$$

That is, $R_a^2 = I$ and R_a is an involution. The map R_a is also self-adjoint and unitary for all $a \in \mathbb{D}$. That is $R_a^* = R_a$ and $R_a R_a^* = R_a^* R_a = R_a^2 = I$ for all $a \in \mathbb{D}$. Notice that R_a can also be defined on $(L^2(\mathbb{U}_+), d\tilde{A})$. Further $R_a(L^2_a(\mathbb{U}_+)) \subset L^2_a(\mathbb{U}_+)$ and $R_a((L^2_a(\mathbb{U}_+))^\perp) \subset (L^2_a(\mathbb{U}_+))^\perp$. Thus $P_+R_a = R_aP_+$, for all $a \in \mathbb{D}$.

Theorem 1. *Let $a \in \mathbb{D}, \varphi \in L^\infty(\mathbb{U}_+)$. The following hold:*

(i) $R_a T_\varphi R_a = T_{\varphi \circ \tau_a}$.

(ii) $R_a H_\varphi R_a = H_{\varphi \circ \tau_a}$.

(iii) $R_a h_\varphi R_a = h_{\varphi\circ\tau_a}$.

Proof. (i) Let $f \in L_a^2(\mathbb{U}_+)$. Then

$$R_a T_\varphi R_a f = R_a P_+(\varphi R_a f) = P_+ R_a M_\varphi R_a f,$$

where $M_\varphi f = \varphi f$ and $R_a M_\varphi R_a f = R_a M_\varphi[(f\circ\tau_a)t_a] = R_a[\varphi(f\circ\tau_a)t_a] = (\varphi\circ\tau_a)(f\circ\tau_a\circ\tau_a)(t_a\circ\tau_a)t_a = (\varphi\circ\tau_a)f$. Thus $R_a T_\varphi R_a f = P_+[(\varphi\circ\tau_a)f] = T_{\varphi\circ\tau_a} f$.

(ii) Let $f \in L_a^2(\mathbb{U}_+)$. Then

$$\begin{aligned} R_a H_\varphi R_a f &= R_a H_\varphi[(f\circ\tau_a)t_a] \\ &= R_a(I-P_+)[\varphi(f\circ\tau_a)t_a] \\ &= (I-P_+)R_a[\varphi(f\circ\tau_a)t_a] \\ &= (I-P_+)[(\varphi\circ\tau_a)(f\circ\tau_a\circ\tau_a)(t_a\circ\tau_a)t_a] \\ &= (I-P_+)[(\varphi\circ\tau_a)f] \\ &= H_{\varphi\circ\tau_a} f. \end{aligned}$$

Thus $R_a H_\varphi R_a = H_{\varphi\circ\tau_a}$.

(iii) Observe that $\overline{P_+} = JP_+J$, where $Jg(s) = g(\overline{s})$, for $g \in L^2(\mathbb{U}_+)$ and $R_a\overline{P_+}g = \overline{P_+}R_a g$, we obtain

$$\begin{aligned} R_a h_\varphi R_a f &= R_a h_\varphi[(f\circ\tau_a)t_a] \\ &= R_a\overline{P_+}[\varphi(f\circ\tau_a)t_a] \\ &= \overline{P_+}R_a[\varphi(f\circ\tau_a)t_a] \\ &= \overline{P_+}[(\varphi\circ\tau_a)(f\circ\tau_a\circ\tau_a)(t_a\circ\tau_a)t_a] \\ &= \overline{P_+}[(\varphi\circ\tau_a)f] \\ &= h_{\varphi\circ\tau_a} f, \text{ for all } f \in L_a^2(\mathbb{U}_+). \end{aligned}$$

Hence $R_a h_\varphi R_a = h_{\varphi\circ\tau_a}$.

□

Theorem 2. *If $T \in \mathcal{L}(L_a^2(\mathbb{U}_+)), T \geq 0, TR_a \geq 0$ for some $a \in \mathbb{D}$ then $TR_a \leq T$.*

Proof. Since $TR_a \geq 0$, it follows that $TR_a = (TR_a)^* = R_a^* T^* = R_a T$ and $(TR_a)^2 = TR_aTR_a = TR_aR_aT = T^2$. From Löwner-Heinz inequality [10], it follows that $TR_a \leq T$. □

3 Numerical radius of R_a

In this section we established that the numerical radius of the operators R_a, $a \in \mathbb{D}$ are equal to 1 and calculated the fixed points of these unitary operators that are also involutions. We characterized the reducing subspaces of the operators $T \in \mathcal{L}(L^2_a(\mathbb{U}_+))$ that commutes with R_a for some $a \in \mathbb{D}$. Let $\mathcal{L}(\mathcal{H})$ be the space of all bounded linear operators from the Hilbert space $\mathcal{H}$ into itself and $\mathcal{LC}(\mathcal{H})$ be the space of all compact operators in $\mathcal{L}(\mathcal{H})$.

Definition 1. *For a bounded linear operator T on a Hilbert space $\mathcal{H}$, the numerical range $NR(T)$ is the image of the unit sphere of $\mathcal{H}$ under the quadratic form $x \to \langle Tx, x\rangle$ associated with the operator. More precisely, $NR(T) := \{\langle Tx, x\rangle : x \in \mathcal{H}, ||x|| = 1\}$.*

Definition 2. *Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{L}(\mathcal{H})$. The numerical radius of T is defined by*

$$\rho(T) = \sup\left\{|\langle Tx, x\rangle| : x \in \mathcal{H}, ||x|| = 1\right\}.$$

Theorem 3. *For all $a \in \mathbb{D}, \rho(R_a) = 1$.*

Proof. We shall first show that if $T \in \mathcal{L}(L^2_a(\mathbb{U}_+)), m(T) = \inf_{||f||=1} |\langle Tf, f\rangle|$ and $\rho(T) = \sup_{||f||=1} |\langle Tf, f\rangle|$ then the following inequality holds:

$$\frac{1}{2}\sqrt{||\ |T|^2 + |T^*|^2\ || + 2m(T^2)} \leq \rho(T) \leq \frac{1}{2}\sqrt{||\ |T|^2 + |T^*|^2\ || + 2\rho(T^2)}.$$

Let f be a unit vector in $L^2_a(\mathbb{D})$ and let $\theta \in \mathbb{R}$ be such that

$$e^{2i\theta}\langle T^2 f, f\rangle = |\langle T^2 f, f\rangle|.$$

Then we obtain

$$\begin{aligned}
\rho(T) \geq ||Re(e^{i\theta}T)|| &= \frac{1}{2}||e^{i\theta}T + e^{-i\theta}T^*|| \\
&= \frac{1}{2}||(e^{i\theta}T + e^{-i\theta}T^*)^2||^{\frac{1}{2}} \\
&= \frac{1}{2}\sqrt{||\ |T|^2 + |T^*|^2 + 2Re(e^{2i\theta}T^2)||} \\
&\geq \frac{1}{2}\sqrt{|\langle(|T|^2 + |T^*|^2 + 2Re(e^{2i\theta}T^2))\, f, f\rangle|} \\
&= \frac{1}{2}\sqrt{|\langle(|T|^2 + |T^*|^2)f, f\rangle + 2\langle Re(e^{2i\theta}T^2)f, f\rangle|}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\sqrt{|\langle(|T|^2+|T^*|^2)f,f\rangle + 2Re(e^{2i\theta}\langle T^2 f,f\rangle)|}\\
&= \frac{1}{2}\sqrt{\langle(|T|^2+|T^*|^2)\,f,f\rangle + 2\,|\langle T^2 f,f\rangle|}\\
&\geq \frac{1}{2}\sqrt{\langle(|T|^2+|T^*|^2)\,f,f\rangle + 2m(T^2)}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\rho(T) &\geq \frac{1}{2}\sup_{||f||=1}\sqrt{\langle(|T|^2+|T^*|^2)\,f,f\rangle + 2m(T^2)}\\
&= \frac{1}{2}\sqrt{||\ |T|^2+|T^*|^2\ || + 2m(T^2)},
\end{aligned}
$$

which establishes the first part of the inequality.

To prove the second part of the inequality, notice that,

$$\rho(T) = \sup_{\psi\in\mathbb{R}} ||Re(e^{i\psi}T)||.$$

Thus we get,

$$
\begin{aligned}
\rho(T) &= \sup_{\psi\in\mathbb{R}} ||Re(e^{i\psi}T)||\\
&= \frac{1}{2}\sup_{\psi\in\mathbb{R}} ||e^{i\psi}T + e^{-i\psi}T^*||\\
&= \frac{1}{2}\sup_{\psi\in\mathbb{R}} ||(e^{i\psi}T + e^{-i\psi}T^*)^2||^{\frac{1}{2}}\\
&= \frac{1}{2}\sup_{\psi\in\mathbb{R}} ||\ |T|^2+|T^*|^2 + 2Re(e^{2i\psi}T^2)||^{\frac{1}{2}}\\
&\leq \frac{1}{2}\sqrt{||\ |T|^2+|T^*|^2\ || + 2\sup_{\psi\in\mathbb{R}}||Re(e^{2i\psi}T^2)||}\\
&= \frac{1}{2}\sqrt{||\ |T|^2+|T^*|^2\ || + 2\rho(T^2)},
\end{aligned}
$$

which proves the second half of the inequality.
Since $R_a^* = R_a$ and $R_a^2 = I$, we obtain $\rho(R_a) = \frac{1}{2}\sqrt{||\ |R_a|^2 + |R_a^*|^2\ || + 2} = 1$. □

For any $a \in \mathbb{D}$, let ϕ_a be the analytic mapping on $\mathbb{D}$ defined by $\phi_a(w) = \frac{a-w}{1-\bar{a}w}, w \in \mathbb{D}$. Let Aut $(\mathbb{D})$ be the Lie group of all automorphisms of $\mathbb{D}$ and $G_0 = \{\psi \in \text{Aut }(\mathbb{D}) : \psi(0) = 0\}$. For any $a \in \mathbb{D}$, let γ_a be the unique geodesic (all geodesics are taken in the Bergman metric [19] on $\mathbb{D}$) such that $\gamma_a(0) = 0, \gamma_a(1) = a$. Since $\mathbb{D}$ is Hermitian symmetric, there exists a unique $\varphi_a \in Aut(\mathbb{D})$ such that $\varphi_a \circ \varphi_a(z) \equiv z$ and $\gamma_a(\frac{1}{2})$ is an isolated fixed point of φ_a and φ_a is the geodesic symmetry at $\gamma_a(\frac{1}{2})$. In particular, $\varphi_a(0) = a$ and $\varphi_a(a) = 0$. If $a = 0$, then we have $\varphi_a(z) = -z$ for all z in $\mathbb{D}$. We denote by ς_a the geodesic midpoint $\gamma_a(\frac{1}{2})$ of 0 and a. Given $\psi \in Aut(\mathbb{D})$, let $a = \psi^{-1}(0)$, then we have

$$(\psi \circ \varphi_a)(0) = \psi(a) = 0,$$

thus $\psi \circ \varphi_a \in G_0$ and so there exists a unitary matrix U such that $\psi = U\varphi_a(U \in G_0)$. If $\psi \in Aut(\mathbb{D})$ has an isolated fixed point in $\mathbb{D}$, then ψ has a unique fixed point and each φ_a has ς_a as a unique fixed point. It is also not difficult to see that for any a and b in $\mathbb{D}$, there exists a unitary $U \in G_0$ such that $\varphi_b \circ \varphi_a = U\varphi_{\varphi_a(b)}$. This can be verified as follows: let $U = \varphi_b \circ \varphi_a \circ \varphi_{\varphi_a(b)}$. Then $U(0) = \varphi_b \circ \varphi_a(\varphi_a(b)) = \varphi_b(b) = 0$, thus $U \in G_0$ is unitary. It is also not difficult to check that if $a \in \mathbb{D}$, then $\varsigma_a = \frac{1-\sqrt{1-|a|^2}}{|a|^2}a$. One can also check that $k_a(\varsigma_a) = 1$ for all $a \in \mathbb{D}, U_a k_{\varsigma_a} = 1$ for all $a \in \mathbb{D}$ and $\varphi_\lambda(\varsigma_a) = \varsigma_{\varphi_\lambda(a)}$ for any $\lambda \in \mathbb{D}$ and $a \in \mathbb{D}$.

Lemma 1. *let $a \in \mathbb{D}$ and $f, g \in L^2_a(\mathbb{U}_+)$. Then*

(i) $\langle f \circ \tau_a, g \circ \tau_a \rangle = \langle t_a f, t_a g \rangle$.

(ii) The eigenvectors of R_a corresponding to distinct eigenvalues are orthogonal.

(iii) $(L \circ \tau_{\varsigma_a} \circ \tau_a)(s) = -(L \circ \tau_{\varsigma_a})(s)$.

(iv) $(L' \circ \tau_{\varsigma_a} \circ \tau_a)(t_{\varsigma_a} \circ \tau_a)t_a = (L' \circ \tau_{\varsigma_a})t_{\varsigma_a}$.

(v) There does not exist $\lambda \in \mathbb{C}$ such that $R_a = \lambda I$.

Proof. (i) Let $f, g \in L^2_a(\mathbb{U}_+)$. Then

$$\begin{aligned}\langle f \circ \tau_a, g \circ \tau_a \rangle &= \int_{\mathbb{U}_+} (f \circ \tau_a)(w)\overline{(g \circ \tau_a)(w)} d\widetilde{A}(w) \\ &= \int_{\mathbb{U}_+} f(w)\overline{g(w)}| - t_a(w)|^2 d\widetilde{A}(w) \\ &= \langle t_a f, t_a g \rangle.\end{aligned}$$

(ii) Let λ and μ be distinct eigenvalues of R_a. Suppose $R_a f = \lambda f$ and $R_a g = \mu g$. Then

$$\begin{aligned} 0 &= \langle (R_a^{*^2} R_a - 2R_a^* R_a + I) f, g\rangle \\ &= \langle R_a^2 f, R_a^2 g\rangle - 2\langle R_a f, R_a g\rangle + \langle f, g\rangle \\ &= (\lambda^2 \overline{\mu}^2 - 2\lambda\overline{\mu} + 1)\langle f, g\rangle. \end{aligned}$$

Since $\lambda \neq \mu$ with $|\lambda| = 1 = |\mu|$, we obtain $\lambda^2\overline{\mu}^2 - 2\lambda\overline{\mu} + 1 = (\frac{\lambda}{\mu} - 1)^2 \neq 0$. This leads to $\langle f, g\rangle = 0$, which proves the claim.

(iii) Notice that $\varphi_{\varsigma_a} \circ \varphi_a = -\varphi_{\varsigma_a}$, for all $a \in \mathbb{D}$. Now since $\tau_a = L^{-1} \circ \varphi_a \circ L$, we obtain

$$\begin{aligned} (L \circ \tau_{\varsigma_a} \circ \tau_a)(s) &= (L \circ L^{-1} \circ \varphi_{\varsigma_a} \circ L \circ L^{-1} \circ \varphi_a \circ L)(s) \\ &= (\varphi_{\varsigma_a} \circ \varphi_a \circ L)(s) \\ &= -(\varphi_{\varsigma_a} \circ L)(s), \text{for all } s \in \mathbb{U}_+. \end{aligned}$$

On the other hand,

$$\begin{aligned} -(L \circ \tau_{\varsigma_a})(s) &= -(L \circ L^{-1} \circ \varphi_{\varsigma_a} \circ L)(s) \\ &= -(\varphi_{\varsigma_a} \circ L)(s), \text{for all } s \in \mathbb{U}_+. \end{aligned}$$

Thus we establish (iii) for all $s \in \mathbb{U}_+$.

To prove (iv) notice that $\varphi_a^{'}(z) = -k_a(z)$, for all $z \in \mathbb{D}$ and $(k_{\varsigma_a} \circ \varphi_a)k_a = k_{\varsigma_a}$. That is, $U_a k_{\varsigma_a} = k_{\varsigma_a}$ for all $a \in \mathbb{D}$. Thus $U_a U_{\varsigma_a} 1 = U_{\varsigma_a} 1$. This implies

$$(\mathcal{W} U_a \mathcal{W}^{-1})(\mathcal{W} U_{\varsigma_a} \mathcal{W}^{-1})(L^{'}) = (\mathcal{W} U_{\varsigma_a} \mathcal{W}^{-1})(L^{'}).$$

Hence $R_a R_{\varsigma_a} L^{'} = R_{\varsigma_a} L^{'}$. Now since for all $a \in \mathbb{D}, R_a d_{\overline{w}} = \mathcal{W} U_a k_a = \mathcal{W} 1 = L^{'}$ where $w = L\overline{a}$, we obtain $V_{\varsigma_a} d_{L\varsigma_a} = L^{'}$. That is,

$$d_{L\varsigma_a} = R_{\varsigma_a}^{-1}(L^{'}) = R_{\varsigma_a} L^{'}.$$

Hence $R_a d_{L\varsigma_a} = d_{L\varsigma_a}$ and $R_a \Big((L^{'} \circ \tau_{\varsigma_a}) t_{\varsigma_a} \Big) = (L^{'} \circ \tau_{\varsigma_a}) t_{\varsigma_a}$. That is,

$$(L^{'} \circ \tau_{\varsigma_a} \circ \tau_a)(t_{\varsigma_a} \circ \tau_a) t_a = (L^{'} \circ \tau_{\varsigma_a}) t_{\varsigma_a}.$$

(iv) Suppose $R_a = \lambda I$, for some constant $\lambda \in \mathbb{C}$ for some $a \in \mathbb{D}$. Then since $R_a^2 = I$, hence $\lambda = \pm 1$. But there exists $f \in L_a^2(\mathbb{U}_+)$ such that

$R_a f \neq f$ and there exists $f \in L^2_a(\mathbb{U}_+)$ such that $R_a f \neq -f$.

Let $g = \mathcal{W}k_{\varsigma_a}$, where ς_a is the geodesics midpoint between 0 and $\frac{1}{2}$. Then $g \in L^2_a(\mathbb{U}_+)$ and it is not difficult to check that $R_a g = g$ and $R_a M' = d_{\overline{w}}, R_a d_{\overline{w}} = L', d_{\overline{w}} \neq -L'$, and $d_{\overline{w}} \neq L'$, where $d_{\overline{w}}(s) = \dfrac{1}{\sqrt{\pi}}\dfrac{w+i}{\overline{w}-i}\dfrac{(-2i)\text{Im } w}{(s+w)^2}$, for $s, w \in \mathbb{U}_+$.

□

Theorem 4. *Let $a \in \mathbb{D}$ and $f \in L^2_a(\mathbb{U}_+)$. Then*

(i) $R_a f = f$ if and only if there exists an even function $g \in L^2_a(\mathbb{D})$ such that $f = (L' \circ \tau_{\varsigma_a})t_{\varsigma_a}(g \circ L \circ \tau_{\varsigma_a})$.

(ii) $R_a f = -f$ if and only if there exists an odd function $g \in L^2_a(\mathbb{D})$ such that $f = (L' \circ \tau_{\varsigma_a})t_{\varsigma_a}(g \circ L \circ \tau_{\varsigma_a})$.

Proof. We shall only establish (i). The proof of (ii) is similar. Suppose g is even and $g \in L^2_a(\mathbb{D})$. That is, $g(z) = g(-z)$ and $f = (L' \circ \tau_{\varsigma_a})t_{\varsigma_a}(g \circ L \circ \tau_{\varsigma_a})$. Then

$$R_a f = (f \circ \tau_a)t_a = (L' \circ \tau_{\varsigma_a} \circ \tau_a)(t_{\varsigma_a} \circ \tau_a)(g \circ L \circ \tau_{\varsigma_a} \circ \tau_a)t_a$$

Since by Lemma 1, $L \circ \tau_{\varsigma_a} \circ \tau_a = -(L \circ \tau_{\varsigma_a})$ and $t_a(L' \circ \tau_{\varsigma_a} \circ \tau_a)(t_{\varsigma_a} \circ \tau_a) = (L' \circ \tau_{\varsigma_a})t_{\varsigma_a}$, we obtain

$$R_a f = (L' \circ \tau_{\varsigma_a})t_{\varsigma_a} g(-(L \circ \tau_{\varsigma_a})) = (L' \circ \tau_{\varsigma_a})t_{\varsigma_a} g(L \circ \tau_{\varsigma_a}) = f.$$

Conversely, suppose $R_a f = f$. We need to find an even function g such that

$$f = (L' \circ \tau_{\varsigma_a})t_{\varsigma_a}(g \circ L \circ \tau_{\varsigma_a}).$$

Let $g(Ls) = ((L^{-1})' \circ L)(s)t_{\varsigma_a}(s)(f \circ \tau_{\varsigma_a})(s)$. Since $t_{\varsigma_a}(s)t_{\varsigma_a}(\tau_{\varsigma_a}(s)) = 1$, we have

$$g(Ls)t_{\varsigma_a}(\tau_{\varsigma_a}(s)) = ((L^{-1})' \circ L)(s)f(\tau_{\varsigma_a}(s)).$$

Thus replacing s by $\tau_{\varsigma_a}(s)$, we obtain

$$(g \circ L \circ \tau_{\varsigma_a})(s)t_{\varsigma_a}(s) = ((L^{-1})' \circ L)(\tau_{\varsigma_a}(s))f(s)$$

and hence

$$(g \circ L \circ \tau_{\varsigma_a})(s)t_{\varsigma_a}(s)L'(\tau_{\varsigma_a}(s)) = f(s).$$

We shall now show that g is even. For any $s \in \mathbb{U}_+$,

$$\begin{aligned}
&(g \circ L \circ \tau_{\varsigma_a})(s) = ((L^{-1})' \circ L \circ \tau_{\varsigma_a})(s) t_{\varsigma_a}(\tau_{\varsigma_a}(s)) f(s) \\
&= ((L^{-1})' \circ L \circ \tau_{\varsigma_a})(s) t_{\varsigma_a}(\tau_{\varsigma_a}(s)) t_a(s)(f \circ \tau_a)(s) \\
&= ((L^{-1})' \circ L \circ \tau_{\varsigma_a})(s) t_{\varsigma_a} \\
&\quad \cdot (\tau_{\varsigma_a}(s)) t_a(s)(L' \circ \tau_{\varsigma_a} \circ \tau_a)(s) t_{\varsigma_a}(\tau_a(s))(g \circ L \circ \tau_{\varsigma_a} \circ \tau_a)(s) \\
&= ((L^{-1})' \circ L \circ \tau_{\varsigma_a})(s) t_{\varsigma_a}(\tau_{\varsigma_a}(s))(L' \circ \tau_{\varsigma_a})(s) t_{\varsigma_a}(s) g(-(L \circ \tau_{\varsigma_a}))(s) \\
&= g(-(L \circ \tau_{\varsigma_a}))(s), \qquad (1)
\end{aligned}$$

since $t_{\varsigma_a}(\tau_{\varsigma_a}(s)) t_{\varsigma_a}(s) = 1$, for all $s \in \mathbb{U}_+$ and

$$\begin{aligned}
((L^{-1})' \circ L \circ \tau_{\varsigma_a})(s)(L' \circ \tau_{\varsigma_a})(s) &= \left(\left[((L^{-1})' \circ L)L'\right] \circ \tau_{\varsigma_a}\right)(s) \\
&= \left[(1 \circ \tau_{\varsigma_a})\right](s) = 1.
\end{aligned}$$

Replacing s by $\tau_{\varsigma_a}(s)$ in (1), we get

$$g(Ls) = g(-Ls) \text{for all } s \in \mathbb{U}_+.$$

That is, $g(z) = g(-z)$ for all $z \in \mathbb{D}$. Hence g is an even function. □

Corollary 1. *Suppose $a \in \mathbb{D}$ and $f \in L^2_a(\mathbb{U}_+)$. Then $R_a f = f$ if and only if $f = (L' \circ \tau_{\varsigma_a})(g_1 \circ L \circ \tau_{\varsigma_a}) t_{\varsigma_a}$, where*

$$\begin{aligned}
(g_1 \circ L)(s) = \frac{1}{2} \Big[&((L^{-1})' \circ L)(s)(f \circ \tau_{\varsigma_a})(s) t_{\varsigma_a}(s) \\
&+ ((L^{-1})' \circ L)(-s)(f \circ \tau_{\varsigma_a})(-s) t_{\varsigma_a}(-s) \Big]
\end{aligned}$$

and $R_a f = -f$ if and only if $f = (L' \circ \tau_{\varsigma_a})(g_2 \circ L \circ \tau_{\varsigma_a}) t_{\varsigma_a}$, where

$$\begin{aligned}
(g_2 \circ L)(s) = \frac{1}{2} \Big[&((L^{-1})' \circ L)(s)(f \circ \tau_{\varsigma_a})(s) t_{\varsigma_a}(s) \\
&- ((L^{-1})' \circ L)(-s)(f \circ \tau_{\varsigma_a})(-s) t_{\varsigma_a}(-s) \Big].
\end{aligned}$$

Proof. Let $R_a = \mathcal{P}_a - \mathcal{P}_a^+$ be the spectral decomposition of R_a. Then $R_a f = f$ if and only if $\mathcal{P}_a f = f$ for any $f \in L^2_a(\mathbb{U}_+)$. Thus if M_a is the range space of $\mathcal{P}_a$, we have

$$M_a = \left\{ (L' \circ \tau_{\varsigma_a})(g \circ L \circ \tau_{\varsigma_a}) t_{\varsigma_a} : g \text{ is even} \right\}.$$

Suppose $f \in L^2_a(\mathbb{U}_+)$, then the even function g_1 satisfying $\mathcal{P}_a f = (L^{'} \circ \tau_{\varsigma_a})(g_1 \circ L \circ \tau_{\varsigma_a})t_{\varsigma_a} = f$ is given by the formula

$$(g_1 \circ L)(s) = \frac{1}{2}\Big[((L^{-1})^{'} \circ L)(s)(f \circ \tau_{\varsigma_a})(s)t_{\varsigma_a}(s) \\ +((L^{-1})^{'} \circ L)(-s)(f \circ \tau_{\varsigma_a})(-s)t_{\varsigma_a}(-s)\Big].$$

and the odd function g_2 with $\mathcal{P}^+_a f = (L^{'} \circ \tau_{\varsigma_a})(g_2 \circ L \circ \tau_{\varsigma_a})t_{\varsigma_a} = f$ is given by the formula

$$(g_2 \circ L)(s) = \frac{1}{2}\Big[((L^{-1})^{'} \circ L)(s)(f \circ \tau_{\varsigma_a})(s)t_{\varsigma_a}(s) \\ -((L^{-1})^{'} \circ L)(-s)(f \circ \tau_{\varsigma_a})(-s)t_{\varsigma_a}(-s)\Big].$$

These formulas are obtained by using the identity $\mathcal{P}_a = \frac{1}{2}(I + R_a)$ and Theorem 4. □

Theorem 5. *Let $T \in \mathcal{L}(L^2_a(\mathbb{U}_+))$. If $TR_a = R_aT$ for some $a \in \mathbb{D}$, then $M_a = \Big\{(L^{'} \circ \tau_{\varsigma_a})t_{\varsigma_a}(g \circ L \circ \tau_{\varsigma_a}) : g$ is even$\Big\}$ is a reducing subspace of T.*

Proof. Let $TR_a = R_aT$ for some $a \in \mathbb{D}$. Let $R_a = \mathcal{P}_a - \mathcal{P}^{\perp}_a$ be the spectral decomposition of R_a. Then $R_a f = f$ if and only if $\mathcal{P}_a f = f$ for $f \in L^2_a(\mathbb{U}_+)$. It follows from Theorem 4 that $R_a f = f$ if and only if there exists an even function $g \in L^2_a(\mathbb{D})$ such that $f = (L^{'} \circ \tau_{\varsigma_a})t_{\varsigma_a}(g \circ L \circ \tau_{\varsigma_a})$. Thus if M_a is the range space of $\mathcal{P}_a$, we have $M_a = \Big\{(L^{'} \circ \tau_{\varsigma_a})t_{\varsigma_a}(g \circ L \circ \tau_{\varsigma_a}) : g \text{ is even}\Big\}$. Now $TR_a = R_aT$ for some $a \in \mathbb{D}$ if and only if $T\mathcal{P}_a = \mathcal{P}_aT$. This is true if and only if M_a is a reducing subspace of T. □

4 Toeplitz operators and their distance from R_a

In this section we showed that the Toeplitz operator T_G defined on $L^2_a(\mathbb{U}_+)$ with symbol $G \in h^\infty(\mathbb{U}_+)$ is positive if and only if the symbol $G \geq 0$. We showed that if $\varphi \geq 0$ and $\varphi \in h^\infty(\mathbb{U}_+)$ and $||R_a - T_\varphi|| < 1$, for some $a \in \mathbb{D}$, then T_G is invertible. Further we showed that if $\varphi \geq 0, \varphi \in h^\infty(\mathbb{U}_+)$ and $||R_a - T_\varphi|| \leq 1$, for some $a \in \mathbb{D}$ then T_φ is not invertible if and only if $||I - T_\varphi|| = ||I - \frac{T_\varphi}{2}|| = 1$.

Let $Wh^\infty(\mathbb{D}) = h^\infty(\mathbb{U}_+)$, where $h^\infty(\mathbb{D})$ is the space of all bounded harmonic functions on $\mathbb{D}$.

Theorem 6. *Let $G \in h^{\infty}(\mathbb{U}_+)$. Then $T_G \geq 0$ if and only if $G \geq 0$.*

Proof. First we shall show that if $u \in h^{\infty}(\mathbb{D})$, then $u(\mathbb{D}) \subset NR(\mathcal{T}_u)$, the numerical range of the Toeplitz operator $\mathcal{T}_u$ defined on $L_a^2(\mathbb{D})$.

Let k_a be the normalized reproducing kernel of $L_a^2(\mathbb{D})$. Now $\langle \mathcal{T}_u k_a, k_a\rangle \in NR(\mathcal{T}_u)$ for all $a \in \mathbb{D}$. Thus $\langle \mathcal{T}_u k_a, k_a\rangle = \langle u k_a, k_a\rangle = \int_{\mathbb{D}} u|k_a|^2 dA = \int_{\mathbb{D}} (u \circ \varphi_a) dA = u(a)$ for all $a \in \mathbb{D}$. Hence $u(\mathbb{D}) \subset NR(\mathcal{T}_u)$. Now we proceed to verify that if $u \in h^{\infty}(\mathbb{D})$, then $\mathcal{T}_u \geq 0$ if and only if $u \geq 0$. The operator $\mathcal{T}_u \geq 0$ if and only if $\langle \mathcal{T}_u f, f\rangle \geq 0$ for all $f \in L_a^2(\mathbb{D})$. Thus if $\mathcal{T}_u \geq 0$ then $NR(\mathcal{T}_u) \subset [0,\infty)$. From the first part of the proof, it follows that $u(\mathbb{D}) \subset NR(\mathcal{T}_u) \subset [0,\infty)$. Hence $u \geq 0$. Now assume $u \geq 0$. Then $\langle \mathcal{T}_u f, f\rangle = \langle P(uf), f\rangle = \langle uf, f\rangle = \int_{\mathbb{D}} u|f|^2 dA \geq 0$, for every $f \in L_a^2(\mathbb{D})$. Hence $\mathcal{T}_u \geq 0$. We shall now verify that if $G \in h^{\infty}(\mathbb{U}_+)$, the Toeplitz operator T_G defined on $L_a^2(\mathbb{U}_+)$ with symbol G is unitarily equivalent to the Toeplitz operator $\mathcal{T}_\varphi$ defined on $L_a^2(\mathbb{D})$ with symbol $\varphi(z) = G\left(i\frac{1-z}{1+z}\right) = (G \circ L^{-1})(z)$.

The operator $\mathcal{W}$ maps $\sqrt{n+1}z^n$ to the function $\frac{2i}{\sqrt{\pi}}\sqrt{n+1}\left(\frac{i-s}{i+s}\right)^n \frac{1}{(i+s)^2}$ which belongs to $L_a^2(\mathbb{U}_+)$. The Toeplitz operator T_G maps this vector to $P_+\left(G(s)\frac{2i}{\sqrt{\pi}}\sqrt{n+1}\left(\frac{i-s}{i+s}\right)^n \frac{1}{(i+s)^2}\right)$ which is equal to

$$\mathcal{W}P\mathcal{W}^{-1}(G(s)\frac{2i}{\sqrt{\pi}}\sqrt{n+1}\left(\frac{i-s}{i+s}\right)^n \frac{1}{(i+s)^2}).$$

Now

$$\begin{aligned}
&\mathcal{W}P\mathcal{W}^{-1}\left(G(s)\frac{2i}{\sqrt{\pi}}\sqrt{n+1}\left(\frac{i-s}{i+s}\right)^n \frac{1}{(i+s)^2}\right)\\
&= \mathcal{W}P\left(\mathcal{W}^{-1}\left(G(s)\frac{2i}{\sqrt{\pi}}\sqrt{n+1}\left(\frac{i-s}{i+s}\right)^n \frac{1}{(i+s)^2}\right)\right)\\
&= \frac{2i}{\sqrt{\pi}}\sqrt{n+1}\mathcal{W}P\left(2i\sqrt{\pi}G(L^{-1}z)\frac{1}{(1+z)^2}\left(L(L^{-1}z)\right)^n \frac{1}{(i+L^{-1}z)^2}\right)\\
&= 2i\sqrt{\pi}\frac{2i}{\sqrt{\pi}}\sqrt{n+1}\mathcal{W}P\left(G\left(i\frac{1-z}{1+z}\right)z^n\frac{1}{(1+z)^2}\frac{1}{\left(i+i\frac{1-z}{1+z}\right)^2}\right)
\end{aligned}$$

$$
\begin{aligned}
&= (-4)\mathcal{W}P\left(G\left(i\frac{1-z}{1+z}\right)z^n\sqrt{n+1}\frac{1}{(1+z)^2}\frac{(1+z)^2}{(i(1+z)+i(1-z))^2}\right)\\
&= \mathcal{W}P\left(G\left(i\frac{1-z}{1+z}\right)z^n\sqrt{n+1}\right)\\
&= \mathcal{W}\mathcal{T}_{\varphi}\left(z^n\sqrt{n+1}\right),
\end{aligned}
$$

where $\varphi(z) = G\left(i\frac{1-z}{1+z}\right) = \left(G \circ L^{-1}\right)(z)$. Since the sequence of vectors $\{\sqrt{n+1}z^n\}_{n=0}^{\infty}$ forms an orthonormal basis for $L_a^2(\mathbb{D})$, this proves our claim. Thus $\mathcal{T}_{G\circ L^{-1}}$ is unitarily equivalent to T_G defined on $L_a^2(\mathbb{U}_+)$. We have already shown $\mathcal{T}_{G\circ L^{-1}} \geq 0$ if and only if $G \circ L^{-1} \geq 0$ on $\mathbb{D}$. Now $\mathcal{T}_{G\circ L^{-1}} \geq 0$ if and only if $\langle \mathcal{T}_{G\circ L^{-1}}g, g\rangle \geq 0$ for all $g \in L_a^2(\mathbb{D})$. That is, if $\langle P(G\circ L^{-1})g, g\rangle \geq 0$ for all $g \in L_a^2(\mathbb{D})$. But

$$
\begin{aligned}
\langle P(G \circ L^{-1})g, g\rangle &= \langle (G \circ L^{-1})g, g\rangle\\
&= \int_{\mathbb{D}} (G \circ L^{-1})(z)|g(z)|^2 dA(z)\\
&= \int_{\mathbb{U}_+} G(s)|(g \circ L)(s)|^2 |L^{'}(s)|^2 d\tilde{A}(s)\\
&= \int_{\mathbb{U}_+} G(s)|\mathcal{W}g(s)|^2 d\tilde{A}(s)\\
&= \langle G\mathcal{W}g, \mathcal{W}g\rangle\\
&= \langle Gf, f\rangle, \text{where } f = \mathcal{W}g \in L_a^2(\mathbb{U}_+)\\
&= \langle T_G f, f\rangle
\end{aligned}
$$

Thus $\langle \mathcal{T}_{G\circ L^{-1}}g, g\rangle \geq 0$ for all $g \in L_a^2(\mathbb{D})$ if and only if $T_G \geq 0$ on $L_a^2(\mathbb{U}_+)$. On the other hand $(G \circ L^{-1})(z) \geq 0$ for all $z \in \mathbb{D}$ if and only if $G(s) \geq 0$ for all $s \in \mathbb{U}_+$. Thus we showed that $T_G \geq 0$ if and only if $G \geq 0$ on $\mathbb{U}_+$. □

Lemma 2. *Suppose $\varphi \in L^{\infty}(\mathbb{U}_+)$. The following hold:*

(i) If $\varphi \geq 0, \varphi \in h^{\infty}(\mathbb{U}_+)$ and $||R_a - T_{\varphi}|| < 1$, for some $a \in \mathbb{D}$ then T_{φ} is invertible and $||I - T_{\varphi}|| \leq ||R_a - T_{\varphi}|| \leq ||I + T_{\varphi}||$, where $I \in \mathcal{L}(L_a^2(\mathbb{U}_+))$ is the identity operator on $L_a^2(\mathbb{U}_+)$.

(ii) If $\varphi \geq 0, \varphi \in h^{\infty}(\mathbb{U}_+)$ and $||R_a - T_{\varphi}|| \leq 1$, for some $a \in \mathbb{D}$ then T_{φ} is not invertible if and only if $||I - T_{\varphi}|| = ||I - \frac{T_{\varphi}}{2}|| = 1$.

(iii) If for some $a \in \mathbb{D}, ||R_a - T_{\varphi}|| \leq 1$ then $||T_{\varphi}|| \leq \rho(R_a T_{\varphi}) + \frac{1}{2}$.

(iv) Let $T \in \mathcal{L}(L_a^2(\mathbb{U}_+))$. If for some $a \in \mathbb{D}, ||T - R_a|| \leq \epsilon < 1$, then the operator T is invertible and $||T - T|T|^{-1}|| < \frac{\epsilon(1+\epsilon)}{1-\epsilon}$.

Proof. (i) If $f \in L^2_a(\mathbb{U}_+)$ and $||f|| = 1$, then

$$\begin{aligned}
||(R_a - T_\varphi)f||^2 &= \langle (R_a - T_\varphi)f, (R_a - T_\varphi)f\rangle \\
&= ||R_a f||^2 + ||T_\varphi f||^2 - \langle R_a f, T_\varphi f\rangle - \langle T_\varphi f, R_a f\rangle \\
&= ||f||^2 + ||T_\varphi f||^2 - \langle f, R_a T_\varphi f\rangle - \langle R_a T_\varphi f, f\rangle \\
&= 1 + ||T_\varphi f||^2 - \langle f, R_a T_\varphi f\rangle - \langle R_a T_\varphi f, f\rangle \\
&= 1 + ||T_\varphi f||^2 - \overline{\langle R_a T_\varphi f, f\rangle} - \langle R_a T_\varphi f, f\rangle \\
&= 1 + ||T_\varphi f||^2 - 2Re\langle R_a T_\varphi f, f\rangle \\
&\geq 1 + ||T_\varphi f||^2 - 2|\langle R_a T_\varphi f, f\rangle| \\
&\geq 1 + ||T_\varphi f||^2 - 2||R_a T_\varphi f||\ ||f|| \\
&= 1 + ||T_\varphi f||^2 - 2||T_\varphi f||\ ||f|| \\
&= 1 + ||T_\varphi f||^2 - 2||T_\varphi f|| \\
&= (1 - ||T_\varphi f||)^2 .
\end{aligned}$$

Now since $T_\varphi \geq 0$,

$$\inf_{||f||=1} \langle T_\varphi f, f\rangle = \inf_{||f||=1} ||T_\varphi f|| \text{ and } \sup_{||f||=1} \langle T_\varphi f, f\rangle = \sup_{||f||=1} ||T_\varphi f||$$

and we have

$$\begin{aligned}
||R_a - T_\varphi|| &= \sup_{||f||=1} ||(R_a - T_\varphi)f|| \\
&\geq \sup_{||f||=1} |\ 1 - ||T_\varphi f||\ | \\
&= \sup_{||f||=1} |\ 1 - \langle T_\varphi f, f\rangle\ | \\
&= \sup_{||f||=1} |\ \langle (I - T_\varphi)f, f\rangle\ | \\
&= ||I - T_\varphi||.
\end{aligned}$$

Thus if $||R_a - T_\varphi|| < 1$, then $||I - T_\varphi|| < 1$ and this implies T_φ is invertible [2].
Further,

$$\begin{aligned}
||R_a - T_\varphi|| &= \sup_{||f||=1} ||R_a f - T_\varphi f|| \\
&\leq \sup_{||f||=1} (1 + ||T_\varphi f||) \\
&= \sup_{||f||=1} ||\langle (I + T_\varphi)f, f\rangle|| \\
&= ||I + T_\varphi||.
\end{aligned}$$

(ii) Since $||R_a - T_\varphi|| \leq 1$, hence from the argument of (i) it follows that, $||I - T_\varphi|| \leq 1$ and $||I - \frac{T_\varphi}{2}|| \leq ||\frac{I}{2}|| + ||\frac{I}{2} - \frac{T_\varphi}{2}|| \leq \frac{1}{2} + \frac{1}{2} = 1$, by convexity. Now if T_φ is not invertible then it follows from Neumann Lemma [18] that $||I - T_\varphi|| = 1$ and $||I - \frac{T_\varphi}{2}|| = 1$. Conversely, if $||I - T_\varphi|| = ||I - \frac{T_\varphi}{2}|| = 1$, then by the parallelogram law [14], we obtain

$$\left|\left|\frac{1}{2}T_\varphi f\right|\right|^2 + \left|\left|\left(I - \frac{T_\varphi}{2}\right) f\right|\right|^2 = 2\left|\left|\frac{1}{2}f\right|\right|^2 + 2\left|\left|\frac{1}{2}(I - T_\varphi)f\right|\right|^2 \leq 1$$

for $||f|| \leq 1$. Hence $I - \frac{T_\varphi}{2}$ approximately achieves its norm at some norm one vector f with $||T_\varphi f||$ as close as we wish to 0. Hence T_φ is not invertible.

(iii) The condition $||R_a - T_\varphi|| \leq 1$ is equivalent to the inequality

$$||T_\varphi f||^2 + 1 \leq 2Re\langle (R_a T_\varphi)f, f\rangle + 1,$$

for any $f \in L^2_a(\mathbb{U}_+)$ with $||f|| = 1$. Since $||R_a g||^2 = ||g||^2$, for $g \in L^2_a(\mathbb{U}_+)$, and $Re\langle (R_a T_\varphi)f, f\rangle \leq |\langle (R_a T_\varphi)f, f\rangle|$, hence

$$||T_\varphi f||^2 + 1 \leq 2|\langle (R_a T_\varphi)f, f\rangle| + 1$$

for any $f \in L^2_a(\mathbb{U}_+)$ with $||f|| = 1$. Hence

$$\sup_{f \in L^2_a(\mathbb{U}_+), ||f||=1} ||T_\varphi f||^2 + 1 \leq 2 \sup_{f \in L^2_a(\mathbb{U}_+), ||f||=1} \langle (R_a T_\varphi)f, f\rangle + 1.$$

Thus $||T_\varphi||^2 + 1 \leq 2\rho(R_a T_\varphi) + 1$. Since $2||T|| \leq ||T||^2 + 1$, for $T \in \mathcal{L}(L^2_a(\mathbb{U}_+))$, we obtain

$$2||T_\varphi|| \leq ||T_\varphi||^2 + 1 \leq 2\rho(R_a T_\varphi) + 1.$$

That is,

$$||T_\varphi|| \leq \rho(R_a T_\varphi) + \frac{1}{2}.$$

(iv) If $T = R_a + S$ with $||S|| \leq \epsilon$ then

$$(1 - \epsilon)^2 I \leq T^* T \leq (1 + \epsilon)^2 I.$$

Thus it follows that T is invertible. Since $(1 + \epsilon)^{-1} I \leq |T|^{-1} \leq (1 - \epsilon)^{-1} I$ and $||T|| \leq (1 + \epsilon)$, we obtain

$$\begin{aligned} ||T - T|T|^{-1}|| &\leq (1 + \epsilon)||I - |T|^{-1}|| \\ &\leq (1 + \epsilon)\epsilon(1 - \epsilon)^{-1}. \end{aligned}$$

$\square$

Theorem 7. *If $\varphi \in h^\infty(\mathbb{U}_+)$, then $\varphi \geq 0, ||\varphi||_\infty \leq 1$ if and only if $||I - T_\varphi^n|| \leq 1$ for all $n \in \mathbb{N}$.*

Proof. If $\varphi \geq 0$, then by Theorem 6 T_φ is positive. Now since $\varphi \in h^\infty(\mathbb{U}_+)$ and $||T_\varphi|| = ||\varphi||_\infty \leq 1$, hence T_φ is a contraction. That is, $0 \leq T_\varphi \leq I$. Hence $||I - T_\varphi^n|| \leq 1$ for all $n \in \mathbb{N}$. To prove the converse, let $f \in L_a^2(\mathbb{U}_+)$ and $||f|| = 1$. Thus $|1 - \langle T_\varphi^n f, f\rangle| = |\langle (I - T_\varphi^n) f, f\rangle| \leq ||I - T_\varphi^n|| \leq 1$. So $NR(T_\varphi^n)$ is contained in the closed right half plane for all $n \in \mathbb{N}$. Hence T_φ is positive. Now if there exists a positive number $r \in \sigma(T_\varphi)$ (the spectrum of T_φ) such that $r > 1$, then there exists a positive integer n such that $r^n > 2$. Thus the spectral radius of $I - T_\varphi^n$ is greater than 1. This contradicts the assumption that $||I - T_\varphi^n|| \leq 1$. Hence $0 \leq T_\varphi \leq I$. □

Theorem 8. *If T_φ^N and T_φ^{N+1} are positive contractions for some positive integer N then $||I - T_\varphi^n|| \leq 1$ for all $n \geq N; n \in \mathbb{N}$.*

Proof. Notice that ker $T_\varphi^N =$ ker T_φ^{N+1}, as ker $T_\varphi^N \subset$ ker $T_\varphi^{N+1} \subset \cdots \subset$ ker $T_\varphi^{2N} =$ ker $(T_\varphi^N)^* T_\varphi^N =$ ker T_φ^N. Decompose the Bergman space $L_a^2(\mathbb{U}_+)$ as $L_a^2(\mathbb{U}_+) = \ker T_\varphi^N \oplus (\ker T_\varphi^N)^\perp = \ker T_\varphi^N \oplus \overline{\text{Range } T_\varphi^N}$. Then T_φ, T_φ^N and T_φ^{N+1} have the form $T_\varphi = \begin{pmatrix} T_1^\varphi & T_2^\varphi \\ T_3^\varphi & T_4^\varphi \end{pmatrix}, T_\varphi^N = \begin{pmatrix} 0 & 0 \\ 0 & T_5^\varphi \end{pmatrix}$ and $T_\varphi^{N+1} = \begin{pmatrix} 0 & 0 \\ 0 & T_6^\varphi \end{pmatrix}$ with respect to the above decompositions of $L_a^2(\mathbb{U}_+)$. Thus $T_\varphi^{N+1} = T_\varphi T_\varphi^N = \begin{pmatrix} 0 & T_2^\varphi T_5^\varphi \\ 0 & T_4^\varphi T_5^\varphi \end{pmatrix} = T_\varphi^N T_\varphi = \begin{pmatrix} 0 & 0 \\ T_5^\varphi T_3^\varphi & T_5^\varphi T_4^\varphi \end{pmatrix}$. Hence $T_2^\varphi T_5^\varphi = 0, T_5^\varphi T_3^\varphi = 0$ and $T_4^\varphi T_5^\varphi = T_5^\varphi T_4^\varphi = T_6^\varphi = T_6^{\varphi^*} = (T_5^\varphi T_4^\varphi)^* = T_4^{\varphi^*} T_5^\varphi$. Since $\overline{\text{Range } T_5^\varphi} = (\ker T_5^\varphi)^\perp = (\ker T_\varphi^N)^\perp$, we obtain $T_2^\varphi = 0, T_3^\varphi = 0$, and $T_4^\varphi = T_4^{\varphi^*}$. Thus $(T_1^\varphi)^N = 0$. Since T_φ^N and T_φ^{N+1} are positive contractions, we obtain T_φ is a positive contraction. □

Theorem 9. *Let T be a proper contraction on $L_a^2(\mathbb{U}_+)$ with $TR_a = R_a T$ for some $a \in \mathbb{D}$. Then*

$$\left|\left|R_a(I+T)(I-T)^{-1} - \frac{1+r^2}{1-r^2} R_a\right|\right| \leq \frac{2r}{1-r^2} \tag{2}$$

if and only if $||T|| \leq r$.

Proof. Assume $TR_a = R_a T$ for some $a \in \mathbb{D}$. Notice that R_a is unitary and the inequality (2) is equivalent to

$$\left|\left|(I+T)(I-T)^{-1} - \frac{1+r^2}{1-r^2}\right|\right| \leq \frac{2r}{1-r^2}. \tag{3}$$

Let $S = (I+T)(I-T)^{-1} - \frac{1+r^2}{1-r^2}I$ and $t = \frac{2r}{1-r^2}$. Then if (2) holds, that is, if $||S|| \leq t$ then $||SR_a|| \leq ||S||\ ||R_a|| \leq t$ as $||R_a|| = 1$. Therefore (2) holds. Conversely, if (2) holds, i.e. $||SR_a|| \leq t$, we get (3) using the fact that R_a is unitary. Thus (2) and (3) are equivalent. Since (3) is equivalent to

$$\left[(I-T^*)^{-1}(I+T^*) - \frac{1+r^2}{1-r^2}I\right]\left[(I+T)(I-T)^{-1} - \frac{1+r^2}{1-r^2}I\right] \leq \frac{4r^2}{(1-r^2)^2}I,$$

we obtain

$$(1-r^2)^2(I+T^*)(I+T) + (1+r^2)^2(I-T^*)(I-T) - 2(1-r^4)(I-T^*T) \leq 4r^2(I-T^*)(I-T).$$

Thus

$$(1-r^2)(I+T^*T) \leq (1+r^2)(I-T^*T).$$

That is, $T^*T \leq r^2 I$ and hence $||T|| \leq r$. The result follows. □

5 Schatten Class and Frechet Derivative

In this section we deal with Schatten class operators defined on $L_a^2(\mathbb{U}_+)$. We showed that if $\varphi \in L^\infty(\mathbb{U}_+)$ and T_φ is invertible with the polar decomposition $T_\varphi = U|T_\varphi|$, then for all $a \in \mathbb{D}$ and for every $A \in \mathcal{L}(L_a^2(\mathbb{U}_+))$ with $UA = AU$, the inequality $||(U-T_\varphi)A||_2 \leq ||(R_a - T_\varphi)A||_2 \leq ||(U+T_\varphi)A||_2$ holds. Further if $T \in \mathcal{L}(L_a^2(\mathbb{U}_+))$ is positive and $R_a - T$ is compact for some $a \in \mathbb{D}$, then $I - T$ is compact and if $R_a - T \in S_p(0 < p \leq \infty)$ for some $a \in \mathbb{D}$, then $I - T \in S_p$. We also established that if for some $a \in \mathbb{D}, R_a$ is a local maximum or a local minimum of $O_p = ||U-T||_p^p, p > 1$ where U is unitary, $T > 0$, then $M_a = \{(L' \circ \tau_{\varsigma_a})t_{\varsigma_a}(g \circ L \circ \tau_{\varsigma_a}) : g \in L_a^2(\mathbb{D}),\ \text{g is even}\}$ is a reducing subspace of T and $||I-T||_p < ||R_a - T||_p$.

For any non-negative integer n, the nth singular value of $T \in \mathcal{LC}(\mathcal{H})$ is defined by

$$s_n(T) = \inf\{||T-K||, K \in \mathcal{LC}(\mathcal{H}),\ rank\ K \leq n\}.$$

Here $||.||$ is the operator norm. Clearly $s_0(T) = ||T||$ and $s_0(T) \geq s_1(T) \geq s_2(T) \geq \cdots \geq 0$. The Schatten Von Neumann class $S_p = S_p(\mathcal{H}), 0 < p < \infty$, consists of all operators $T \in \mathcal{LC}(\mathcal{H})$ such that

$$||T||_{S_p} = \left(\sum_{n=0}^{\infty}(s_n(T))^p\right)^{\frac{1}{p}} < \infty.$$

If $1 \leq p < \infty$, then $||.||_{S_p}$ is a norm, which makes S_p a Banach space. For $p < 1, ||.||_{S_p}$ does not satisfy the triangle inequality, it is a quasinorm (i.e., $||T_1+T_2||_{S_p} \leq C(||T_1||_{S_p}+||T_2||_{S_p})$ for $T_1, T_2 \in S_p$ and C, a constant), which makes S_p a quasi-Banach space. The space S_1 is called the trace-class of $\mathcal{H}$ and S_2 is called the Hilbert-Schmidt class.

If $T \in S_1$ and $\{\varepsilon_i\}$ is any orthonormal basis for the Hilbert space $\mathcal{H}$ then the quantity trace(T) defined by trace$(T) = \sum_{i=1}^{\infty}\langle T\varepsilon_i, \varepsilon_i\rangle$ is independent of the choice of $\{\varepsilon_i\}$ and is called the trace of T. The Hilbert schmidt norm of T is defined as,

$$||T||_2 = \left(\sum_{n=0}^{\infty}||Te_n||^2\right)^{\frac{1}{2}} = \left(\sum_{n,m=0}^{\infty}|\langle Te_n, f_m\rangle|^2\right)^{\frac{1}{2}},$$

where $\{e_n\}_{n=0}^{\infty}$ and $\{f_m\}_{m=0}^{\infty}$ are any two orthonormal bases for $L_a^2(\mathbb{U}_+)$. Notice that if $T \in S_p$ then $||T||_p^p = \text{trace}(|T|^p)$. It has been shown by McCarthy [15] that $S_p(1 < p < \infty)$ is uniformly convex. It also follows from [16] that S_p has Frechet differentiable norm and the map $T \mapsto ||T||_p^p$ is differentiable.

Let S be any bounded linear operator on a Hilbert space $\mathcal{H}$. Then S can be expressed uniquely [2](polar decomposition) as $S = U_1P_1$ where P_1 is a positive operator and U_1 is a partial isometry and kerU_1 = kerP_1. If S is self-adjoint then U_1 is self-adjoint and commutes with P_1.

In the following result we show that the nearest and farthest unitary operators to and from an arbitrary positive Toeplitz operator are I and $-I$ respectively.

Theorem 10. *Let $\varphi \geq 0, \varphi \in h^{\infty}(\mathbb{U}_+)$. Then for every unitary operator $R_a \in \mathcal{L}(L_a^2(\mathbb{U}_+)), a \in \mathbb{D}$, and for every $A \in \mathcal{L}(L_a^2(\mathbb{U}_+))$,*

$$||(I - T_\varphi)A||_2 \leq ||(R_a - T_\varphi)A||_2 \leq ||(I + T_\varphi)A||_2. \tag{4}$$

Proof. Since $\varphi \geq 0$, the Toeplitz operator T_φ is positive. If T_φ is a positive diagonal operator and if R_a is unitary diagonal operator then (4) follows from the following scalar inequalities

$$|a-1| \leq |a-z| \leq |a+1|$$

for every $a \geq 0$ and for every z with $|z| = 1$. Now the general case as claimed in the theorem follows using Voiculescu pertubation theorem [1]. $\square$

The following result shows that the nearest and farthest unitary operators to an arbitrary invertible operator T are U and $-U$, respectively, where U is the unitary factor occurring in the polar decomposition of T.

Theorem 11. *Let $\varphi \in L^\infty(\mathbb{U}_+)$ and suppose $T_\varphi \in \mathcal{L}(L^2_a(\mathbb{U}_+))$ is invertible with the polar decomposition $T_\varphi = U|T_\varphi|$. Then for all $a \in \mathbb{D}$, and for every $A \in \mathcal{L}(L^2_a(\mathbb{U}_+))$ with $UA = AU$, the inequality $||(U - T_\varphi)A||_2 \leq ||(R_a - T_\varphi)A||_2 \leq ||(U + T_\varphi)A||_2$ holds.*

Proof. Applying Theorem 10 to the positive operator $|T_\varphi|$ and the unitary operator U^*R_a, we obtain

$$||\ |T_\varphi|A - A\ ||_2 \leq ||\ |T_\varphi|A - AU^*R_a\ ||_2 \leq ||\ |T_\varphi|A + A\ ||_2.$$

Since $||.||_2$ is unitarily invariant and since $UA = AU$ and $U^*A = AU^*$, it follows that

$$||T_\varphi A - AU||_2 = ||U|T_\varphi|A - AU||_2 = ||\ |T_\varphi|A - U^*AU\ ||_2 = ||\ |T_\varphi|A - A\ ||_2,$$

$$\begin{aligned} ||T_\varphi A - AR_a||_2 &= ||U|T_\varphi|A - AR_a||_2 = ||\ |T_\varphi|A - U^*AR_a\ ||_2 \\ &= ||\ |T_\varphi|A - AU^*R_a\ ||_2, \end{aligned}$$

and

$$||T_\varphi A + AU||_2 = ||U|T_\varphi|A + AU||_2 = ||\ |T_\varphi|A + U^*AU\ ||_2 = ||\ |T_\varphi|A + A\ ||_2.$$

Thus,

$$||T_\varphi A - AU||_2 \leq ||T_\varphi A - AR_a||_2 \leq ||T_\varphi A + AU||_2.$$

This completes the proof of the theorem. $\square$

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and $f : \mathbb{T} \to \mathbb{C}$ be a sufficiently smooth function of the form

$$f(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n)z^n,\ z \in \mathbb{T},$$

where $\hat{f}(n) = \frac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(e^{i\theta})e^{-in\theta}d\theta$.

Theorem 12. *Let f be a complex-valued function defined on $\mathbb{T}$ such that $m = \displaystyle\sum_{n=-\infty}^{\infty} |n\hat{f}(n)| < \infty$. Then*

$$||f(R_{a_1}) - f(R_{a_2})||_p \leq m\ ||R_{a_1} - R_{a_2}||_p,$$

for all $a_1, a_2 \in \mathbb{D}$ and $1 \leq p \leq \infty$.

Proof. Clearly, if $n > 0$, then

$$R_{a_1}^n - R_{a_2}^n = \sum_{k=0}^{n-1} R_{a_1}^n (R_{a_1} - R_{a_2}) R_{a_2}^{n-1-k},$$

and so

$$||R_{a_1}^n - R_{a_2}^n||_p \leq \sum_{k=0}^{n-1} ||R_{a_1}^k||\ ||R_{a_1} - R_{a_2}||_p\ ||R_{a_2}^{n-1-k}|| = n||R_{a_1} - R_{a_2}||_p.$$

For $n < 0$,

$$\begin{aligned} ||f(R_{a_1}) - f(R_{a_2})||_p &\leq \sum_{n=-\infty}^{\infty} |\hat{f}(n)|\ ||R_{a_1}^n - R_{a_2}^n||_p \\ &\leq \sum_{n=-\infty}^{\infty} |n\hat{f}(n)|\ ||R_{a_1} - R_{a_2}||_p = m||R_{a_1} - R_{a_2}||_p. \end{aligned}$$

The result follows. $\square$

Lemma 3. *If $p > 1$, define the map $\Xi : S_p \to \mathbb{R}$ as $\Xi(S) = ||S||_p^p$. The map Ξ is Frechet differentiable with derivative D_S at S and is given by*

$$\begin{aligned} D_S(T) &= \frac{1}{2} p\ trace\left(|S|^{p-1}U^*T + T^*U|S|^{p-1}\right) \\ &= p\ Re\left[trace\left(|S|^{p-1}U^*T\right)\right] \end{aligned}$$

where $|S|$ is the positive square root of S^*S and $S = U|S|$ is the polar decomposition of S.

Proof. Let $S, T \in S_p, 1 < p < \infty$. Let Q be a projection such that Range Q is a reducing subspace of S. Notice that the operator $2Q - I$ is unitary and

$$(2Q - I)[S + QT(I - Q)](2Q - I) = S - QT(I - Q).$$

Hence

$$||S + QT(I - Q)||_p^p = ||S - QT(I - Q)||_p^p.$$

Thus

$$D_S[QT(I - Q)]^p = p\ Re\left(||\ |S|^{p-2}QS^*T - |S|^{p-2}QS^*TQ\ ||_p^p\right) = 0.$$

Now let $S \geq 0$. Then $S = \sum_{i=1}^{\infty} \lambda_i(\varepsilon_i \otimes \varepsilon_i)$ where $\lambda_i \geq 0$ and (ε_i) is an orthonormal basis for H. Let Q_i be the projection onto sp$\{\varepsilon_i\}$ and let $C_n = I - \sum_{i=1}^{n} Q_i$. Since $D_S(Q_1TC_1) = D_S(C_1TQ_1) = 0$, we obtain $D_S(T) = D_S(Q_1TQ_1) + D_S(C_1TC_1)$.

Repeating the above argument and using mathematical induction, it is not difficult to see that for any integer n,

$$D_S(T) = \sum_{i=1}^{n} D_S(Q_iTQ_i) + D_S(F_nTF_n).$$

Further, notice that

$$\begin{aligned} D_S(Q_iTQ_i) &= p\ Re[\lambda_i^{p-1}\langle T\varepsilon_i, \varepsilon_i\rangle] \\ &= p\ Re[\langle S^{p-1}T\varepsilon_i, \varepsilon_i\rangle]. \end{aligned}$$

This can be verified by observing that $||S + tQ_iTQ_i||_p^p = |\lambda_i + t\langle T\varepsilon_i, \varepsilon_i\rangle|^p + \sum_{j\neq i} \lambda_j^p$ and evaluating $\frac{d}{dt}\,||S + tQ_iTQ_i||_p^p\ \Big|_{t=0}$. Thus

$$D_S(T) = p\sum_{i=1}^{n} Re\langle S^{p-1}T\varepsilon_i, \varepsilon_i\rangle + D_S(C_nTC_n).$$

Since C_n converges strongly to 0 as $n \to \infty$, it follows from [5] that (C_nTC_n) converges to 0 in S_p. Since D_S is continuous, $(D_S(C_nTC_n)) \to 0$ and we see that

$$\begin{aligned} D_S(T) &= p\sum_{i=1}^{\infty} Re\langle S^{p-1}T\varepsilon_i, \varepsilon_i\rangle \\ &= p\ Re\ \text{trace}(S^{p-1}T)\ ,\text{when } S \geq 0. \end{aligned}$$

Now let $S \in S_p$ and $S = U|S|$ be its polar decomposition. By definition of partial isometry there exists K such that either K or K^* is an isometry and such that K and U coincide on $(\ker|S|)^{\perp}$. Thus $S = K|S|$. If K^* is an isometry then for any $T \in S_p$ we have

$$|S+T|^2 = |S|^2 + |S|K^*T + T^*K|S| + T^*KK^*T = |\ |S| + K^*T|^2\,,$$

and so

$$||S+T||_p^p = ||\ |S| + K^*T\ ||_p^p.$$

Therefore $D_S(T) = D_{|S|}(K^*T)$. Hence

$$D_S(T) = D_{|S|}(K^*T) = p\ Re\ \text{trace}(|S|^{p-1}K^*T).$$

When K is an isometry, taking the adjoint and proceeding similarly we obtain

$$D_S(T) = D_{|S^*|}(KT^*) = p\ Re\ \text{trace}(|S^*|^{p-1}KT^*).$$

Since $|S^*|^{p-1} = K|S|^{p-1}K^*$, the result follows. □

Theorem 13. *Let $T \in \mathcal{L}(L_a^2(\mathbb{U}_+))$ and $T \geq 0$. If $R_a - T \in \mathcal{LC}(L_a^2(\mathbb{U}_+))$ for some $a \in \mathbb{D}$, then $I - T \in \mathcal{LC}(L_a^2(\mathbb{U}_+))$. Further if $R_a - T \in S_p(0 < p \leq \infty)$ for some $a \in \mathbb{D}$, then $I - T \in S_p$.*

Proof. Notice that $R_aT - TR_a = (R_a - T)R_a - R_a(R_a - T) \in \mathcal{LC}(L_a^2(\mathbb{U}_+))$. Since $R_a^2 = I$, hence $I - T^2 = (R_a - T)(R_a + T) + TR_a - R_aT \in \mathcal{LC}(L_a^2(\mathbb{U}_+))$. Now $T \geq 0$ implies $I + T$ is invertible and so

$$I - T = (I - T^2)(I + T)^{-1} \in \mathcal{LC}(L_a^2(\mathbb{U}_+)).$$

A similar argument shows that if $R_a - T \in S_p(0 < p \leq \infty)$ for some $a \in \mathbb{D}$, then $I - T \in S_p$. □

Let $T \in \mathcal{L}(L_a^2(\mathbb{U}_+))$. Suppose $1 < p < \infty$ and

$$L_T = \{U \in \mathcal{L}\left(L_a^2(\mathbb{U}_+)\right) : \text{U is unitary and } U - T \in S_p\}.$$

If $L_T \neq \phi$, define $O_p(U) = ||U - T||_p^p, p > 1$.

Theorem 14. *If $T \in \mathcal{L}(L_a^2(\mathbb{U}_+))$ and $T > 0$, then the following hold:*

(i) If R_a is a local maximum or a local minimum of O_p, for some $a \in \mathbb{D}$, then $M_a = \{(L' \circ \tau_{\varsigma_a})t_{\varsigma_a}(g \circ L \circ \tau_{\varsigma_a}) : g \in L_a^2(\mathbb{D}),\ g \text{ is even}\}$ is a reducing subspace of T and if $p > 1$ then $||I - T||_p < ||R_a - T||_p$.

(ii) If $E \in \mathcal{L}(L_a^2(\mathbb{U}_+))$ is a local extremum (either a local maximum or a local minimum) of O_p then E is a symmetry and $ET = TE$. If further $T \geq 0$, then ker T is a reducing subspace of E and $E\big|_{(ker\ T)^{\perp}}$ is a symmetry.

Proof. Let $f \in \mathcal{L}(L_a^2(\mathbb{U}_+))$ with $||f|| = 1$ and $\eta \in \mathbb{R}$. Define $G_f(\eta)$ on $L_a^2(\mathbb{U}_+)$ as follows:

$$G_f(\eta)g = e^{i\eta}\langle g, f\rangle f + g - \langle g, f\rangle f, g \in L_a^2(\mathbb{U}_+).$$

The map $G_f(\eta)$ is an unitary operator on $L_a^2(\mathbb{U}_+)$.

For $p > 1$, the derivative of O_p exists everywhere. Further if O_p has a local extremum at E, then for each f, $\dfrac{dO_p}{d\eta}(EG_f(\eta))$ vanishes at $\eta = 0$. Let $E - T = U|E - T|$ be the polar decomposition of $E - T$. Then

$$\frac{d}{d\eta}O_p\left[EG_f(\eta)\right]\big|_{\eta=0} = p\ Re \text{ trace}\left[|E-T|^{p-1}U^*Ei(f\otimes f)\right] = 0.$$

Evaluating the trace using an orthonormal basis containing f, we obtain $\langle |U - T|^{p-1}U^*Ef, f\rangle \in \mathbb{R}$. Since this holds for any f, it follows that $|E - T|^{p-1}U^*E$ is self adjoint. Further, since E^*U is a partial isometry and ker $(E^*U) =$ ker $U =$ ker $|E - T| =$ ker $|E - T|^{p-1}$. Hence $E^*U|E - T|^{p-1}$ is the unique polar decomposition of a self adjoint operator. Hence E^*U is self adjoint and commutes with $|E - T|^{p-1}$. Therefore E^*U commutes with every power of $|E - T|^{p-1}$, in particular with $|E - T|$. Thus

$$\begin{aligned} E^*(E-T) &= E^*U|E-T| \\ &= |E-T|E^*U \\ &= |E-T|U^*E \\ &= (E^*-T)E \end{aligned}$$

and so $E^*T = TE$ showing that E^*T is self-adjoint. Now since $T > 0$, we obtain $0 =$ ker $T =$ ker E^* and it follows that E is a symmetry and $ET = TE$. Now if $T \geq 0$, then it is not difficult to verify that $E^*T = TE$. Let Q be the orthogonal projection onto (ker $T)^{\perp}$. Then E^*QT is the unique polar decomposition of a self-adjoint operator. Thus E^*Q is self-adjoint, that is $E^*Q = QE$. This implies $EE^*QE^* = EQEE^*$. Thus $QE^* = EQ$. Thus $QEQ = (E^*Q)Q = E^*Q = QE$ and $QEQ = Q(QE^*) = QE^* = EQ$. Hence Q commutes with E and ker T reduces E. Now since $(EQ)^2 = E(QEQ) = EE^*Q = Q$, we obtain that E restricted to (ker $T)^{\perp}$ is a symmetry. This proves (ii) and the first part of (i) follows from Theorem 5. Now we shall show that if $p > 1$ then the function $O_p(U) = ||U - T||_p^p$ has a unique local minimum which occurs at $U = I$ and which is also a global minimum and in particular,

$$||I - T||_p < ||R_a - T||_p.$$

From Theorem 13, if $L_T \neq \phi$ then $I \in L_T$. Also, since a global minimum is also a local minimum, from the first part it follows that the local minimum can only be attained at some symmetry E, commuting with T. But then $I-E$ and $I-T$ are commuting compact normal operators and so have a common orthonormal basis $\{\varepsilon_i\}$ of eigenvectors. Let $\alpha_i = \langle T\varepsilon_i, \varepsilon_i\rangle, \beta_i = \langle E\varepsilon_i, \varepsilon_i\rangle$. Then $|\beta_i| = 1$ and

$$||E - T||_p^p = \sum_{i=1}^{\infty} |\beta_i - \alpha_i|^p \geq \sum_{i=1}^{\infty} |1 - \alpha_i|^p = ||I - T||_p^p. \tag{5}$$

Equality holds in (5), only when $\beta_i = 1$ for all i. That is, only if $E = I$. Thus from Lemma 1, it follows that $||I - T||_p < ||R_a - T||_p$ as $R_a \neq I$ for all $a \in \mathbb{D}$. □

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