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INITIAL SIMPLICIAL COMPLEXES ASSOCIATED TO SOME TORIC VARIETIES*

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Abstract

We study the triangulation of the initial simplicial complex arising from the toric deformation of the Grassmann variety $\mathbb{G}(1,n)$ and the Hankel variety $H(1,n)$, for $n = 3, 4$.

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1 Introduction

Simplicial complexes on a finite set of vertices arise in different ways in commutative algebra, as described in ([14], [16]). In particular, if we consider a graded ideal I of the polynomial ring $S = K[x_1, \ldots, x_n]$, K any field, and a total order on the monomials of S, let $in_{\prec}(I)$ be the initial ideal of I. The ideal $J = \sqrt{(in_{\prec}(I))}$ is a monomial squarefree ideal of S and it defines a simplicial complex Δ, whose the Stanley-Reisner ideal is J. If we have a semigroup ring $R \subset S$, generated on $\mathbb{K}$ by monomials of S its definition ideal is a binomial ideal $I_{\mathcal{A}}$ and $J_{\mathcal{A}} = \sqrt{(in_{\prec}(I_{\mathcal{A}}))}$ defines a simplicial complex $\Delta_{\mathcal{A}}$, where $\mathcal{A}$ is the set of lattice points generating the semigroup subtended by R. We call such a simplicial complex the simplicial complex arising from the semigroup ring R. In this paper we are interested to semigroup

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rings that arise from a toric deformation ([14]) of the coordinate ring of the Grassman variety $\mathbb{G}(1,n)$ of all lines of the projective space $\mathbb{P}^n$ and of the coordinate ring of the Hankel variety $H(1,n)$ of all Hankel lines of $\mathbb{P}^n$, a subvariety of $\mathbb{G}(1,n)$, studied by many authors ([4] [1] [2] [10] [6] [7] [12] [5]). More precisely, we will compute the initial complexes in correspondence of the $\mathbb{G}(1,n)$ and $H(1,n)$ and we study the triangulations of the representing polytopes. Combinatoric properties of the triangulations are founded [14]. In particular, in section 2 we recall some definitions and known results about deformations. In section 3, we concentrate on the case $n = 4, 5$. For $n = 4$, any total order on the monomials of the presentation ring of the deformation of $\mathbb{G}(1,3) = H(1,3)$ builds to two distinct initial simplicial complexes, hence to two distinct triangulations. For $n = 5$, we study a specific initial simplicial complex associated to the deformation of $\mathbb{G}(1,4)$, fixing a particular term order on the set of variables $[i,j]$, $1 \leq i < j \leq 5$,depending from the partial order on the set of the minors of a generic 2×5 matrix([14]). As a consequence, the Gröbner basis of the toric ideal $I_{\mathcal{A}}$ is quadratic. In general, this is not the case. Adopting the same term order, we study the corresponding initial simplicial complex of the deformation of $H(1,4)$, by considering a generic 2×5 Hankel matrix. The corresponding triangulations are investigated.

2 Notations and known results

Let K be a field and let $S = K[x_1, x_2, \dots, x_n]$ be the polynomial ring in n variables $x_1, x_2, \dots, x_n$. We denote the set of the power products of S by: $M^n = \{x_1^{a_1}\dots x_n^{a_n} \,|\, a_i \in \mathbb{N}, i = 1,\dots,n\}$, where $x_1^{a_1}\cdots x_n^{a_n} = \mathbf{x}^{\mathbf{a}}$, $\mathbf{a} = (a_1, a_2, ..., a_n) \in \mathbb{N}^n$.

Definition 1. *A term order on M^n is a total order $\prec$ on M^n satisfying:*

(1) $1 \prec \mathbf{x}^{\mathbf{b}}\, \forall\, \mathbf{x}^{\mathbf{b}} \in M^n$, $\mathbf{x}^{\mathbf{b}} \neq 1$*;*

(2) If $\mathbf{x}^{\mathbf{a}} \prec \mathbf{x}^{\mathbf{b}}$*, then* $\mathbf{x}^{\mathbf{a}}\mathbf{x}^{\mathbf{c}} \prec \mathbf{x}^{\mathbf{b}}\mathbf{x}^{\mathbf{c}}\, \forall\, \mathbf{x}^{\mathbf{c}} \in M^n$.

Definition 2. (*[14]*) *We define the lexicographical order on M^n with order of variables $x_1 > x_2 > \dots > x_n$ as follows:*
for $\mathbf{a} = (a_1, a_2, \dots, a_n), \mathbf{b} = (b_1, b_2, \dots, b_n)$ *with* $\mathbf{x}^{\mathbf{a}} < \mathbf{x}^{\mathbf{b}} \iff \exists h \,|\, a_1 = b_1, \dots,$
$a_h = b_h, a_{h+1} = b_{h+1}$, $h = 0, \dots n-1$.

Let I be any ideal of $S = K[x_1, \dots, x_n]$ and let $\prec$ be a term order on the monomials of S.

Definition 3. *The initial complex $\Delta_{\prec}(I)$ of I with respect to $\prec$ is the simplicial complex on the vertex set $\{1, 2, \dots, n\}$, whose the Stanley-Reisner ideal is the radical of the monomial ideal $in_{\prec}(I)$.*

Let $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$, where $\mathbf{a}_i \in \mathbb{N}^n$, be a set of lattice points. Assume that the set $\mathcal{A}$ spans an affine hyperplane. This means that the toric ideal $I_{\mathcal{A}}$ is a homogeneous ideal of the polynomial ring S.

Definition 4. *([14, chapter 8]) Let σ be a subset of lattice points, $\sigma \subseteq \mathcal{A}$, and $pos(\sigma)$ be the polyhedral cone generated by σ. A triangulation of $\mathcal{A}$ is a collection Δ of subsets of $\mathcal{A}$, such that $\{pos(\sigma), \sigma \in \mathcal{A}\}$ is the set of cones of a simplicial fan whose support is the same as $pos(\mathcal{A})$, $supp\{pos(\sigma), \sigma \in \Delta\} = pos(\mathcal{A})$.*

Let R be a monomial subring of S and let $\mathcal{A}$ be the set of lattice points of the semigroup under R. We will identify the given set of lattice points $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_n\} \subseteq \mathbb{N}^n$ with the index set $\{1, \dots, t\}$.

In our paper we consider a $r \times s$ matrix (t_{ij}) of variables $r \leq s$. Let R be the subalgebra of $K[t_{11}, t_{12}, \dots, t_{rs}]$ generated by the $r \times r$ minors of the matrix (t_{ij}). Its projective spectrum $Proj(R)$ is the Grassmann variety $G(r-1, s-1)$ of r-dimensional linear subspaces in the s-dimensional vector space K^s, K algebraic closed, presented in its usual Plücker embedding.

We associate a new variable $[i_1, i_2]$ to the 2×2 minor M_{i_1,i_2} with column indices $i_1 < i_2$ and consider the ring $K[[i_1, i_2], 1 \leq i_1 < i_2 \leq n]$ generated by these $\binom{n}{2}$ brackets.

Let $I_{2,n}$ be the ideal of $K[[i_1, i_2], 1 \leq i_1 < i_2 \leq n]$ generated by the algebraic relations among the 2×2 minors. Note with $I_{2,n}$ is the kernel of the map:

$$K[[i_1, i_2], 1 \leq i_1 < i_2 \leq n] \to K[M_{i_1,i_2}, 1 \leq i_1 < i_2 \leq n]$$

$$[i_1, i_2] \to M_{i_1,i_2} = t_{1i_1}t_{2i_2} - t_{1i_2}t_{2i_1}.$$

The ideal $I_{2,n}$ is called the Grassmann-Plücker ideal.

The toric ideal $I_{\mathcal{A}_{2,n}}$ is the kernel of the map

$$\begin{aligned} K[[i_1, i_2], 1 \leq i_1 < i_2 \leq n] \to K[&t_{11}t_{22}, t_{11}t_{23}, \dots, t_{1(n-1)}t_{2,n}] \\ &\subset K[t_{11}, t_{12}, \dots, t_{2n}] \end{aligned}$$

$$[i_1, i_2] \to t_{1i_1}t_{2i_2}.$$

The following theorem is a well known result, proved by many authors in different ways.

Theorem 1. *Let* $\begin{pmatrix} t_{11} & \cdots & t_{1n} \\ t_{21} & \cdots & t_{2n} \end{pmatrix}$ *be a generic matrix of variables. Let* $R = K[M_{1,2}, M_{1,3}, \ldots, M_{n-1,n}]$ *be the subalgebra of* $K[t_{11}, \ldots, t_{2n}]$ *generated by the minors and* $in_{\prec}(R) = K[t_{11}t_{22}, \ldots, t_{1(n-1)}t_{2n}]$ *the subalgebra of* $K[t_{11}, t_{12}, \ldots, t_{2n}]$ *generated by the main diagonal terms of the minors. Then* $I_{\mathcal{A}_{2,n}}$ *is generated by a Gröbner basis of quadrics.*

Consider now a generic $2 \times n$ Hankel matrix

$$H = \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ t_{12} & t_{13} & \cdots & t_{1(n+1)} \end{pmatrix}.$$

As before, we associate a new variable $[i_1, i_2]$ to the 2×2 minor with columns i_1, i_2, $i_1 < i_2$, in the Hankel matrix and consider the polynomial ring $K[[i_1, i_2], 1 \leq i_1 < i_2 \leq n]$ and let $I^H_{2,n}$ be the ideal generated by the algebraic relations among the 2×2 minors.

Proposition 1. *The toric ideal* $I^H_{2,n}$ *is the kernel of the map*

$$K[[i_1, i_2], 1 \leq i_1 < i_2 \leq n] \to K[t_{11}t_{13}, t_{11}t_{14}, \ldots, t_{1(n-1)}t_{1(n+1)}]$$

$$[i_1, i_2] \to t_{1i_1}t_{1(i_2+1)}.$$

Proof. By definition of toric ideal given in [14]. □

Then we have:

Theorem 2. *The set of* 2×2 *minors of a Hankel* $2\times n$ *matrix of variables is a canonical basis (Sagbi basis) for the subalgebra they generate, with respect to any diagonal term order on* $K[t_{11}, t_{12}, \ldots, t_{1n}, t_{1(n+1)}]$.

Proof. See [1]. □

Theorem 3. *The ideal* $I^H_{2,n}$ *has a Gröbner basis of quadrics given by the following set of polynomials of* $K[[1,2], \ldots, [n-1,n]]$ *with respect to any diagonal term order and the lex order on the variables,* $[1,2] \prec [1,3] \prec \ldots \prec [n-1,n]$*:*

(a) $[i,j][h,k] - [i,k][h,j] + [i,h][k,j] \quad 1 \leq i < h < k < j \leq n$

(b) $[i,j][h,k] - [i,h-1][j+1,k] - [i,j+1][h,k-1] - [i+1,j][h-1,k] + [i+1,j+1][h-1,k-1] + [i,h][j+1,k-1] + [i+1,h-1][j,k] - [i+1,h][j,k-1]$ $1 \leq i < j < h < k \leq n$, $h - j > 1$.

Proof. See [1, Corollary 2.4]. □

Theorem 4. *The toric ideal* $I_{H_{2,n}}$ *is generated by a Gröbner basis of quadrics given by the following set of binomials of* $K[[1,2],\dots,[n-1,n]]$*:*

(a) $[i,j][h,k]-[i,k][h,j] \quad 1 \le i < h < k < j \le n$

(b) $[i,j][h,k]-[i,h-1][j+1,k] \quad 1 \le i < j < h < k \le n,\ h-j>1.$

Proof. See [1, Section 3]. □

3 Main Results

The simplicial complex that we will study have as Stanley-Reisner ideal the initial ideal of the toric ideal , with respect to a given order on the variables and a total order on the monomials according to [14]. We denote such simplicial complex $\Delta_{\prec}(I_{\mathcal{A}})$. We denote the triangulation of the simplicial complex $\Delta_{\prec}(I_{\mathcal{A}})$, where $I_{\mathcal{A}}$ is the toric ideal of the semigroup associated to the toric deformation of $\mathbb{G}(1,n)$ and $H(1,n)$ with respect to a weight matrix, respectively by $T_{\mathbb{G}(1,n)}$ and $T_{H(1,n)}$. The following theorem describes the triangulations for $n=3,4$.

Theorem 5. *Let* $K[M_{ij}]$ *be the coordinates ring of* $\mathbb{G}(1,3)=H(1,3)$*. Then, the set* $\mathcal{A}$ *has two distinct triangulations* $T^{(1)}_{\mathbb{G}(1,3)}$ *and* $T^{(2)}_{\mathbb{G}(1,3)}$ *with maximal faces set given respectively by the sets*

$$\{\{1,2,3,4,6\},\{2,3,4,5,6\}\},\{\{1,2,3,5,6\},\{1,2,4,5,6\}\}$$

Proof. Let $T^{(i)}_{\mathbb{G}(1,3)}, i=1,2$, be the triangulation associated to the initial simplicial complex $\Delta^{(i)}_{\prec}(I_{\mathcal{A}}), i=1,2$. Denote by T_{ij} the variable $[i,j], 1 \le i < j \le 4$, and fix the order on the variables:

$$T_{12} > T_{13} > T_{14} > T_{23} > T_{24} > T_{34}.$$

Then the Gröbner basis of $I_{\mathcal{A}}$ is

$$\{T_{13}T_{24} - T_{14}T_{23}\}$$

and we have two initial ideals $in_{\prec}(I_{\mathcal{A}}) = (T_{13}T_{24})$ and $in_{\prec}(I_{\mathcal{A}}) = (T_{14}T_{23})$, hence two triangulations. Fixing the order on the old variables $t_{11} > t_{12} > t_{13} > t_{14} > t_{21} > t_{22} > t_{23} > t_{24}$, and the lex order on the monomials $t_{11}t_{22} > t_{11}t_{23} > t_{11}t_{24} > t_{12}t_{23} > t_{12}t_{24} > t_{13}t_{24}$, we can order the set of lattice points as:

$$(1,0,0,0,0,1,0,0) > (1,0,0,0,0,0,1,0) > (1,0,0,0,0,0,0,1) >$$

$$(0,1,0,0,0,0,0,1) > (0,0,1,0,0,0,0,1) > (0,0,0,1,0,0,0,1)$$

and

$$\mathbf{a}_6 < \mathbf{a}_5 < \mathbf{a}_4 < \mathbf{a}_3 < \mathbf{a}_2 < \mathbf{a}_1.$$

In the correspondence $\mathbf{a}^i \to \mathbf{t}^{\mathbf{a}_i} \to z_i$, $t = t_{11}t_{12}t_{13}t_{14}t_{21}t_{22}t_{23}t_{24}$ and $in_\prec(I_\mathcal{A}) = (T_{13}T_{24})$, $in_\prec(I_\mathcal{A}) = (T_{13}T_{23})$ can be written respectively as $in_\prec(I_\mathcal{A}) = (z_2z_5)$ and $in_\prec(I_\mathcal{A}) = (z_3z_4)$ so the set $\{2,5\}$ does not belong to $\Delta_\prec^{(1)}(I_\mathcal{A})$ and the set $\{3,4\}$ does not belong to $\Delta_\prec^{(2)}(I_\mathcal{A})$. It follows that the set of maximal faces of $\Delta_\prec^{1}(I_\mathcal{A})$ and $\Delta_\prec^{(2)}(I_\mathcal{A})$ is respectively:

$$\{\{1,2,3,4,6\},\{2,3,4,5,6\}\},\{\{1,2,3,5,6\},\{1,2,4,5,6\}\}.$$

□

Remark 1. *Starting from a generic* 2×4 *Hankel matrix*

$$H = \begin{pmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{12} & t_{13} & t_{14} & t_{24} \end{pmatrix}$$

under the $K-$*algebras homomorphism* $[1,2] \to t_{11}t_{13}$, $[1,3] \to t_{11}t_{14}$, $[1,4] \to t_{11}t_{24}$, $[2,3] \to t_{11}t_{14}$, $[2,4] \to t_{12}t_{24}$, *that gives the semigroups isomorphism*

$$\begin{aligned}&K[t_{11}t_{22}, t_{11}t_{23}, t_{11}t_{24}, t_{12}t_{23}, t_{12}t_{24}, t_{13}t_{24}]\\ &\cong K[t_{11}t_{13}, t_{11}t_{14}, t_{11}t_{24}, t_{12}t_{14}, t_{12}t_{24}, t_{13}t_{24}],\end{aligned}$$

one obtains 6 *lattice points and* 2 *triangulations where, as in the generic case, only one chord does not belong to each simplicial complex. Then, we have the same geometric realizations, module an exchange of the vertices.*

For $n = 5$, *the toric ideal of* $\mathbb{G}(1,4)$ *has different Gröbner bases for different term orders. In our theorem we will study that case for a specific Gröbner basis. More precisely, we have:*

Theorem 6. *Let* $K[M_{ij}]$ *be the coordinates ring of* $\mathbb{G}(1,4)$. *The set* $\mathcal{A}$ *has a triangulation with maximal faces set given by*

$$\{\{1,2,3,4,7,9,10\},\{1,2,4,5,6,9,10\}\}.$$

Proof. Let T be the triangulation associated to the initial simplicial complex $\Delta_\prec(I_\mathcal{A})$. Denote by T_{ij} the variable $[i,j], 1 \le i < j \le 5$. The toric ideal $I_\mathcal{A}$

$$\begin{aligned}I_\mathcal{A} =&(T_{14}T_{23} - T_{13}T_{24}, T_{15}T_{23} - T_{13}T_{25},\\ &T_{15}T_{24} - T_{14}T_{25}, T_{15}T_{34} - T_{14}T_{35}, T_{25}T_{34} - T_{24}T_{35}).\end{aligned}$$

It is knows that I admits a quadratic Gröbner basis given by the set of binomials

$$\{T_{14}T_{23} - T_{13}T_{24}, T_{15}T_{23} - T_{13}T_{25}, T_{15}T_{24} - T_{14}T_{25},\\ T_{15}T_{34} - T_{14}T_{35}, T_{25}T_{34} - T_{24}T_{35}\},$$

where the initial monomials are all the products of two incomparable elements of the partially ordered set

$$\{[1,2[,[1,3],[1,4],[1,5],[2,3],[2,4],[2,5],[3,4],[3,5],[4,5,]\}.$$

Then, $in_{\prec}(I_{\mathcal{A}}) = (T_{14}T_{23}, T_{15}T_{23}, T_{15}T_{24}, T_{15}T_{34}, T_{25}T_{34})$ Fixing the order on the old variables $t_{11} > t_{12} > t_{13} > t_{14} > t_{15} > t_{21} > t_{22} > t_{23} > t_{24} > t_{25}$, and the lex order on the monomials $t_{11}t_{22} > t_{11}t_{23} > t_{11}t_{24} > t_{11}t_{25} > t_{12}t_{23} > t_{12}t_{24} > t_{13}t_{24} > t_{13}t_{25} > t_{14}t_{25}$, we can order the set of lattice points as:

$$(1,0,0,0,0,0,1,0,0,0) > (1,0,0,0,0,0,0,1,0,0) > (1,0,0,0,0,0,0,0,1,0) >$$

$$(1,0,0,0,0,0,0,0,0,1) > (0,1,0,0,0,0,0,1,0,0) > (0,1,0,0,0,0,0,0,1,0) >$$

$$(0,1,0,0,0,0,0,0,0,1) > (0,0,1,0,0,0,0,1,0,1,0) > (0,0,1,0,0,0,0,0,0,1)$$

$$> (0,0,0,1,0,0,0,0,0,1)$$

and

$$\mathbf{a}_{10} < \mathbf{a}_9 < \mathbf{a}_8 < \mathbf{a}_7 < \mathbf{a}_6 < \mathbf{a}_5 < \mathbf{a}_4 < \mathbf{a}_3 < \mathbf{a}_2 < \mathbf{a}_1.$$

In the correspondence $\mathbf{a}^i \to \mathbf{t}^{\mathbf{a}_i} \to z_i$, $\mathbf{t} = t_{11}t_{12}t_{13}t_{14}t_{15}t_{21}t_{22}t_{23}t_{24}t_{25}$ and $in_{\prec}(I_{\mathcal{A}})$ can be written respectively as $in_{\prec}(I_{\mathcal{A}}) = (z_2z_5, z_3z_5, z_3z_8, z_4z_6, z_4z_8)$ and the set $\{\{2,5\},\{3,5\},\{3,8\},\{4,6\},\{4,8\}\}$ does not belong to $\Delta_{\prec}(I_{\mathcal{A}})$. It follows that the set of maximal faces of is $\Delta^1_{\prec}(I_{\mathcal{A}})$ is:

$$\{\{1,2,3,4,7,9,10\},\{1,2,4,5,6,9,10\}\}.$$

Now, we consider the Hankel case for $n = 5$. We have the toric ideal generated by 6 quadratic binomials, obtaining a triangulation of the set of lattice points which is distinct from the triangulation of the same points in the grassmann case. $\square$

Theorem 7. *Let $K[M_{ij}]$ be the coordinates ring of $H(1,4)$. The set $\mathcal{A}$ has a triangulation with maximal faces set given by*

$$\{\{1,2,3,4,7,9\},\{2,3,4,7,9,10\},\{1,2,5,6,7,9\},\{2,5,6,8,9,10\}\}$$

Proof. Consider the unique Hankel relation $T_{12}T_{45}$-$T_{13}T_{35}$. Since the initial monomial coming from the product of two incomparable elements $[1,2],[4,5]$ is coprime with the other initial monomials of the elements of the Gröbner basis of $I_{\mathcal{A}}$ we do not have new relations coming from the S-couples. Then $I_{\mathcal{A}}$ admits a quadratic Gröbner basis given by

$$\{T_{14}T_{23}-T_{13}T_{24}, T_{15}T_{23}-T_{13}T_{25}, T_{15}T_{24}-T_{14}T_{25},$$
$$T_{15}T_{34}-T_{14}T_{35}, T_{25}T_{34}-T_{24}T_{35}, T_{12}T_{45}-T_{13}T_{35}\}.$$

Then, $in_{\prec}(I_{\mathcal{A}}) = (T_{14}T_{23}, T_{15}T_{23}, T_{15}T_{24}, T_{15}T_{34}, T_{25}T_{34})$. Fixing the order on the old variables $t_{11} > t_{12} > t_{13} > t_{14} > t_{15} > t_{21} > t_{22} > t_{23} > t_{24} > t_{25}$, and the lex order on the monomials $t_{11}t_{13} > t_{11}t_{14} > t_{11}t_{15} > t_{11}t_{25} > t_{12}t_{14} > t_{12}t_{15} > t_{12}t_{25} > t_{13}t_{15} > t_{14}t_{25}$, we can order the set of lattice points as:

$$(1,0,1,0,0,0,0,0,0,0) > (1,0,0,1,0,0,0,0,0,0) > (1,0,0,0,1,0,0,0,0,0) >$$
$$(1,0,0,0,0,0,0,0,0,1) > (0,1,0,1,0,0,0,0,0,0) > (0,1,0,0,1,0,0,0,0,0) >$$
$$((0,1,0,0,0,0,0,0,0,1) > (0,0,1,0,1,0,0,0,0,0) > (0,0,1,0,1,0,0,0,0,1)$$
$$> (0,0,0,1,0,0,0,0,0,1)$$

and

$$\mathbf{a}_{10} < \mathbf{a}_9 < \mathbf{a}_8 < \mathbf{a}_7 < \mathbf{a}_6 < \mathbf{a}_5 < \mathbf{a}_4 < \mathbf{a}_3 < \mathbf{a}_2 < \mathbf{a}_1.$$

In the correspondence $\mathbf{a}^i \to \mathbf{t}^{\mathbf{a}_i} \to z_i$, $\mathbf{t} = t_{11}t_{12}t_{13}t_{14}t_{15}t_{21}t_{22}t_{23}t_{24}t_{25}$, $in_{\prec}(I_{\mathcal{A}}) = (z_1z_{10}, z_3z_5, z_4z_5, z_4z_6, z_4z_8, z_7z_8)$ and the set

$$\{\{1,10\},\{3,5\},\{4,5\},\{4,6\},\{4,8\},\{7,8\}\}$$

does not belong to $\Delta_{\prec}(I_{\mathcal{A}})$. It follows that the set of maximal faces of $\Delta_{\prec}(I_{\mathcal{A}})$ is:

$$\{\{1,2,3,4,7,9\},\{2,3,4,7,9,10\},\{1,2,5,6,7,9\},\{2,5,6,8,9,10\}\}.$$

□

Remark 2. It is an open problem to study all triangulations of the set $\mathcal{A}$. Since we can have initial terms of degree > 2 for a generic term order, sets of cardinality > 2 do not belong to the initial complex and the resulting triangulation can be more complicated. In the same direction triangulations arising from the semigroups studied in ([11], [7], [8]) can be investigated. Some results for linear orders ([16]) are contained in [3]. Moreover, by working directly on Stanley Reisner ideals, simplicial complexes are studied in [15].

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