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VISCOSITY APPROXIMATION METHOD FOR SOLVING MINIMIZATION PROBLEM AND FIXED POINT PROBLEM FOR NONEXPANSIVE MULTIVALUED MAPPING IN CAT(0) SPACES*

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Abstract

In this paper, we propose and study some viscosity-type proximal point algorithms for approximating a common solution of minimization problem and fixed point problem in a CAT(0) space. Using our algorithms, we prove that the proposed implicit iteration net and sequence both converge strongly to a common solution of minimization problem and fixed point problem for nonexpansive multivalued mappings which is also a unique solution of some variational inequalities. Furthermore, numerical examples of our algorithm are given to show its advantage over existing algorithms in the literature. Our theorems extend and improve some related results in literature.

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1 Introduction

Let (X,d) be a metric space, a geodesic path joining $x \in X$ to $y \in X$ is a mapping $c : [0,l] \subset \mathcal{R} \to X$ (where $\mathcal{R}$ is the set of real numbers) such that $c(0) = x$, $c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0,l]$. In this case, the mapping c is called an isometry (i.e., a distance-preserving mapping) and $l = d(x,y)$. The image of c is called a geodesic segment joining x to y. When this image is unique, it is denoted by $[x,y]$. The space (X,d) is said to be a geodesic space, if every two points of X are joined by a geodesic and (X,d) is said to be uniquely geodesic, if there is exactly one geodesic joining x to y for each $x, y \in X$. A geodesic triangle $\triangle(x_1, x_2, x_3)$ is a geodesic metric space (X,d) that consists of three points $x_1, x_2, x_3 \in X$ (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A comparison triangle for the geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X,d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in the Euclidean plane $\mathcal{R}^2$ such that $d(x_i, x_j) = d_{\mathcal{R}^2}(\overline{x_i}, \overline{x_j})$ for all $i, j \in \{1,2,3\}$.
A geodesic metric space is said to be a CAT(0) space if all geodesic triangles satisfy the following CAT(0) comparison axiom.
CAT(0): Let $\triangle$ be a geodesic triangle in X and $\overline{\triangle} \subset \mathcal{R}^2$ be a comparison triangle for $\triangle$. Then, $\triangle$ is said to satisfy the CAT(0) inequality if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$,

$$d(x,y) \leq d_{\mathcal{R}^2}(\overline{x}, \overline{y}).$$

If x, y, z are points in a CAT(0) space X and y_0 is the midpoint of the segment $[y,z]$, then the CAT(0) inequality implies

$$d^2(x, y_0) \leq \frac{1}{2}d^2(x,y) + \frac{1}{2}d^2(x,z) - \frac{1}{4}d(y,z). \tag{1}$$

Inequality (1) is known as the (CN) inequality of Bruhat and Titis [9], and it was later extended by Khamsi and Kirk [23] to the following:

$$d^2(z, tx \oplus (1-t)y) \leq td^2(z,x) + (1-t)d^2(z,y) - t(1-t)d^2(x,y), \tag{2}$$

where $t \in [0,1]$, $x, y, z \in X$ and $tx \oplus (1-t)y$ denotes the unique point in $[x,y]$. Recall that, if X is a CAT(0) space and $x, y \in X$, then for any $t \in [0,1]$, $tx \oplus (1-t)y$ can be written as the unique point $z \in [x,y]$ such that $d(z,x) = (1-t)d(x,y)$ and $d(z,y) = td(x,y)$. Let $[x,y] := \{(1-t)x \oplus ty : t \in [0,1]\}$, then a subset C of X is convex if $[x,y] \subseteq C$ for all $x, y \in C$.

For a metric space X, a subset C of X is called proximinal, if for each $x \in X$, there exists $z \in C$ such that $d(x,z) = \inf\{d(x,y) : y \in C\}$. We shall denote by $P(X)$, the family of all nonempty proximinal subsets of X, $CB(X)$ the family of all nonempty closed and bounded subsets of X and 2^X the family of all nonempty subsets of X. Let H denote the Hausdorff metric induced by the metric d, then for all $A, B \in 2^X$,

$$H(A,B) = \max\{\sup_{a\in A} d(a,B),\ \sup_{b\in B} d(b,A)\},$$

where $d(a,B) = \inf_{b\in B} d(a,b)$ is the distance from the point a to the subset B. Let $T : X \to 2^X$ be a multivalued mapping. A point $x \in X$ is called a fixed of T, if $x \in Tx$ while $x \in X$ is called a strict fixed point of T, if $Tx = \{x\}$. We denote by $F(T)$ the fixed points set of T.
A mapping $T : X \to 2^X$ is called a contraction, if there exists $\rho \in (0,1)$ such that

$$H(Tx,Ty) \le \rho d(x,y), \text{ for all } x,y \in X.$$

If $\rho = 1$, then T is called nonexpansive.

Let $\{x_n\}$ be a bounded sequence in X and $r(.,\{x_n\}) : X \to [0,\infty)$ be a continuous functional defined by $r(x,\{x_n\}) = \limsup_{n\to\infty} d(x,x_n)$. The asymptotic radius of $\{x_n\}$ is given by $r(\{x_n\}) := \inf\{r(x,\{x_n\}) : x \in X\}$ while the asymptotic center of $\{x_n\}$ is the set $A(\{x_n\}) = \{x \in X : r(x,\{x_n\}) = r(\{x_n\})\}$. It is generally known that in a complete CAT(0) space, $A(\{x_n\})$ consists of exactly one point. A sequence $\{x_n\}$ in X is said to be Δ-convergent to a point $x \in X$, if $A(\{x_{n_k}\}) = \{x\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_{n\to\infty} x_n = x$ (see [13]). Δ-convergence in metric spaces was introduced and studied by Lim [29], which was later introduced in CAT(0) spaces and was shown to be very similar to the weak convergence in Banach space setting by Kirk and Panyanak [26]. In 2003, Kirk [25] proved that every nonexpansive sigle-valued mapping defined on a nonempty closed convex and bounded subset of a complete CAT(0) space has a fixed point. Later in 2009, Saejung [35] introduced the following Halpern iterative algorithm and proved its strong convergece to a fixed point of a nonexpansive single-valued mapping in a complete CAT(0) space: For arbitrary $u, x_1 \in C$, let $\{x_n\}$ be generated by

$$x_{n+1} = \alpha_n u \oplus (1-\alpha_n)Tx_n,\ \forall n \ge 1, \tag{3}$$

where C is a nonempty closed and convex subset of a CAT(0) space, T is a nonexpansive single-valued mapping on C and $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying some conditions.

In 2014, BO and Yi [8] introduced a viscosity iterative algorithm to approximate a fixed point of a nonexpansive multivalued mapping in a complete CAT(0) space. They proved that their proposed implicit iteration net and sequence both converge strongly to a fixed point of nonexpansive multivalued mappings which is also a unique solution of some variational inequalities. More precisely, they proved the following theorems.

Theorem 1. *Let C be a nonempty closed and convex subset of a complete CAT(0) space X and $T : C \to CB(C)$ be a nonexpansive multivalued mapping. Let f be a contraction on C with coefficient $0 < \alpha < 1$. For each $t \in (0, 1]$, let $\{x_t\}$ be given by the following implicit iteration*

$$x_t = tf(x_t) \oplus (1-t)u(x_t), \quad u(x_t) \in T(x_t). \tag{4}$$

Suppose $F(T) \neq \emptyset$, then $\{x_t\}$ converges strongly as $t \to 0$ to $\bar{x} = P_{F(T)}f(\bar{x})$ which also solves the following variational inequality

$$\langle \overrightarrow{\bar{x}f(\bar{x})}, \overrightarrow{x\bar{x}} \rangle \geq 0, \quad x \in F(T). \tag{5}$$

Theorem 2. *Let C be a nonempty bounded closed and convex subset of a complete CAT(0) space X. Let f be a contraction on C with coefficient $\alpha \in (0, 1)$, and let $T : C \to CB(C)$ be nonexpansive multi-valued mappings. Let $x_1 \in C$ be arbitrary and the sequence $\{x_n\}$ be generated by*

$$x_{n+1} = t_n f(x_n) \oplus (1-t_n)u(x_n), \quad u(x_n) \in Tx_n, \quad n \geq 1, \tag{6}$$

where $\{t_n\}$ is a sequence in $(0, 1)$ satisfying the following conditions:

(i) $\lim_{n\to\infty} t_n = 0$,

(ii) $\sum_{n=1}^{+\infty} t_n = +\infty$,

(iii) $\sum_{n=0}^{+\infty} |t_{n+1} - t_n| < +\infty$.

Suppose $F(T) \neq \emptyset$, then the sequence $\{x_n\}$ converges strongly to some point $\bar{x} \in F(T)$ which also solves (5).

In 2008, Berg and Nikolaev [7] introduced the concept of quasilinearization mapping in CAT(0) spaces. They denoted a pair $(a, b) \in X \times X$ by $\overrightarrow{ab}$ and called it a vector. Using this, they defined quasilinearization as a mapping $\langle ., . \rangle : (X \times X) \times (X \times X) \to \mathcal{R}$ given by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}\left(d^2(a,d) + d^2(b,c) - d^2(a,c) - d^2(b,d)\right), \ (a,b,c,d \in X).$$

Note that $\langle\overrightarrow{ab},\overrightarrow{ab}\rangle = d^2(a,b)$, $\langle\overrightarrow{ba},\overrightarrow{cd}\rangle = -\langle\overrightarrow{ab},\overrightarrow{cd}\rangle$, $\langle\overrightarrow{ab},\overrightarrow{cd}\rangle = \langle\overrightarrow{ae},\overrightarrow{cd}\rangle + \langle\overrightarrow{eb},\overrightarrow{cd}\rangle$ and $\langle\overrightarrow{ab},\overrightarrow{cd}\rangle = \langle\overrightarrow{cd},\overrightarrow{ab}\rangle$ for all $a,b,c,d,e \in X$. Recall that a geodesic space X is said to satisfy the Cauchy-Schwartz inequality if $\langle\overrightarrow{ab},\overrightarrow{cd}\rangle \leq d(a,b)d(c,d)\ \forall a,b,c,d \in X$. It was shown in [7] that a geodesically connected metric space is a CAT(0) space if and only if it satisfies the Cauchy-Schwartz inequality.

One of the most important problems in optimization theory and nonlinear analysis is the problem of approximating solutions of Minimization Problem (MP), which is defined as follows: Find $x \in X$ such that

$$f(x) = \min_{y\in X} f(y). \tag{7}$$

The solution set of MP (7) is denote by $\arg\min_{y\in X} f(y)$. Let X be a CAT(0) space. The function $f : X \to (-\infty, +\infty]$ is said to be convex, if

$$f(tx \oplus (1-t)y) \leq tf(x) + (1-t)f(y)\ \forall x, y \in X,\ t \in (0,1).$$

f is called proper, if $D(f) := \{x \in X : f(x) < +\infty\} \neq \emptyset$, where $D(f)$ denotes the domain of f. The function $f : D(f) \to (-\infty, \infty]$ is said to be lower semi-continuous at a point $x \in D(f)$ if

$$f(x) \leq \liminf_{n\to\infty} f(x_n), \tag{8}$$

for each sequence $\{x_n\}$ such that $\{x_n\}$ converges strongly to x. f is said to be lower semi-continuous on $D(f)$ if it is lower semi-continuous at any point in $D(f)$. For any $\lambda > 0$, the Moreau-Yosida resolvent of f in a complete CAT(0) space is defined in [4] as follows:

$$J_\lambda^f(x) = arg\min_{y\in X}\left[f(y) + \frac{1}{2\lambda}d^2(y,x)\right], \tag{9}$$

for all $x \in X$. The mapping J_λ^f is nonexpansive and well defined for all $\lambda > 0$ (see [31]). If f is a proper convex and lower semi-continuous function, then the set $F(J_\lambda^f)$ of fixed points of J_λ^f coincides with the set $argmin_{y\in X} f(y)$ of minimizers of f (see [31, 38]).

In 1970, Martinet [30] introduced the well known Proximal Point Algorithm (PPA) which is a powerful and one of the most popular tool for finding solutions of MP (7). Rockafellar [34] further studied the PPA in Hilbert spaces for approximating solutions of (7). The PPA and their modifications have been studied extensively in both Hilbert and Banach space settings by numerous others (see [1, 14, 18, 19, 20, 33, 36] and the references therein).

The PPA was later introduced and studied in CAT(0) spaces by Bačák [5] for approximating a solution of MP (7), using the following algorithm: For arbitrary $x_1 \in X$, the sequence $\{x_n\}$ is defined by

$$x_{n+1} = \arg\min_{y \in X} \left(f(y) + \frac{1}{2\lambda_n} d^2(y, x_n) \right), \tag{10}$$

for each $n \geq 1$, where $\lambda_n > 0$ for all $n \geq 1$. Bačák [5] proved that $\{x_n\}$ Δ-converges to a minimizer of f under the conditions that f has a minimizer in X and $\sum_{n=1}^{\infty} \lambda_n = \infty$. In 2014, Bačák [6] employed a split version of the PPA for minimizing a sum of convex functions in complete CAT(0) spaces. Very recently, Suparatulatorn *et al.* [38] proposed a new algorithm by combining the PPA and the Halpern's iteration process (resulting into a Halpern-type PPA) to approximate a common solution of minimization problem and fixed point problem for nonexpansive single-valued mapping in a complete CAT(0) space. They proved the following strong convergence result.

Theorem 3. *Let (X, d) be a complete CAT(0) space and $f : X \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Let T be a nonexpansive mapping on X such that $\Omega := F(T) \cap argmin_{y \in X} f(y)$ is nonempty. Assume that $\{\lambda_n\}$ is a sequence such that $\lambda_n \geq \lambda > 0$ for some λ and for all $n \geq 1$. Suppose that $u, x_1 \in X$ are arbitrarily chosen and $\{x_n\}$ is generated in the following manner:*

$$\begin{cases} y_n = argmin_{y \in X} \left[f(y) + \frac{1}{2\lambda_n} d^2(y, x_n) \right], \\ x_{n+1} = \alpha_n u \oplus (1 - \alpha_n) T y_n, \end{cases} \tag{11}$$

for each $n \geq 1$, where $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying
(I)$\lim_{n \to \infty} \alpha_n = 0$,
(II) $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(III) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$,
(IV) $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$.
Then $\{x_n\}$ strongly converges to $z \in \Omega$ which is the nearest point of Ω to u.

Halpern-type PPAs have also been studied in CAT(0) spaces by many authors (see [2, 21, 32, 39]), and have shown to converge strongly. However, we point out here that viscosity-type algorithms which also converge strongly, have higher rate of convergence than Halpern-type algorithms (see [17, Remark 3.7]). Moreover, Halpern-type convergence theorems implies viscosity-type convergence theorems for contractions (see [17, 37]).

Motivated by this, and inspired by the works of Bo and Yi [8], Suparatulatorn *et al.* [38] and other related works in literature, we introduce a viscosity-type PPA for approximating a common solution of MP and fixed point problem for nonexpansive multivalued mapping in CAT(0) spaces. We prove that the proposed implicit iteration net and sequence both converge strongly to a common solution of MP and fixed point problem for nonexpansive multivalued mappings which is also a unique solution of some variational inequalities. Furthermore, numerical examples of our algorithm are given to show its advantage over existing algorithms in the literature. Our results extend and complement the results of Saejung [35], Liu and Yi [8], Suparatulatorn *et al.* [38] and a host of other results in this direction.

2 Preliminaries

We shall need the following results in the proof of our main results.

Definition 1. *[11] Let C be a nonempty closed and convex subset of a CAT(0) space X. The metric projection of X onto C is a mapping $P_C : X \to C$ which assigns to each $x \in X$, the unique point $P_C x$ in C such that $d(x, P_C x) = \inf\{d(x, y) : y \in C\}$.*

Lemma 1. *[12] Let X be a CAT(0) space. Then, for all $x, y, z \in X$ and $t, s \in [0, 1]$, the following hold:*

(i) $d(tx \oplus (1-t)y, z) \leq td(x, z) + (1-t)d(y, z)$,

(ii) $d^2(tx \oplus (1-t)y, z) \leq td^2(x, z) + (1-t)d^2(y, z) - t(1-t)d^2(x, y)$,

(iii) $d(tx \oplus (1-t)y, sx \oplus (1-s)y) \leq |t-s|d(x, y)$,

(iv) $d(tx \oplus (1-t)y, tx \oplus (1-t)z) \leq (1-t)d(y, z)$.

Lemma 2. *[31]. Let X be a complete CAT(0) space and $f : X \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. For each $x \in X$ and $\lambda \geq \mu > 0$, the following identity holds:*

$$J_\lambda^f x = J_\mu^f \left(\frac{\lambda - \mu}{\lambda} J_\lambda^f x \oplus \frac{\mu}{\lambda} x \right),$$

where J_λ^f is the resolvent of f.

Lemma 3. *[40] Let X be a complete CAT(0) space. For any $t \in [0,1]$ and $u, v \in X$, let $u_t = tu \oplus (1-t)v$. Then, for all $x, y \in X$,*

(i) $\langle \overrightarrow{u_t x}, \overrightarrow{u_t y} \rangle \leq t\langle \overrightarrow{ux}, \overrightarrow{u_t y} \rangle + (1-t)\langle \overrightarrow{vx}, \overrightarrow{u_t y} \rangle$;

(ii) $\langle \overrightarrow{u_t x}, \overrightarrow{uy} \rangle \leq t\langle \overrightarrow{ux}, \overrightarrow{uy} \rangle + (1-t)\langle \overrightarrow{vx}, \overrightarrow{uy} \rangle$ *and* $\langle \overrightarrow{u_t x}, \overrightarrow{vy} \rangle + (1-t)\langle \overrightarrow{vx}, \overrightarrow{vy} \rangle$;

(iii) $d^2(x,u) \leq d^2(y,u) + 2\langle \overrightarrow{xy}, \overrightarrow{xu} \rangle$.

Lemma 4. *[3] Let X be a complete CAT(0) space and $f : X \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Then, for all $x, y \in X$ and $\lambda > 0$, we have*

$$\frac{1}{2\lambda} d^2(J_\lambda^f x, y) - \frac{1}{2\lambda} d^2(x,y) + \frac{1}{2\lambda} d^2(x, J_\lambda^f x) + f(J_\lambda^f x) \leq f(y).$$

Lemma 5. *[12] Every bounded sequence in a complete CAT(0) space always have a $\triangle$-convergent subsequence.*

Lemma 6. *[22] Let X be a complete CAT(0) space and $\{x_n\}$ be a bounded sequence in X. Then $\{x_n\}\Delta$-converges to x if and only if $\limsup_{n\to\infty} \langle \overrightarrow{x_n x}, \overrightarrow{yx} \rangle \leq 0$ for all $y \in X$.*

Lemma 7. *[11, Theorem 2.4] Let C be a nonempty closed and convex subset of a complete CAT(0) space X, $x \in X$ and $u \in C$. Then, $u = P_C x$ if and only if $\langle \overrightarrow{yu}, \overrightarrow{ux} \rangle \geq 0 \ \forall y \in C$.*

Lemma 8. *[8] Let C be a closed and convex subset of a complete CAT(0) space X and $T : C \to CB(C)$ be a nonexpansive multivalued mapping, then the conditions that $\{x_n\}$ Δ-converges to x and $\{d(x_n, z_n)\}$ converges strongly to 0 (where $z_n \in Tx_n$), imply that $x \in Tx$.*

Lemma 9. *[41] Let $\{a_n\}$ be a sequence of non-negative real numbers satisfying*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n \delta_n + \gamma_n, \quad n \geq 0,$$

where $\{\alpha_n\}$, $\{\delta_n\}$ and $\{\gamma_n\}$ satisfy the following conditions:
(i) $\{\alpha_n\} \subset [0,1]$, $\sum_{n=0}^{\infty} \alpha_n = \infty$,
(ii) $\limsup_{n\to\infty} \delta_n \leq 0$,
(iii) $\gamma_n \geq 0 (n \geq 0)$, $\sum_{n=0}^{\infty} \gamma_n < \infty$.
Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 10. *[16] Let X be a metric space and $A, B \in P(X)$. Then, for all $a \in A$, there exists $b \in B$ such that $d(a,b) \leq H(A,B)$.*

3 Main Results

The following lemma will be very useful for our study.

Lemma 11. . *Let X be a complete CAT(0) space and $f : X \to (-\infty, \infty]$ be proper convex and lower semi-continuous function. Then,* $d^2(J_\lambda^f x, x) \leq d^2(J_\mu^f x, x)$ *for* $0 < \lambda < \mu$ *and* $x \in X$.

Proof:
Let $x, y \in X$, then we obtain from (9) that

$$f(J_\mu^f x) + \frac{1}{2\mu} d^2(J_\mu^f x, x) \leq f(y) + \frac{1}{2\mu} d^2(y, x).$$

In particular, we have that

$$f(J_\mu^f x) + \frac{1}{2\mu} d^2(J_\mu^f x, x) \leq f(J_\lambda^f x) + \frac{1}{2\mu} d^2(J_\lambda^f x, x). \tag{10}$$

Similarly, we obtain

$$f(J_\lambda^f x) + \frac{1}{2\lambda} d^2(J_\lambda^f x, x) \leq f(J_\mu^f x) + \frac{1}{2\lambda} d^2(J_\mu^f x, x). \tag{11}$$

Adding (10) and (11), we obtain that

$$d^2(J_\lambda^f x, x) - \frac{\lambda}{\mu} d^2(J_\lambda^f x, x) \leq d^2(J_\mu^f x, x) - \frac{\lambda}{\mu} d^2(J_\mu^f x, x).$$

That is,

$$\left(1 - \frac{\lambda}{\mu}\right) d^2(J_\lambda^f x, x) \leq \left(1 - \frac{\lambda}{\mu}\right) d^2(J_\mu^f x, x).$$

Since, $0 < \lambda < \mu$, we obtain that

$$d^2(J_\lambda^f x, x) \leq d^2(J_\mu^f x, x).$$

In what follows, we propose our implicit iterative net for our first convergence theorem: For each $t \in (0, 1]$, let the net $\{x_t\}$ be generated by

$$\begin{cases} u_t = J_{\lambda_t}^f (x_t), \\ x_t = tg(u_t) \oplus (1-t)v_t, \quad v_t \in Tu_t, \end{cases} \tag{12}$$

where $J_{\lambda_t}^f$ is the resolvent of a proper convex and lower semi-continmuous function f, g is a contraction mapping and T is a nonexpansive multivalued mapping.

The implicit iterative net (12) clearly generalizes the following implicit iteration studied by Saejung [35]: For $t \in (0,1)$ and fixed $u \in C$, $\{x_t\}$ is defined by

$$x_t = tu \oplus (1-t)Tx_t, \tag{13}$$

where T is a nonexpansive sigle-valued mapping defined on C.
Observe that the implicit iteration (13) is of Halpern-type, and as mentioned earlier, the rate of convergence of Halpern-type iterations is relatively lower than that of viscosity-type iterations.
Note also that, if $J_{\lambda_t}^f \equiv I$ in (12) (where I is the identity mapping on X), then (12) reduces to (4) studied by Bo and Yi [8]. Thus, iterative net (12) extends the implicit iteration of Bo and Yi [8] to an implicit proximal point iteration.

We now present our strong convergence theorem for the implicit proximal point iteration (12).

Theorem 4. *Let X be a complete CAT(0) space and $f : X \to (-\infty, +\infty]$ be a proper convex and lower semi-continuous function. Let $T : X \to P(X)$ be a nonexpansive multivalued mapping such that $Tp = \{p\}$, for each $p \in F(T)$ and $\Gamma := F(T) \cap argmin_{y\in X} f(y)$ is nonempty. Suppose that g is a contraction mapping defined on X with coefficient $\rho \in (0,1)$ and $\lambda_t \geq \lambda > 0$ for some λ. Let the net $\{x_t\}$ be defined by (12) such that Lemma 10 holds. Then, the net $\{x_t\}$ converges strongly to $\overline{x} = P_\Gamma g(\bar{x})$ (where P_Γ is the metric projection of X onto Γ) which also solves the variational inequality*

$$\langle \overrightarrow{\overline{x}g(\overline{x})}, \overrightarrow{x\overline{x}} \rangle \geq 0, \ \forall x \in \Gamma. \tag{14}$$

Proof: By similar argument as in the proof of [8, Page 53], we obtain that (12) is well-defined.
Let $p \in \Gamma$, then from (12), Lemma 1 and Lemma 10, we obtain

$$\begin{aligned} d(x_t,p) &= d(tg(u_t) \oplus (1-t)v_t, p) \\ &\leq td(g(u_t),p) + (1-t)d(v_t,p) \\ &\leq td(g(u_t),p)) + (1-t)H(Tu_t, Tp) \\ &\leq t(d(g(u_t),g(p)) + d(g(p),p)) + (1-t)d(u_t,p) \\ &\leq t(\rho d(u_t,p)) + d(g(p),p)) + (1-t)d(u_t,p) \\ &= t(\rho d(J_{\lambda_t}^f x_t,p)) + d(g(p),p)) + (1-t)d(J_{\lambda_t} x_t,p) \\ &\leq (t\rho + (1-t))d(x_t,p) + td(g(p),p), \end{aligned}$$

which implies

$$d(x_t, p) \leq \frac{1}{1-\rho} d(g(p), p).$$

Hence, $\{x_t\}$ is bounded. Consequently, $\{u_t\}$, $\{v_t\}$ and $\{g(u_t)\}$ are also bounded.

From (12), we have

$$\begin{aligned} \lim_{t\to 0} d(x_t, v_t) &= \lim_{t\to 0} d(tg(u_t) \oplus (1-t)v_t, v_t) \\ &\leq \lim_{t\to 0} td(g(u_t), v_t) = 0. \end{aligned} \tag{15}$$

From Lemma 4, we have that

$$\frac{1}{2\lambda_t} d^2(u_t, p) - \frac{1}{2\lambda_t} d^2(x_t, p) + \frac{1}{2\lambda_t} d^2(x_t, u_t) \leq f(p) - f(u_t).$$

Since $f(p) \leq f(u_t)$ for all $n \geq 1$, we obtain

$$d^2(u_t, p) \leq d^2(x_t, p) - d^2(x_t, u_t). \tag{16}$$

Thus, using (16) and Lemma 1, we have

$$\begin{aligned} d^2(x_t, p) &= d^2(tg(u_t) \oplus (1-t)v_t, p) \\ &\leq td^2(g(u_t), p) + (1-t)d^2(v_t, p) \\ &\leq td^2(g(u_t), p) + (1-t)H^2(Tu_t, Tp) \\ &\leq td^2(g(u_t), p) + (1-t)d^2(u_t, p) \\ &\leq td^2(g(u_t), p) + d^2(x_t, p) - d^2(x_t, u_t), \end{aligned}$$

which implies that

$$d^2(x_t, u_t) \leq td^2(g(u_t), p) \to 0, \text{ as } t \to 0.$$

That is,

$$\lim_{t\to 0} d^2(x_t, u_t) = 0. \tag{17}$$

From (15) and (17), we obtain

$$\lim_{t\to 0} d^2(u_t, v_t) = 0. \tag{18}$$

Since $\lambda_t \geq \lambda > 0$, we obtain from (17) and Lemma 11 that

$$\begin{aligned} d(x_t, J_\lambda^f x_t) &\leq d(x_t, J_{\lambda_t}^f x_t) \\ &= d(x_t, u_t) \to 0 \text{ as } t \to 0. \end{aligned} \tag{19}$$

Let $x_m := x_{t_m}$ for all $m \geq 1$, with $t_m \in (0,1]$ and $t_m \to 0$, as $m \to \infty$. Since $\{x_m\}$ is bounded and X is a complete CAT(0) space, then from Lemma 5, we may assume that $\triangle\text{-}\lim_{m\to\infty} x_m = \overline{x}$. Since T is a nonexpansive multivalued mapping, it follows from (17), (18) and Lemma 8 that $\overline{x} \in F(T)$. Also, since J_λ^f is a nonexpansive mapping, it follows from (19) and Lemma 8 that $\overline{x} \in F(J_\lambda^f)$. Therefore $\{x_m\}$ Δ-converges to $\overline{x} \in \Gamma$. Thus, by Lemma 6, we obtain

$$\limsup_{m\to\infty} \langle \overrightarrow{g(\bar{x})\bar{x}}, \overrightarrow{x_m\bar{x}} \rangle \leq 0. \tag{20}$$

We now show that $\lim_{m\to\infty} x_m = \bar{x}$. From Lemma 3, we obtain

$$\begin{aligned} d^2(x_m, \bar{x}) &= \langle \overrightarrow{x_m\bar{x}}, \overrightarrow{x_m\bar{x}} \rangle \\ &\leq t_m \langle \overrightarrow{g(u_m)\bar{x}}, \overrightarrow{x_m\bar{x}} \rangle + (1-t_m)\langle \overrightarrow{v_m\bar{x}}, \overrightarrow{x_m\bar{x}} \rangle \\ &\leq t_m \langle \overrightarrow{g(u_m)\bar{x}}, \overrightarrow{x_m\bar{x}} \rangle + (1-t_m) d(v_m, \bar{x}) d(x_m, \bar{x}) \\ &\leq t_m \langle \overrightarrow{g(u_m)\bar{x}}, \overrightarrow{x_m\bar{x}} \rangle + (1-t_m) H(Tu_m, \bar{x}) d(x_m, \bar{x}) \\ &\leq t_m \langle \overrightarrow{g(u_m)\bar{x}}, \overrightarrow{x_m\bar{x}} \rangle + (1-t_m) d(J_{\lambda_m}^f x_m, \bar{x}) d(x_m, \bar{x}) \\ &\leq t_m \langle \overrightarrow{g(u_m)\bar{x}}, \overrightarrow{x_m\bar{x}} \rangle + (1-t_m) d^2(x_m, \bar{x}) \\ &\leq t_m \langle \overrightarrow{g(u_m)g(\bar{x})}, \overrightarrow{x_m\bar{x}} \rangle + t_m \langle \overrightarrow{g(\bar{x})\bar{x}}, \overrightarrow{x_m\bar{x}} \rangle + (1-t_m) d^2(x_m, \bar{x}) \\ &\leq t_m \rho d^2(x_m, \bar{x}) + t_m \langle \overrightarrow{g(\bar{x})\bar{x}}, \overrightarrow{x_m\bar{x}} \rangle + (1-t_m) d^2(x_m, \bar{x}), \end{aligned}$$

which implies

$$d^2(x_m, \bar{x}) \leq \frac{1}{1-\rho} \langle \overrightarrow{g(\bar{x})\bar{x}}, \overrightarrow{x_m\bar{x}} \rangle. \tag{21}$$

Thus, from (20) and (21), we obtain that

$$\lim_{m\to\infty} x_m = \bar{x}. \tag{22}$$

Next, we show that $\bar{x} \in \Gamma$ solves the variational inequality (14). From (12), Lemma 1 and Lemma 10, we obtain for any $z \in \Gamma$ that

$$\begin{aligned} d^2(x_t, z) &= d^2(tg(u_t) \oplus (1-t)v_t, z) \\ &\leq td^2(g(u_t), z) + (1-t)d^2(v_t, z) - t(1-t)d^2(g(u_t), v_t) \\ &\leq td^2(g(u_t), z) + (1-t)d^2(x_t, z) - t(1-t)d^2(g(u_t), v_t), \end{aligned}$$

which implies

$$d^2(x_t, z) \leq d^2(g(u_t), z) - (1-t)d^2(g(u_t), v_t).$$

So that

$$d^2(x_m, z) \leq d^2(g(u_m), z) - (1-t_m)d^2(g(u_m), v_m). \tag{23}$$

Taking limit as $m \to \infty$, we obtain from (15), (17) and (23) that

$$d^2(\bar{x}, z) \leq d^2(g(\bar{x}), z) - d^2(g(\bar{x}), \bar{x}). \tag{24}$$

From (24) and by the definition of quasilinearization mapping, we obtain

$$\langle \overrightarrow{\bar{x}g(\bar{x})}, \overrightarrow{z\bar{x}} \rangle = \frac{1}{2}\left(d^2(g(\bar{x}), z) - d^2(g(\bar{x}), \bar{x}) - d^2(\bar{x}, z)\right) \geq 0, \ \forall z \in \Gamma. \tag{25}$$

Thus, $\bar{x} \in \Gamma$ solves the variational inequality (14).

We now show that the net $\{x_t\}$ converges strongly to $\bar{x}$. We may assume that $x_{s_m} \to x^* \in \Gamma$, where $s_m \to 0$ as $m \to \infty$. Then by same argument as above, we obtain that x^* also solves the variational inequality (14). That is,

$$\langle \overrightarrow{\bar{x}g(\bar{x})}, \overrightarrow{\bar{x}x^*} \rangle \leq 0, \quad \langle \overrightarrow{x^*g(x^*)}, \overrightarrow{x^*\bar{x}} \rangle \leq 0.$$

Thus,

$$\begin{aligned}
0 &\geq \langle \overrightarrow{\bar{x}g(\bar{x})}, \overrightarrow{\bar{x}x^*} \rangle - \langle \overrightarrow{x^*g(x^*)}, \overrightarrow{\bar{x}x^*} \rangle \\
&= \langle \overrightarrow{\bar{x}g(x^*)}, \overrightarrow{\bar{x}x^*} \rangle + \langle \overrightarrow{g(x^*)g(\bar{x})}, \overrightarrow{\bar{x}x^*} \rangle - \langle \overrightarrow{x^*\bar{x}}, \overrightarrow{\bar{x}x^*} \rangle - \langle \overrightarrow{\bar{x}g(x^*)}, \overrightarrow{\bar{x}x^*} \rangle \\
&= \langle \overrightarrow{\bar{x}x^*}, \overrightarrow{\bar{x}x^*} \rangle - \langle \overrightarrow{g(x^*)g(\bar{x})}, \overrightarrow{x^*\bar{x}} \rangle \\
&\geq (1-\rho)d^2(\bar{x}, x^*),
\end{aligned}$$

which implies that $d^2(\bar{x}, x^*) = 0$. Thus, $\bar{x} = x^*$. Hence, the net $\{x_t\}$ converges to $\bar{x} \in \Gamma$ which also solves the variational inequality (14).

Next, we present the following strong convergence theorem for our proposed viscosity-type PPA.

Theorem 5. *Let X be a complete CAT(0) space and $f : X \to (-\infty, +\infty]$ be a proper convex and lower semi-continuous function. Let $T : X \to P(X)$ be a nonexpansive multivalued mapping such that $Tp = \{p\}$, for each $p \in F(T)$ and $\Gamma := F(T) \cap argmin_{y \in X} f(y)$ is nonempty. Suppose that g is a contraction mapping defined on X with coefficient $\rho \in (0, 1)$ and $\lambda_n \geq \lambda > 0$*

for some λ. Let $x_1 \in X$ be arbitrarily chosen and the sequence $\{x_n\}$ be generated by

$$\begin{cases} u_n = J^f_{\lambda_n}(x_n), \\ x_{n+1} = t_n g(u_n) \oplus (1-t_n)v_n, \quad \text{where } v_n \in T(u_n) \forall n \geq 1, \end{cases} \tag{26}$$

such that Lemma 10 holds and $\{t_n\}$ is a sequence in $(0,1)$ satisfying
(i) $\lim_{n\to\infty} t_n = 0$,
(ii) $\sum_{n=1}^{\infty} t_n = \infty$,
(iii) $\sum_{n=1}^{\infty} |t_{n+1} - t_n| < \infty$,
(iv) $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$.
Then the sequence $\{x_n\}$ converges strongly to $\overline{x} \in \Gamma$ which also solves the variational inequality (14).

Proof: Let $p \in \Gamma$, then from (26), Lemma 1 and Lemma 10, we obtain

$$\begin{aligned} d(x_{n+1}, p) &= d(t_n g(u_n) \oplus (1-t_n)v_n, p) \\ &\leq t_n d(g(u_n), p) + (1-t_n) d(v_n, p) \\ &\leq t_n d(g(u_n), p)) + (1-t_n) H(Tu_n, Tp) \\ &\leq t_n(d(g(u_n), g(p)) + d(g(p), p)) + (1-t_n) d(u_n, p) \\ &\leq t_n(\rho d(u_n, p)) + d(g(p), p)) + (1-t_n) d(u_n, p) \\ &\leq (t_n \rho + (1-t_n)) d(x_n, p) + t_n d(g(p), p), \end{aligned}$$

that is

$$d(x_{n+1}, p) \leq \max\{d(x_n, p), \frac{1}{1-\rho} d(g(p), p)\}.$$

By induction, we obtain that

$$d(x_{n+1}, p) \leq \max\{d(x_1, p), \frac{1}{1-\rho} d(g(p), p)\}.$$

Hence, $\{x_n\}$ is bounded. Consequently, $\{u_n\}$, $\{v_n\}$ and $\{g(u_n)\}$ are also bounded.

Next, we show that $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$. Without loss of generality, let us assume that $\lambda_n \geq \lambda_{n-1}$. Since $\lambda_n \geq \lambda > 0\ \forall n \geq 1$, then from Lemma

2, we obtain

$$\begin{aligned}
&d(u_n, u_{n-1}) \leq d(u_n, J^f_{\lambda_n} x_{n-1}) + d(J^f_{\lambda_n} x_{n-1}, u_{n-1}) \\
&= d(J^f_{\lambda_n} x_n, J^f_{\lambda_n} x_{n-1}) + d(J^f_{\lambda_n} x_{n-1}, J^f_{\lambda_{n-1}} x_{n-1}) \\
&\leq d(x_n, x_{n-1}) + d\Bigg(J^f_{\lambda_{n-1}} \big(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n} J^f_{\lambda_n} x_{n-1} \oplus \frac{\lambda_{n-1}}{\lambda_n} x_{n-1} \big), J^f_{\lambda_{n-1}} x_{n-1} \Bigg) \\
&\leq d(x_n, x_{n-1}) + d\Bigg(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n} J^f_{\lambda_n} x_{n-1} \oplus \frac{\lambda_{n-1}}{\lambda_n} x_{n-1}, x_{n-1} \Bigg) \\
&= d(x_n, x_{n-1}) + \frac{\lambda_n - \lambda_{n-1}}{\lambda_n} d\big(J^f_{\lambda_n} x_{n-1}, x_{n-1}\big) \\
&= d(x_n, x_{n-1}) + \frac{|\lambda_n - \lambda_{n-1}|}{\lambda_n} d\big(J^f_{\lambda_n} x_{n-1}, x_{n-1}\big) \\
&\leq d(x_n, x_{n-1}) + \frac{|\lambda_n - \lambda_{n-1}|}{\lambda} d\big(J^f_{\lambda_n} x_{n-1}, x_{n-1}\big). \qquad (27)
\end{aligned}$$

Also, from (26) and (27), we obtain

$$\begin{aligned}
&d(x_{n+1}, x_n) = d(t_n g(u_n) \oplus (1-t_n) v_n, t_{n-1} g(u_{n-1}) \oplus (1-t_{n-1}) v_{n-1}) \\
&\leq d(t_n g(u_n) \oplus (1-t_n) v_n, t_n g(u_n) \oplus (1-t_n) v_{n-1}) \\
&+ d(t_n g(u_n) \oplus (1-t_n) v_{n-1}, t_n g(u_{n-1}) \oplus (1-t_n) v_{n-1}) \\
&+ d(t_n g(u_{n-1}) \oplus (1-t_n) v_{n-1}, t_{n-1} g(u_{n-1}) \oplus (1-t_{n-1}) v_{n-1}) \\
&\leq (1-t_n) d(v_n, v_{n-1}) + t_n d(g(u_n), g(u_{n-1})) \\
&+ |t_n - t_{n-1}| d(g(u_{n-1}), v_{n-1}) \\
&\leq (1-t_n) d(u_n, u_{n-1}) + t_n d(g(u_n), g(u_{n-1})) \\
&+ |t_n - t_{n-1}| d(g(u_{n-1}), v_{n-1}) \\
&\leq ((1-t_n) + t_n \rho) d(u_n, u_{n-1}) + |t_n - t_{n-1}| d(v_{n-1}, g(u_{n-1})) \\
&\leq ((1-t_n) + t_n \rho) \Bigg(d(x_n, x_{n-1}) + \frac{|\lambda_n - \lambda_{n-1}|}{\lambda} d\big(J^f_{\lambda_n} x_{n-1}, x_{n-1}\big) \Bigg) \\
&+ |t_n - t_{n-1}| d(v_{n-1}, g(u_{n-1})) \\
&= (1 - t_n(1-\rho)) d(x_n, x_{n-1}) \\
&+ (1 - t_n(1-\rho)) \frac{|\lambda_n - \lambda_{n-1}|}{\lambda} d(J^f_{\lambda_n} x_{n-1}, x_{n-1}) \\
&+ |t_n - t_{n-1}| d(v_{n-1}, g(u_{n-1})).
\end{aligned}$$

Using conditions (ii), (iii) and (iv), we obtain by Lemma 9 that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \qquad (28)$$

From Lemma 4, we have that

$$\frac{1}{2\lambda_n}d^2(u_n,p)-\frac{1}{2\lambda_n}d^2(x_n,p)+\frac{1}{2\lambda_n}d^2(x_n,u_n)\leq f(p)-f(u_n).$$

Since $f(p)\leq f(u_n)$ for all $n\geq 1$, we obtain

$$d^2(u_n,p)\leq d^2(x_n,p)-d^2(x_n,u_n). \tag{29}$$

Thus, using (29) and Lemma 1, we obtain

$$\begin{aligned} d^2(x_{n+1},p) &= d^2(t_ng(u_n)\oplus(1-t_n)v_n,p)\\ &\leq t_nd^2(g(u_n),p)+(1-t_n)d^2(v_n,p)\\ &\leq t_nd^2(g(u_n),p)+(1-t_n)d^2(u_n,p)\\ &\leq t_nd^2(g(u_n),p)+d^2(x_n,p)-d^2(x_n,u_n), \end{aligned}$$

which implies that

$$\begin{aligned} d^2(x_n,u_n) &\leq t_nd^2(g(u_n),p)+d^2(x_n,p)-d^2(x_{n+1},p)\\ &\leq t_nd^2(g(u_n),p)+d^2(x_n,x_{n+1})+2d(x_n,x_{n+1})d(x_{n+1},p). \end{aligned}$$

It then follows from (28) and condition (i) that

$$\lim_{n\to\infty}d^2(x_n,u_n)=0. \tag{30}$$

Since $\lambda_n\geq\lambda>0$, we obtain from (30) and Lemma 11 that

$$\begin{aligned} d(x_n,J_\lambda^f x_n) &\leq d(x_n,J_{\lambda_n}x_n)\\ &= d(x_n,u_n)\to 0 \text{ as } n\to\infty. \end{aligned} \tag{31}$$

Again,

$$\begin{aligned} d(u_n,v_n) &\leq d(u_n,x_n)+d(x_n,x_{n+1})+d(x_{n+1},v_n)\\ &\leq d(u_n,x_n)+d(x_n,x_{n+1})+d(t_ng(u_n)\oplus(1-t_n)v_n,v_n)\\ &\leq d(x_n,x_{n+1})+d(u_n,x_n)+t_nd(g(u_n),v_n), \end{aligned}$$

which implies from (28), (30) and condition (i), that

$$\lim_{n\to\infty}d(u_n,v_n)=0. \tag{32}$$

Since $\{x_n\}$ is bounded and X is a complete CAT(0) space, then from Lemma 5, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\triangle\text{-}\lim_{k\to\infty}x_{n_k}=$

$\overline{x}$. It follows from (30) that there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\triangle$-$\lim_{k\to\infty} u_{n_k} = \overline{x}$. Since T is a nonexpansive multivalued mapping, it follows from (32) and Lemma 8 that $\overline{x} \in F(T)$. Also, since J_λ^f is a nonexpansive mapping, it follows from (31) and Lemma 8 that $\overline{x} \in F(J_\lambda^f)$. Therefore $\overline{x} \in \Gamma$. Following similar argument as in Theorem 4, we can show that $\bar{x}$ also solves the variational inequality (14). Thus, we conclude that $\bar{x} \in \Gamma$ also solves the variational inequality (14).
Next, we show that

$$\limsup_{n\to\infty}\langle\overrightarrow{g(\overline{x})\overline{x}}, \overrightarrow{x_n\overline{x}}\rangle \leq 0.$$

Observe that

$$\limsup_{n\to\infty}\langle\overrightarrow{g(\overline{x})\overline{x}}, \overrightarrow{x_n\overline{x}}\rangle = \limsup_{k\to\infty}\langle\overrightarrow{g(\overline{x})\overline{x}}, \overrightarrow{x_{n_k}\overline{x}}\rangle. \tag{33}$$

Since $\{x_{n_k}\}$ $\Delta-$ converges to $\overline{x}$, by Lemma 6, we have

$$\limsup_{k\to\infty}\langle\overrightarrow{g(\overline{x})\overline{x}}, \overrightarrow{x_{n_k}\overline{x}}\rangle \leq 0.$$

This together with (33) gives

$$\limsup_{n\to\infty}\langle\overrightarrow{g(\overline{x})\overline{x}}, \overrightarrow{x_n\overline{x}}\rangle \leq 0. \tag{34}$$

Finally, we prove that $\{x_n\}$ converges strongly to $\overline{x}$.
For any $n \geq 1$, let $z_n = t_n\overline{x} \oplus (1-t_n)v_n$. Thus, by Lemma 3, we obtain

$$\begin{aligned}
d^2(x_{n+1}, \overline{x}) &\leq d^2(z_n, \overline{x}) + 2\langle\overrightarrow{x_{n+1}z_n}, \overrightarrow{x_{n+1}\overline{x}}\rangle \\
&\leq (1-t_n)^2 d^2(v_n, \overline{x}) + 2(t_n^2\langle\overrightarrow{g(u_n)\overline{x}}, \overrightarrow{x_{n+1}\overline{x}}\rangle \\
&+ t_n(1-t_n)\langle\overrightarrow{g(u_n)v_n}, \overrightarrow{x_{n+1}\overline{x}}\rangle + t_n(1-t_n)\langle\overrightarrow{v_n\overline{x}}, \overrightarrow{x_{n+1}\overline{x}}\rangle) \\
&\leq (1-t_n)^2 d^2(x_n, \overline{x}) + 2(t_n^2\langle\overrightarrow{g(u_n)\overline{x}}, \overrightarrow{x_{n+1}\overline{x}}\rangle \\
&+ t_n(1-t_n)\langle\overrightarrow{g(u_n)\overline{x}}, \overrightarrow{x_{n+1}\overline{x}}\rangle) \\
&= (1-t_n)^2 d^2(x_n, \overline{x}) + 2t_n\langle\overrightarrow{g(u_n)\overline{x}}, \overrightarrow{x_{n+1}\overline{x}}\rangle \\
&= (1-t_n)^2 d^2(x_n, \overline{x}) + 2t_n(\langle\overrightarrow{g(u_n)g(\overline{x})}, \overrightarrow{x_{n+1}\overline{x}}\rangle + \langle\overrightarrow{g(\overline{x})\overline{x}}, \overrightarrow{x_{n+1}\overline{x}}\rangle) \\
&\leq (1-t_n)^2 d^2(x_n, \overline{x}) + 2t_n\left(\rho d(u_n, \overline{x})d(x_{n+1}, \overline{x}) + \langle\overrightarrow{g(\overline{x})\overline{x}}, \overrightarrow{x_{n+1}\overline{x}}\rangle\right) \\
&\leq (1-t_n)^2 d^2(x_n, \overline{x}) + 2t_n\langle\overrightarrow{g(\overline{x})\overline{x}}, \overrightarrow{x_{n+1}\overline{x}}\rangle \\
&+ \rho t_n(d^2(x_n, \overline{x}) + d^2(x_{n+1}, \overline{x})),
\end{aligned}$$

which implies that

$$d^2(x_{n+1},\overline{x}) \leq \left(1-\frac{2t_n(1-\rho)}{1-t_n\rho}\right)d^2(x_n,\overline{x})+\frac{2t_n}{1-t_n\rho}\langle\overrightarrow{g(\overline{x})\overline{x}},\overrightarrow{x_{n+1}\overline{x}}\rangle+\frac{t_n^2}{(1-t_n\rho)}M,$$

where $M=\sup_{n\geq 1}\{d^2(x_n,\overline{x})\}$. Thus, we have

$$d^2(x_{n+1},\overline{x}) \leq (1-\frac{2t_n(1-\rho)}{1-t_n\rho})d^2(x_n,\overline{x})+\frac{2t_n(1-\rho)}{1-t_n\rho}(\frac{\langle\overrightarrow{g(\overline{x})\overline{x}},\overrightarrow{x_{n+1}\overline{x}}\rangle}{1-\rho}+\frac{t_nM}{2(1-\rho)}).$$

If we let $\gamma_n=\frac{2(1-\rho)t_n}{1-t_n\rho}$ and $\delta_n=\frac{1}{1-\rho}\langle\overrightarrow{g(\overline{x})\overline{x}},\overrightarrow{x_{n+1}\overline{x}}\rangle+\frac{t_n}{2(1-\rho)}M$, we obtain that

$$d^2(x_{n+1},\overline{x}) \leq (1-\gamma_n)d^2(x_n,\overline{x})+\gamma_n\delta_n. \tag{35}$$

It then follows from (28), (34), (35) and Lemma 9 that $\{x_n\}$ converges strongly to $\bar{x}$ which solves the variational inequality (14).

Corollary 1. *Let X be a complete CAT(0) space and $f : X \to (-\infty,+\infty]$ be a proper convex and lower semi-continuous function. Let $T : X \to X$ be a nonexpansive single-valued mapping such that $\Gamma := F(T) \cap argmin_{y\in X} f(y)$ is nonempty. Suppose that g is a contraction mapping defined on X with coefficient $\rho \in (0,1)$ and $\lambda_n \geq \lambda > 0$ for some λ. Let $x_1 \in X$ be arbitrarily chosen and the sequence $\{x_n\}$ be generated by*

$$\begin{cases} u_n = J^f_{\lambda_n}(x_n), \\ x_{n+1} = t_n g(u_n) \oplus (1-t_n)Tu_n, \end{cases} \tag{36}$$

for each $n \geq 1$, where $\{t_n\}$ is a sequence in $(0,1)$ satisfying
(i) $\lim_{n\to\infty} t_n = 0$,
(ii) $\sum_{n=1}^{\infty} t_n = \infty$,
(iii) $\sum_{n=1}^{\infty} |t_{n+1}-t_n| < \infty$,
(iv) $\sum_{n=1}^{\infty} |\lambda_n-\lambda_{n+1}| < \infty$.
Then the sequence $\{x_n\}$ converges strongly to $\overline{x} \in \Gamma$ which also solves the variational inequality

$$\langle\overrightarrow{\overline{x}g(\overline{x})},\overrightarrow{x\overline{x}}\rangle \geq 0,\ \ x \in \Gamma. \tag{37}$$

By setting $g(x) = u$ for arbitrary but fixed $u \in X$ and for all $x \in X$, in Corollary 1, we obtain the following result which coincides with [38, Theorem 3.1].

Corollary 2. *Let X be a complete CAT(0) space and $f : X \to (-\infty, +\infty]$ be a proper convex and lower semi-continuous function. Let $T : X \to X$ be a nonexpansive single-valued mapping such that $\Gamma := F(T) \cap argmin_{y\in X} f(y)$ is nonempty. Suppose that $\lambda_n \geq \lambda > 0$ for some λ. Let $u, x_1 \in X$ be arbitrarily chosen and the sequence $\{x_n\}$ be generated by*

$$\begin{cases} u_n = J^f_{\lambda_n}(x_n), \\ x_{n+1} = t_n u \oplus (1-t_n)Tu_n, \end{cases} \tag{38}$$

for each $n \geq 1$, where $\{t_n\}$ is a sequence in $(0,1)$ satisfying
(i) $\lim_{n\to\infty} t_n = 0$,
(ii) $\sum_{n=1}^{\infty} t_n = \infty$,
(iii) $\sum_{n=1}^{\infty} |t_{n+1} - t_n| < \infty$,
(iv) $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$.
Then the sequence $\{x_n\}$ converges strongly to $\overline{x} \in \Gamma$ which also solves

$$\langle \overrightarrow{\overline{x}u}, \overrightarrow{x\overline{x}} \rangle \geq 0, \;\; x \in \Gamma, \tag{39}$$

which by Lemma 7 implies that $\bar{x} = P_\Gamma u$. In other words, the sequence $\{x_n\}$ converges strongly to $\overline{x} \in \Gamma$ which is the nearest point of Γ to u.

The following corollary of Theorem 5 coincides with Theorem 2.3 of [35].

Corollary 3. *Let X be a complete CAT(0) space and $T : X \to X$ be a nonexpansive single-valued mapping such that $F(T)$ is nonempty. Suppose that $u, x_1 \in X$ are arbitrarily chosen and the sequence $\{x_n\}$ is generated by*

$$x_{n+1} = t_n u \oplus (1-t_n)Tx_n, \tag{40}$$

for each $n \geq 1$, where $\{t_n\}$ is a sequence in $(0,1)$ satisfying
(i) $\lim_{n\to\infty} t_n = 0$,
(ii) $\sum_{n=1}^{\infty} t_n = \infty$,
(iii) $\sum_{n=1}^{\infty} |t_{n+1} - t_n| < \infty$.
Then, the sequence $\{x_n\}$ converges strongly to $\overline{x} \in F(T)$ which is the nearest point of $F(T)$ to u.

The following corollary of Theorem 5 coincides with Theorem 3.2 of [8].

Corollary 4. *Let X be a complete CAT(0) space and $T : X \to P(X)$ be a nonexpansive multivalued mapping such that $Tp = \{p\}$, for each $p \in F(T)$ and $F(T) \neq \emptyset$. Suppose that g is a contraction mapping defined on X with* *coefficient* $\rho \in (0,1)$. *Let* $x_1 \in X$ *be arbitrarily chosen and the sequence* $\{x_n\}$ *be generated by*

$$x_{n+1} = t_n g(x_n) \oplus (1-t_n)v_n, \quad v_n \in Tx_n, \tag{41}$$

for each $n \geq 1$, *where* $\{t_n\}$ *is a sequence in* $(0,1)$ *satisfying*
(i) $\lim_{n\to\infty} t_n = 0$,
(ii) $\sum_{n=1}^{\infty} t_n = \infty$,
(iii) $\sum_{n=1}^{\infty} |t_{n+1} - t_n| < \infty$.
Then the sequence $\{x_n\}$ *converges strongly to* $\overline{x} \in F(T)$ *which also solves the variational inequality*

$$\langle \overrightarrow{\overline{x}g(\overline{x})}, \overrightarrow{x\overline{x}} \rangle \geq 0, \;\; x \in F(T). \tag{42}$$

Remark 1. *(1) Our main results generalize and extend the results of Suparatulatorn et al. [38] from approximating a common solution of minimization problem and fixed point problem for single-valued nonexpansive mapping to approximating a common solution of minimization problem and fixed point problem for multivalued nonexpansive mapping which is also a unique solution of some variational inequalities (see Corollary 2). Furthermore, our algorithm (Algorithm 26) has the potential of converging faster than Algorithm 11 studied by Suparatulatorn et al. [38], since our algorithm is of viscosity-type. Examples are given in Section 4 to further illustrate this (see Figures 1 and 2).*

(2) Our results also extend the results of Bo and Yi [8] from approximating a fixed point of nonexpansive multivalued mapping to approximating a fixed point of nonexpansive multivalued mapping which is also a solution of minimization problem (see Corollary 4).

(3) Our theorem (Theorem 5) extends Theorem 2.3 of Saejung [35] (which is a Halpern's convergence theorem) from approximating a fixed point of a single-valued mapping to approximating a fixed point of a multivalued mapping which is also a minimizer of a proper convex and lower semi-continuous function and a unique solution of some variational inequalities (see Corollary 3).

4 Numerical Example

In this section, we present two numerical examples of our algorithm (Algorithm 26) in 2-dimensional space of real numbers and in a complete CAT(0)

space (in non-Hilbert space), to show its advantage over existing algorithms in the literature.
Throughout this section, we shall take $t_n = \frac{1}{n+1}$ $\forall n \geq 1$ and $g(x) = \frac{1}{2}x$ $\forall x \in X$.

Example 1. *Let $X = \mathcal{R}^2$ be endowed with the Euclidean norm $||.||_2$. Define $T : \mathcal{R}^2 \to \mathcal{R}^2$ by $T(x_1, x_2) = (-x_1, x_2)$. Then, T is a nonexpansive mapping. Let $f : \mathcal{R}^2 \to (-\infty, +\infty]$ be defined by $f(x) = ||x||_1 + \frac{1}{2}||x||_2^2 + (1, -2)^T x + 8$, then f is a proper convex and lower semi-continuous function. Thus, by using the soft thresholding operator (see [15]) and the proximity opperator (see [10]), we obtain that*

$$\begin{aligned} J_1^f(x) &= arg \min_{y \in \mathcal{R}^2} [f(y) + \frac{1}{2}||x-y||^2] \\ &= prox_f x \\ &= prox_{\frac{||.||_1}{2}} \left(\frac{x-(1,-2)^T}{2} \right) \\ &= \left(\max\left\{ \frac{|x_1-1|-1}{2}, 0 \right\} sgn(x_1-1), \right. \\ & \quad \left. \max\left\{ \frac{|x_2+2|-1}{2}, 0 \right\} sgn(x_2+2) \right)^T, \end{aligned}$$

where $sgn(.)$ is the signum function of $\alpha \in \mathcal{R}$ defined by

$$sgn(\alpha) = \begin{cases} 1, & if\ \alpha > 0 \\ 0, & if\ \alpha = 0 \\ -1 & if\ \alpha < 0. \end{cases} \tag{43}$$

Example 2. *Let $X = \mathcal{R}^2$ be endowed with a metric $d_X : \mathcal{R}^2 \times \mathcal{R}^2 \to [0, \infty)$ defined by*

$$d_X(x, y) = \sqrt{(x_1 - y_1)^2 + (x_1^2 - x_2 - y_1^2 + y_2)^2} \ \forall x, y \in \mathcal{R}^2.$$

Then, $(\mathcal{R}^2, d_X)$ is a complete CAT(0) space (see [42, Example 5.2]) with the geodesic joining x to y given by

$$(1-t)x \oplus ty = \left((1-t)x_1 + ty_1, ((1-t)x_1 + ty_1)^2 - (1-t)(x_1^2 - x_2) - t(y_1^2 - y_2)\right).$$

Now define $T : \mathcal{R}^2 \to \mathcal{R}^2$ by $T(x_1, x_2) = (x_1, 2x_1^2 - x_2)$. Clearly, T is not a nonexpansive mapping in the classical sense. However, it is easy to check that T is nonexpansive in $(\mathcal{R}^2, d_X)$. Indeed, for all $x, y \in \mathcal{R}^2$,

$$\begin{aligned} d_X(Tx, Ty) &= \sqrt{(x_1 - y_1)^2 + (x_1^2 - (2x_1^2 - x_2) - y_1^2 + (2y_1^2 - y_2)^2} \\ &= \sqrt{(x_1 - y_1)^2 + (x_1^2 - x_2 - y_1^2 - y_2)^2} \\ &= d_X(x, y). \end{aligned}$$

Again, define $f : \mathcal{R}^2 \to \mathcal{R}$ by $f(x_1, x_2) = 100((x_2 + 1) - (x_1 + 1)^2)^2 + x_1^2$. Then f is a proper convex and lower semi-continuous function in $(\mathcal{R}^2, d_X)$ but not convex in the classical sense (see [42]).

Using Example 1 and 2, we compare our algorithm (Algorithm (26)) with Algorithm (3) of Saejung [35], Algorithm (6) of Bo and Yi [8] and Algorithm (11) of Suparatulatorn *et al.* [38] by considering the following 4 cases (see Figures 1 and 2):
Case 1: $x_1 = (0.5, -0.25)^T$ and $u = (2, 8)^T$,
Case 2: $x_1 = (1, 3)^T$ and $u = (2, 8)^T$,
Case 3: $x_1 = (-1, -3)^T$ and $u = (0.5, 1)^T$,
Case 4: $x_1 = (-1, -3)^T$ and $u = (-0.5, -1)^T$.

Remark 2. *We can see from the graphs above that our viscosity-type algorithm converges faster than the Halpern-type algorithms studied by Saejung [35] and Suparatulatorn et al. [38]. Observe also that, although the algorithm studied by Bo and Yi [8] is also of viscosity-type, our algorithm performs better than it. One possible reason for this could be because of the fact that our viscosity-type iteration is more closer to the proximal point algorithm compared to that of Bo and Yi [8]. In fact, this could also be the reason behind the better performance of Algorithm (11) of Suparatulatorn et al. [38] compared to Algorithm (3) of Saejung [35] as shown by the above numerical results.*

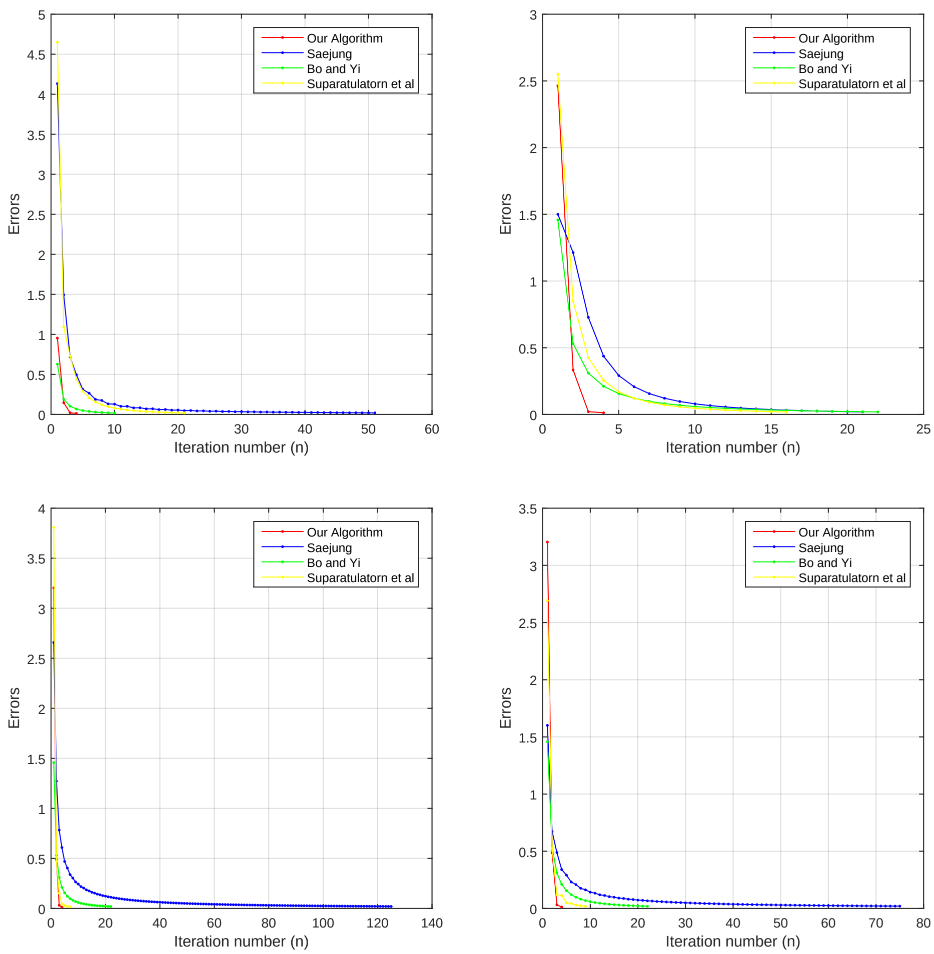

Figure 1: Errors vs Iteration numbers for **Example 1**: **Case 1** (top left); **Case 2** (top right); **Case 3** (bottom left); **Case 4** (bottom right).

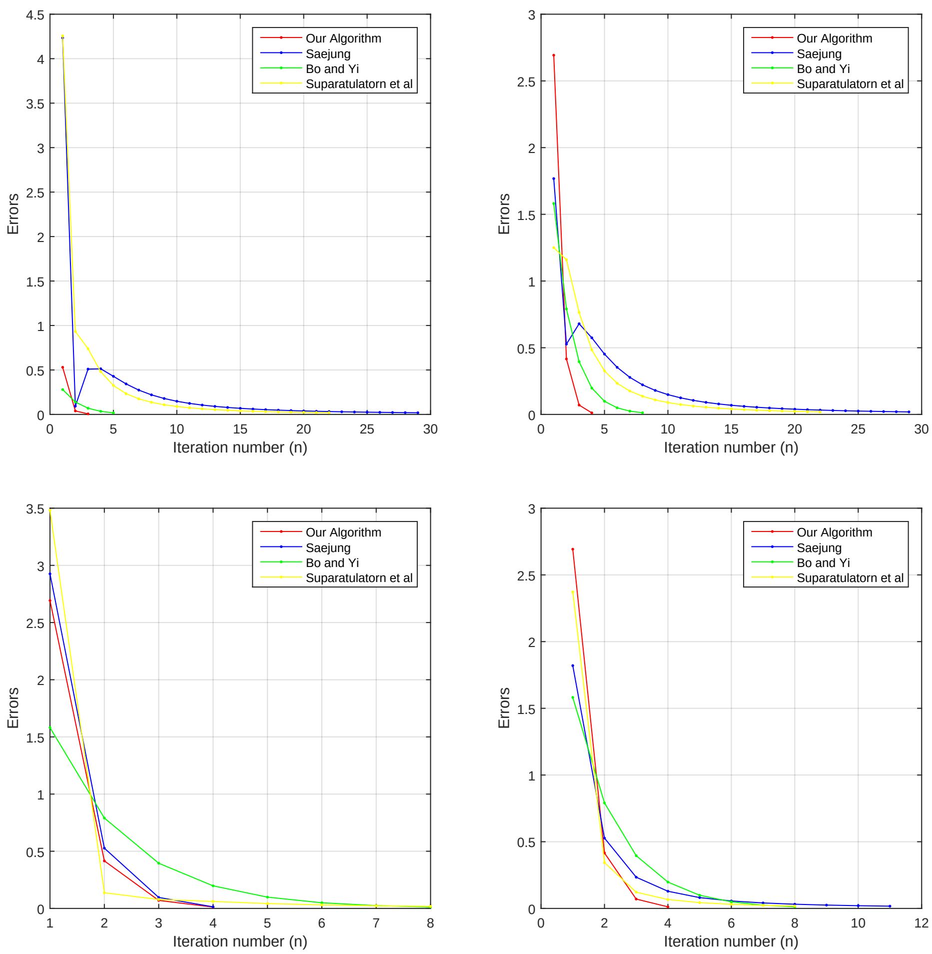

Figure 2: Errors vs Iteration numbers for **Example 2**: **Case 1** (top left); **Case 2** (top right); **Case 3** (bottom left); **Case 4** (bottom right).

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