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ON SEMILOCAL CONVERGENCE ANALYSIS OF THE INVERSE WEIERSTRASS METHOD FOR SIMULTANEOUS COMPUTING OF POLYNOMIAL ZEROS*

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Abstract

In this paper we study the semi-local convergence of the Inverse Weierstrass iterative method for simultaneous approximation of polynomial zeros. We present a semi-local convergence theorem with computationally verifiable initial conditions. Several numerical examples are provided to show the practical applications of the presented theoretical results.

MSC: 65H04, 65H05

keywords: Polynomial zeros, Simultaneous method, Weierstrass method, Inverse Weierstrass method, Local convergence, Semilocal convergence.

1 Introduction

Let $P(z)$ be a monic polynomial of degree $n \geq 2$

$$P(z) = a_0 + a_1 z + \ldots + a_{n-1} z^{n-1} + z^n , \tag{1}$$

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with simple real or complex zeros $\alpha_1, \alpha_2, \ldots, \alpha_n$, and let $z_1^{(0)}, z_2^{(0)}, \ldots, z_n^{(0)}$ be distinct reasonable close approximations of these zeros. Throughout this paper we assume without loss of generality that $a_0 \neq 0$, i.e. $\alpha_i \neq 0$ for $i = 1, \ldots, n$

In this study we consider a simultaneous iterative method defined by

$$z^{(k+1)} = G\left(z^{(k)}\right) = G^{k+1}\left(z^{(0)}\right), \quad k = 0, 1, 2, \ldots, \tag{2}$$

where $G : \mathcal{D} \subset \mathbf{C}^n \to \mathbf{C}^n$ is a vector valued function with components

$$G_i = G_i(z) = \frac{z_i^2}{z_i + W_i(z)}, \; z = (z_1, \ldots, z_n), \; i = 1, \ldots, n, \tag{3}$$

and the *Weierstrass' correction* $W_i : \mathcal{D} \subset \mathbf{C}^n \to \mathbf{C}$ is defined by

$$W_i(z) = \frac{P(z_i)}{\prod_{j\neq i}^n (z_i - z_j)} \quad (i = 1, \ldots, n), \tag{4}$$

where $\mathcal{D}$ is the set of all vectors in $\mathbf{C}^n$ with distinct components. Then we can define the operator $W : \mathcal{D} \subset \mathbf{C}^n \to \mathbf{C}^n$ by $W(z) = (W_1(z), \ldots, W_n(z))$.

The iteration method (2)-(3) was firstly introduced in [3], and some recent results were obtained in [1, 2, 4, 5, 6, 7]. All the obtained results are modifications of local convergence results of classical Weierstrass iterative method presented in [8, 9, 10, 11, 13, 14, 12].

Throughout the paper, we will use only the maximum vector norm defined by

$$\|z\|_\infty = \max_i |z_i|,$$

and we use the function $d : \mathbf{C}^n \to \mathbf{R}_+$ defined by $d(z) = \min\{\delta(z), \gamma(z)\}$, where

$$\delta(z) = min_{i\neq j}|z_i - z_j| \text{ and } \gamma(z) = min_j|z_j| \quad (j = 1, \ldots, n). \tag{5}$$

We denote the set of all polynomials over a field $\mathbf{C}$ by $\mathcal{C}[z]$. In our last work [1] we have proved the following convergence theorem (see also Theorem 1 in [2]).

Theorem 1 *Let $P \in \mathcal{C}[z]$ be a monic polynomial of degree $n \geq 2$, where $\alpha = \{\alpha \in \mathcal{C}^n : \alpha_i \neq 0$ and $\alpha_i \neq \alpha_j$ for $i, j = 1, \ldots, n\}$ is the root vector of P, and let $1 \leq p \leq \infty$. Suppose $z^{(0)} \in \mathcal{C}^n$ is an initial guess satisfying*

$$E(z^{(0)}) = \frac{\|z^{(0)} - \alpha\|}{d(\alpha)} < \bar{R}(n, p) = \frac{2^{\frac{1}{n+1}} - 1}{2.2^{\frac{1}{n+1}} - 1}. \tag{6}$$

Then the following statements hold true.
(i) Convergence. *The Inverse Weierstrass iteration (2)-(3) is well defined and converges quadratically to the root-vector α of P.*
(ii) A posteriori error estimate. *For all $k \geq 0$ we have the estimate*

$$\|z^{(k+1)} - \alpha\| \leq \lambda^{2^k} \|z^{(k)} - \alpha\| , \tag{7}$$

(iii) A priori error estimate. *For all $k \geq 1$ we have the estimate*

$$\|z^{(k)} - \alpha\| \leq \lambda^{2^k - 1} \|z^{(0)} - \alpha\| , \tag{8}$$

where $\lambda = E(z^{(0)})/\bar{R}(n,p)$.

The main purpose of this work is to obtain new semi-local convergence theorems and theorem with computationally verified conditions. The paper is structured as follows: In Section 2, we consider some known and auxiliary results about localization of polynomial zeros. In Section 3, we provide new semi-local convergence theorems and in Section 4, we present several numerical examples.

2 Localization of polynomial zeros

In this section we state some localization lemmas, which play an important role in this study. The next lemmas are due to Proinov [15] in the case when $\|\frac{u}{\delta(z)}\|$ is replaced by $\frac{\|u\|}{d(z)}$.

Lemma 1 *Let $H = \frac{\|u\|}{d(z)}$, where $u, z \in \mathbf{C}^n$ and let $c \geq 0$ be such that $cH < \frac{1}{2}$. Then the closed disks*

$$D_i = \{x \in \mathbf{C} : |x - z_i| \leq c|u_i|\} \ , i = 1, 2, \ldots, n , \tag{9}$$

are mutually disjoint.

Proof. From the definition of $d(x)$ in (5), Holder's inequality and $2cH < 1$, we obtain for $i \neq j$,

$$c(|u_i| + |u_j|) \leq c \left(\frac{|u_i|}{d(z)} + \frac{|u_j|}{d(z)} \right) |z_i - z_j| \leq 2cH|z_i - z_j| < |z_i - z_j| ,$$

which proves that the disks (9) are mutually disjoint.

Lemma 2 *Let $P \in \mathcal{C}[z]$ be a polynomial of degree $n \geq 2$. Suppose there exists $z \in \mathbf{C}^n$ with distinct components and let $c \geq 1$ be such that*

$$cE_p < \frac{1}{2} \quad and \quad \frac{1}{c} + \frac{(n-1)E_p(z)}{1 - cE_p(z)} \leq 1\,, \tag{10}$$

where the function $E_p : \mathcal{D} \to \mathbf{R}_+$ is defined by $E_p(z) = \frac{\|W(z)\|}{d(z)}$. Then P has only simple zeros in $\mathbf{C}$ and the disks

$$D_i = \{x \in \mathbf{C} : |x - z_i| \leq c|W_i|\}\ , i = 1, 2, \ldots, n\,, \tag{11}$$

are mutually disjoint and each of them contains exactly one zero of P.

Proof. The proof is the same as the proof of Proposition 3.3 in [15].

As a corollary of Lemma 2 we can state the following theorem (see also Theorem 3.4 in [15]).

Theorem 2 *Let $P \in \mathcal{C}[z]$ be a polynomial of degree $n \geq 2$. Suppose there exists $z \in \mathbf{C}^n$ with distinct components such that*

$$E_p(z) = \frac{\|W(z)\|}{d(z)} \leq \frac{1}{(1 + \sqrt{n-1})^2}\,. \tag{12}$$

In the case when $n = 2$ we assume that inequality (12) is strict. Then P has only simple zeros in $\mathbf{C}$. Besides, for any real number

$$c \in [\theta(E_p(z)), \phi(E_p(z))],$$

where θ and ϕ are real functions defined by

$$\theta(t) = \frac{2}{1 - (n-2)t + \sqrt{(1-(n-2)t)^2 - 4t}}\ ;\ \phi(t) = \frac{2}{1 - (n-2)t} \tag{13}$$

the disks (11) are mutually disjoint and each of them contains exactly one zero of P.

Proof. It is easy to prove that

$$\frac{1}{(1 + \sqrt{n-1})^2} \leq \frac{1}{n+2}$$

with equality only if $n = 2$. Then it follows from (12) that

$$E_p(z) < \frac{1}{n+2}\,.$$

From this and $c \leq \phi(E_p(z))$, we get

$$cE_p(z) \leq E_p(z)\phi(E_p(z)) < \frac{1}{2}$$

which proves the first inequality in (10).

Note that $1 \leq \theta(t) \leq \phi(t)$ provided that $0 \leq t \leq 1/(1+\sqrt{n-1})^2$. The assumption $c \in [\theta(E_p(z)), \phi(E_p(z))]$ implies the second inequality in (10). Now the statement follows from Lemma 2.

3 Semilocal convergence theorems

In this section we state and prove the main results of the paper. The next theorem improves the results of Theorem 1.

Theorem 3 *Let $P \in \mathcal{C}[z]$ be a polynomial of degree $n \geq 2$, where*

$$\alpha = \{\alpha \in \mathbf{C}^n : \alpha_i \neq 0 \ \textit{and}\ \alpha_i \neq \alpha_j \ \textit{for}\ i,j = 1,\ldots,n\}$$

is the root vector of P. If the initial guess

$$\mathbf{z}^{(0)} = \{\mathbf{z}^{(0)} \in \mathbf{C}^n : z_i^{(0)} \neq 0 \ \textit{and}\ z_i^{(0)} \neq z_j^{(0)} \ \textit{for}\ i,j = 1,\ldots,n\}$$

is such that

$$\frac{\|\mathbf{z}^{(0)} - \alpha\|}{d(z^{(0)})} \leq R(n) := \frac{2^{\frac{1}{n+1}} - 1}{4.2^{\frac{1}{n+1}} - 3}\,, \tag{14}$$

then the iteration (2)-(3) is well defined, converges to α quadratically and the error estimates (7)-(8) hold true.

Proof. First we will prove the following relation

$$d(\mathbf{z}^{(0)}) - 2\|\mathbf{z}^{(0)} - \alpha\| \leq d(\alpha)\,. \tag{15}$$

Let us consider the two cases:

Case 1. Suppose that $d(\alpha) = \delta(\alpha)$, i.e. $\delta(\alpha) \leq \gamma(\alpha)$. Then

$$d(\mathbf{z}^{(0)}) \leq |z_i^{(0)} - z_j^{(0)}| \leq |\alpha_i - \alpha_j| + |z_i^{(0)} - \alpha_i| + |z_j^{(0)} - \alpha_j| \leq |\alpha_i - \alpha_j| + 2\|\mathbf{z}^{(0)} - \alpha\|\,,$$

according that, we have

$$d(\mathbf{z}^{(0)}) \leq d(\alpha) + 2\|\mathbf{z}^{(0)} - \alpha\|\,, \tag{16}$$

which implies (15).

Case 2. Suppose that $d(\alpha) = \gamma(\alpha)$, then we have

$$d(\mathbf{z}^{(0)}) \leq |z_i^{(0)}| \leq |\alpha_i| + |z_i^{(0)} - \alpha_i| \leq |\alpha_i| + \|\mathbf{z}^{(0)} - \alpha\| \leq |\alpha_i| + 2\|\mathbf{z}^{(0)} - \alpha\| \,,$$

which implies (16) and then we get (15) by analogy of Case 1.

It follows from (15) that

$$d(\mathbf{z}^{(0)}) \left(1 - 2\frac{\|\mathbf{z}^{(0)} - \alpha\|}{d(\mathbf{z}^{(0)})}\right) \leq d(\alpha) \,,$$

which can be written in the form

$$\frac{1}{d(\alpha)} \leq \frac{1}{d(\mathbf{z}^{(0)})} \left(1 - 2\frac{\|\mathbf{z}^{(0)} - \alpha\|}{d(\mathbf{z}^{(0)})}\right)^{-1} .$$

Multiplying both sides of this inequality by $\|\mathbf{z}^{(0)} - \alpha\|$, we get

$$\frac{\|\mathbf{z}^{(0)} - \alpha\|}{d(\alpha)} \leq \frac{\|\mathbf{z}^{(0)} - \alpha\|}{d(\mathbf{z}^{(0)})} \left(1 - 2\frac{\|\mathbf{z}^{(0)} - \alpha\|}{d(\mathbf{z}^{(0)}}\right)^{-1} . \tag{17}$$

From the last inequality and using (14), we obtain

$$\frac{\|\mathbf{z}^{(0)} - \alpha\|}{d(\alpha)} \leq \frac{2^{\frac{1}{n+1}} - 1}{2.2^{\frac{1}{n+1}} - 1} .$$

Now the statements of the theorem follows from (6) and Theorem 1.

Although Theorem 3 is an improvement of Theorem 1, the initial conditions in the both theorems depend on the desired root-vector α which is unknown. Which means that these two results are rather of theoretical importance.

The next theorem is the main result of this paper. We shall prove local convergence with computationally verifiable initial condition (see (18)). This initial condition is of significant practical importance since it depends only on available data: the coefficients a_i and the degree n of the polynomial P, and the initial approximation $\mathbf{z}^{(0)}$.

Theorem 4 *Let $P \in \mathcal{C}[z]$ be a polynomial of degree $n \geq 2$ (where $a_0 \neq 0$). Suppose there exists a vector*

$$\mathbf{z}^{(0)} = \{\mathbf{z}^{(0)} \in \mathbf{C}^n : z_i^{(0)} \neq 0 \text{ and } z_i^{(0)} \neq z_j^{(0)} \text{ for } i,j = 1, \ldots, n\}$$

such that

$$E_p(z^{(0)}) = \frac{\|W(z)\|}{d(z^{(0)})} \le \tilde{R}(n) := \frac{R(1-R)}{1+(n-2)R}\,, \tag{18}$$

where

$$R = R(n) = \frac{2^{\frac{1}{n+1}} - 1}{4.2^{\frac{1}{n+1}} - 3}\,. \tag{19}$$

Then $P(z)$ has only simple zeros and the following statements hold true.
(i) Convergence. *The Inverse Weierstrass iteration (2)-(3) is well defined and converges quadratically to the root-vector α of P.*
(ii) A posteriori error estimate. *For all $k \ge 0$ we have the estimate*

$$\|z^{(k+1)} - \alpha\| \le \lambda^{2^k} \|z^{(k)} - \alpha\|\,, \tag{20}$$

(iii) A priori error estimate. *For all $k \ge 1$ we have the estimate*

$$\|z^{(k)} - \alpha\| \le \lambda^{2^k - 1} \|z^{(0)} - \alpha\|\,, \tag{21}$$

where $\lambda = E(z^{(0)})/\bar{R}(n,p)$ and $\bar{R}$ is defined by (8).

Proof. Let $\tau = 1/(1+\sqrt{n-1})$ and $\mu = 1/(1+\sqrt{(n-1)})^2$. It can be shown that

$$0 \le R \le \tau\,. \tag{22}$$

Consider the real function $\sigma : [0,\tau] \to [0,\mu]$ defined by

$$\sigma(t) = \frac{t(1-t)}{1+(n-1)t}\,.$$

Function σ is strictly increasing on $[0,\tau]$ and the inverse function of σ is the function $\varphi : [0,\mu] \to [0,\tau]$ defined by

$$\varphi(t) = \frac{2t}{1-(n-2)t+\sqrt{(1-(n-2)t)^2-4t}} = t\theta(t)\,,$$

where $\theta(t)$ is defined by (13).Then from (22) and (18) it follows that

$$E_p(z^{(0)}) = \frac{\|W(z)\|}{d(z^{(0)})} \le \sigma(R) \le \mu = \frac{1}{(1+\sqrt{(n-1)})^2}\,. \tag{23}$$

Then it follows from Theorem 2 that P has only simple zeros and the disks

$$D_i = \left\{x \in \mathbf{C} : |x - z_i| \le \theta(E_p(z^{(0)}))|W_i|\right\}, i = 1,2,\ldots,n\,,$$

are mutually disjoint and each of them contains exactly one zero of P. This means that there is a root-vector $\alpha \in \mathbf{C}^n$ of P such that

$$|z_i - \alpha_i| \leq \theta(E_p(z^{(0)}))|W_i(z^{(0)})| \,.$$

Last inequality implies

$$\frac{\|\mathbf{z}^{(0)} - \alpha\|}{d(z^{(0)})} \leq \theta(E_p(z^{(0)}))E_p(z^{(0)}) = \varphi\left(E_p(z^{(0)})\right) \,.$$

From this and (23), we conclude that

$$\frac{\|\mathbf{z}^{(0)} - \alpha\|}{d(z^{(0)})} \leq \varphi\left(E_p(z^{(0)})\right) \leq \varphi\left(\sigma\left(E_p(z^{(0)})\right)\right) = R \,.$$

Now all the statements (i)-(iii) of the theorem follow from the Theorem 3.

4 Numerical Examples

In this section, we consider several numerical examples in which we apply Theorem 4 to prove the quadratic convergence of the method (2)-(3).

According to Theorem 4, if there exists an integer $k \geq 0$ such that

$$E_p(z^{(k)}) \leq \tilde{R} = \frac{R(1-R)}{1+(n-2)R} \,, \tag{24}$$

where

$$R = \frac{2^{\frac{1}{n+1}} - 1}{4.2^{\frac{1}{n+1}} - 3} \,.$$

then the iteration (2)-(3) is well defined and converges quadratically to the root vector α of P. We use the following stop criterion

$$\|z^{(i)} - \alpha\| \leq 10^{-15} \,, \quad (i \geq k) \,. \tag{25}$$

Example 1 *Consider the polynomial (see [16])*

$$P(z) = z^9 + 3z^8 - 3z^7 - 9z^6 + 3z^5 + 9z^4 + 99z^3 + 297z^2 - 100z - 300$$

with the root vector $\alpha = (2i, 2+i, -3, -2i, -1, 1, -2+i, 2-i, -2-i)$.

We use Abert's initial approximation vector $z^{(0)}$ given by

$$z_k^{(0)} = -\frac{a_1}{n} + r_0 \exp i\theta_k \,, \;\; \theta_k = \frac{\pi}{n}\left(2k - \frac{3}{2}\right) \,, \;\; k = 1, \ldots, n \,,$$

where $n = 9$ and $r_0 = 10$ (see also [2]). The radius of convergence is

$$\tilde{R} \approx 0.0379$$

and the convergence condition (24) is satisfied for $k = 8$ ($E_p(z^{(8)}) \approx 0.0207$). The stopping criteria is reached after eleven iterations, see Table 1.

Table 1: Numerical results for Example 1.

$iter(i)$	$z_1^{(i)}$	$z_2^{(i)}$	$z_3^{(i)}$
0	-1.263 +1.736i	-4.683 +7.660i	-11.11+10i
8	0.0050 + 1.9960i	1.9847 + 0.9861i	-3.0039 - 0.0003i
11	$2.963 \times 10^{-18} + 2i$	2+i	$-3 + 1.009 \times 10^{-18} i$

$iter(i)$	$z_4^{(i)}$	$z_5^{(i)}$	$z_6^{(i)}$
0	-17.53 +7.660i	-20.95+1.736i	-19.77 -5i
8	- 2.0005i	-1.0003 + 0.0005i	1.0031 - 0.0022i
11	$5.340 \times 10^{-18} - 2i$	-1	$1 + 2.568 \times 10^{-18} i$

$iter(i)$	$z_7^{(i)}$	$z_8^{(i)}$	$z_9^{(i)}$
0	-14.53-9.396i	-7.690 -9.396i	-2.450-5i
8	-1.9999 + 1.0000i	2.0086 - 1.0093i	-1.9971 - 0.9993i
11	$-2 + i$	$2 - i$	$-2 - i$

Example 2 *Consider the polynomial (see [14])*

$$P(z) = z^3 - (2 + 5i)z^2 - (3 - 10i)z + 15i$$

with the root vector $\alpha = (-1, 3, 5i)$. *We use the initial vector* $z^{(0)} = (-1.5, 2.7, 4.5i)$.

The radius of convergence is

$$\tilde{R} \approx 0.0868$$

and the convergence condition (24) is satisfied for $k = 2$ ($E_p(z^{(2)}) \approx 0.0059$). The stopping criteria is reached after five iterations, see Table 2.

Table 2: Numerical results for Example 2.

$iter(i)$	$z_1^{(i)}$	$z_2^{(i)}$	$z_3^{(i)}$
0	-1.5	2.7	4.5i
1	-1.0768 + 0.0092i	3.0196 - 0.0162i	-0.0949 + 5.0545i
2	-1.0060 + 0.0001i	3.0008 - 0.0006i	-0.0018 + 5.0004i
3	-1.0000 + 0.0000i	3.0000 - 0.0000i	-0.0000 + 5.0000i
4	-1.0000 + 0.0000i	3.0000 - 0.0000i	0.0000 + 5.0000i
5	-1.0000	3.0000	5i

Example 3 *Consider the polynomial*

$$P(z) = z^5 - 15z^4 + 22z^3 + 438z^2 - 1175z - 1575$$

with the root vector $\alpha = (-5, -1, 5, 7, 9)$. *We use the initial vector* $z^{(0)} = (-5.7, -1.8, 4.1, 6.2, 9.8)$.

The radius of convergence is

$$\tilde{R} \approx 0.0605$$

and the convergence condition (24) is satisfied for $k = 2$ ($E_p(z^{(2)}) \approx 0.00599$). The stopping criteria is reached after five iterations, see Table 3.

Table 3: Numerical results for Example 3.

$iter(i)$	$z_1^{(i)}$	$z_2^{(i)}$	$z_3^{(i)}$	$z_4^{(i)}$	$z_5^{(i)}$
0	-5.7000	-1.8000	4.1000	6.2000	9.8000
1	-4.8988	-1.2583	5.1844	6.5166	9.3553
2	-5.0094	-1.0396	4.9707	7.0226	9.0623
3	-5.0000	-1.0017	4.9992	7.0012	9.0005
5	-5	-1	5	7	9

5 Conclusion

In this work we investigate convergence analysis of the Inverse Weierstrass iterative method for simultaneous approximation of polynomial zeros. Our goal was to obtain semi-local convergence analysis results. We establish a new convergence theorems with new radiuses of convergence. The main theorem provides also a-priori and a-posteriori error estimates. Numerical results with different examples confirm the theoretical results.

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