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HOPFIELD NETWORKS WITH MULTIPLICATIVE NOISE IN AN ANISOTROPIC NORM SETUP*

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Abstract

A non symmetric version of Hopfield networks subject to state-multiplicative noise, is considered in an anisotropic norm setup. Such networks arise in the context of visuo-motor control loops and may, therefore, be used to mimic their complex behavior. In this paper, we adopt the Lur'e - Postnikov systems approach to generalize a Bounded Real Lemma like result of generalized Hopfield networks, to compute their anisotropic norm.

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1 Introduction

Hopfield networks ([19]) are symmetric recurrent neural networks which exhibit motions in the state space which converge to minima of energy. Symmetric Hopfield networks can be used to solve practical complex problems

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such as implementation of associative memory, linear programming solving and optimal guidance problems solution. Recurrent networks which are non symmetric stochastic versions of Hopfield networks play an important role in understanding human motor tasks involving visual feedback (see ([2], ([3]) and the references therein). When such networks are used to model human motor tasks involving visual feedback ([2]) ([3]) the effects of state-multiplicative noise and pure time delay become dominant. It is mentioned in ([2]) that besides stick balancing at the fingertip, state-multiplicative noise arises at every level of the human nervous system (e.g. reflexes ([4]), motor control ([5]). The deterministic discrete-time version of this network has been considered in ([6]) whereas the stochastic continuous-time version of this network driven by white noise, has been considered in ([7]) where the stochastic stability of a network of form (6) given in the next section has been analyzed. In this latest paper it has been shown that the network is almost surely stable when the time derivative energy $\frac{d\mathcal{E}}{dt} \leq 0$ is replaced by $\mathcal{L}\mathcal{E} \leq 0$ where $\mathcal{L}\mathcal{E}$ is the infinitesimal generator associated with the Itô type stochastic equation describing the continous-time Hoptfield network.

In the present paper, we analyze the anisotropic-norm of discrete-time Hopfield neural networks, which arises when the exogenous signals are neither purely white noise, or band limited. When the exogenous signals are of white noise type, then H_2- norm analysis is required whereas in the case of deterministic bounded energy signals, the framework of the H_∞-norm ([8]) is to be applied . When the input $w(t)$ of a discrete-time system Σ is a sequence of zero mean independent random vectors of unit covariance, its H_2-norm is given, in terms of its output $y(t), t = 0, 1, ...,$ by $||\Sigma||_2 := \sqrt{\lim_{\ell\to\infty} \frac{1}{\ell} \sum_{t=0}^{\ell} E[|y(t)|^2]}$ whereas for $\lim_{\ell\to\infty} \frac{1}{\ell} \sum_{t=0}^{\ell} E[|w(t)|^2] < \infty$, the H_∞ - norm of Σ, is associated with $\lim_{\ell\to\infty} \frac{1}{\ell} \sum_{t=0}^{\ell} E[|y(t)|^2 - \gamma^2 |w(t)|^2] < 0$. However, many practical cases involve a compromise between the H_2 and the H_∞-norm setups since the former may not be suitable when the considered signals are strongly colored (e.g. periodic signals). On the other hand, H_∞-optimization may poorly predict performance when these signals are weakly colored (e.g. white noise). Mixed H_2/H_∞ results have been considered in ([9]) and ([10]). A promising alternative to accomplish such compromise is to use the so-called *a-anisotropic norm* (see e.g. ([11]), ([12]), ([13]) since it offers and intermediate topology between the H_2 and H_∞ norms. More precisely, if the colored signal is generated by an m-dimensional exogenous input, the a-anisotropic norm $|||F|||_a$ of a stable system F has the property (see, for instance ([14]) :

$$\frac{1}{\sqrt{m}}||F||_2 = |||F|||_0 \leq |||F|||_a \leq ||F||_\infty = \lim_{a\to\infty} |||F|||_a.$$

where

$$|||\Sigma|||_a := \sup_{G\in\mathcal{G}_a} \frac{||\Sigma G||_2}{||G||_2} \tag{1}$$

and where it is assumed that the disturbance w_k is generated by a coloring filter G, the input of which is a white noise. The class of admissible filters G with anisotropy less than a, namely $\bar{A}(G) < a$, is denoted by $\mathcal{G}_a$ where the mean anisotropy of G, is defined here as

$$\bar{A}(G) = -\frac{1}{2} \ln \det \left(\frac{mE\left[\tilde{w}(0)\tilde{w}(0)^T\right]}{Tr\left(E\left[w(0)w(0)^T\right]\right)} \right), \tag{2}$$

where $E[\tilde{w}(0)\tilde{w}(0)^T]$ is the covariance of the prediction error $\tilde{w}(0) := w(0) - E[w(0)|w(t), t < 0]$. Note that in cases where G is a linear system without multiplicative noise, then its output w has a Gaussian distribution, the original entropy-theoretic mean anisotropy definition of ([11]) can be used, which is equivalent ([15]) in such a case to (2). When G is, however, corrupted with multiplicative noise as considered in this paper, the above equivalent definition for $\bar{A}(G)$ no longer holds, and the higher moments than just the spectral density are involved. In spite of this fact, in the present paper, we adopt as in ([16]) an anisotropic-norm setup, where the simple definition of (2) in terms of second order moments only of $w(0)$ and its estimate is used. This definition leads to results which are consistent (see ([16])) both with the anisotropic norm-related results of ([14]) and ([17]) for linear systems without multiplicative noise and with the H_∞-norm related results of ([18]) for systems with multiplicative noise.

At this point, it is useful to note again, that the mean anisotropy $\bar{A}(G)$ of $w(t)$ is just a measure of its whiteness. Namely, if $w(t)$ is white, then it can not be estimated from its past values (i.e. its optimal estimate is just zero) and $\tilde{w}(0) = w(0)$ which leads to $\bar{A}(G) = 0$. On the other hand, if $w(t)$ can be perfectly estimated, then $\bar{A}(G)$ by the above definition, tends to infinity. Note that whenever the transfer matrix function corresponding to G is rank deficient (namely w has frequency bands with zero power spectrum) on some finite interval of frequencies, then $\bar{A}(G)$ also tends to infinity. Following ([11]) we denote the class of admissible filters G with $\bar{A}(G) < a$ by $\mathcal{G}_a$. The anisotropic norm $|||F|||_a$ of the system F is then defined by

$$|||F|||_a := \sup_{G\in\mathcal{G}_a} \frac{||FG||_2}{||G||_2} \tag{3}$$

Throughout the paper $\mathcal{R}^n$ denotes the n dimensional Euclidean space, $\mathcal{R}^{n\times m}$ is the set of all $n \times m$ real matrices, $\mathcal{Z}_+$ is the set of all nonnegative integers, Tr stands for the trace of a square matrix and the notation $X>0$, (respectively, $X\geq 0$) for $X \in \mathcal{R}^{n\times n}$ means that X is symmetric and positive definite (respectively, semi-definite). Also $||w||^2$ for $w \in \mathcal{R}^n$ will denote $w^T w$. Throughout the paper $(\Omega, \mathcal{F}, P)$ is a given probability space. Expectation is denoted by $E[\cdot]$ and conditional expectation of x on the event $\theta(t) = i$ is denoted by $E[x|\theta(t) = i]$.

2 Problem formulation

The neural network proposed by Hopfield, can be described by a system of ordinary differential equations of the form

$$\dot{v}_i(t) = a_i v_i(t) + \sum_{j=1}^{n} F_{ij} g_j(v_j(t)) + \bar{c}_i =: \kappa_i(v), 1 \leq i \leq n \tag{4}$$

where v_i represents the voltage on the input of the i-th neuron, $a_i < 0, 1 \leq i \leq n$, $F_{ij} = F_{ji}$ and the activations $g_i(\cdot), i = 1, ..., n$ are C^1–bounded and strictly increasing functions. The stability of this network is analyzed in ([19]) by defining the network energy functional:

$$\begin{aligned}\mathcal{E}(v) \quad = \quad & -\sum_{i=1}^{n} a_i \int_0^{v_i} u \frac{dg_i(u)}{du} du \\ & -\frac{1}{2}\sum_{i,j=1}^{n} F_{ij} g_i(v_i) g_j(v_j) - \sum_{i=1}^{n} \bar{c}_i g_i(v_i)\end{aligned} \tag{5}$$

which is a Lyapunov function if g_i are increasing activation functions since

$$\frac{d\mathcal{E}}{dt} = -\sum \frac{dg_i(v_i)}{dv_i} \kappa_i(v)^2 \leq 0$$

with $\kappa_i(v)$ defined in (4). The zero rate of the energy is obtained only in the equilibrium points, also referred to as attractors, where $\kappa_i(v^0) = 0, 1 \leq i \leq n$. However, the neural network may be subject to environmental noise and to connection matrix perturbations which can be modelled as $\sum_{j=1}^{n} b_{ij} w_j(t)$ added to the right hand side of (4). The network subject to the combination of these two effects can be then described in matrix form as:

$$\dot{v}(t) = Av(t) + \mathcal{F} g(v(t)) + Bw(t) + \bar{C} \tag{6}$$

where

$$\begin{array}{l} A := diag(a_1, ..., a_n), \\ B = [b_{ij}]_{i,j} = 1, ..., n,\ \mathcal{F} = [F_{ij}]_{i,j=1,...,n} \\ \bar{C} := \left[\begin{array}{cccc} \bar{c}_1 & \bar{c}_2 & ... & \bar{c}_n \end{array} \right]^T, v := \left[\begin{array}{cccc} v_1 & v_2 & ... & v_n \end{array} \right]^T \end{array}$$

and where $g(v) := \left[\begin{array}{cccc} g_1(v_1) & g_2(v_2) & ... & g_n(v_n) \end{array} \right]^T$, $y(t)$ denoting the observed output of the network and $w(t) = (w_1(t), ..., w_m(t))^T$ is the driving process, to be specified in the sequel. To analyze the effect of $w(t)$ we first define the error of the Hopfield network output with respect to its equilibrium points by $x(t) = v(t) - v^0$ and assume that the errors vector $x(t)$ satisfy:

$$\dot{x}(t) = Ax(t) + \mathcal{F}f(x(t)) + Bw(t) \tag{7}$$

where by definition, $f(x) = g(x + v_0) - g(v_0)$.

In the present paper a discrete-time version of (7) is used in which multiplicative white noise perturbations are added, namely

$$\begin{array}{rcl} x(t+1) & = & \mathcal{A}(t)x(t) + \mathcal{B}(t)w(t) + \mathcal{F}f(x(t)) \\ y(t) & = & Cx(t) + Dw(t),\ t = 0, 1, ... \end{array} \tag{8}$$

with

$$\begin{array}{rcl} \mathcal{A}(t) & := & A_0 + \sum_{i=1}^{r} \xi_i(t)A_i \\ \mathcal{B}(t) & := & B_0 + \sum_{i=1}^{r} \xi_i(t)B_i\,, \end{array}$$

where $\xi(t) = (\xi_1(t), ..., \xi_r(t))^T$ is a sequence of independent random vectors $\xi : \Omega \to \mathcal{R}^r$ on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and $y(t)$ stands for the network output. It is assumed that $\{\xi(t)\}_{t\geq 0}$ satisfies the conditions $E\left[\xi(t)\right] = 0$ and $E\left[\xi(t)\xi^T(t)\right] = I_r$, $t = 0, 1, \ldots$. The matrices of the state space model (8) have the dimensions as follows: $A_i \in \mathcal{R}^{n\times n}$, $B_i \in \mathcal{R}^{n\times m}$, $i = 0, 1...., r$, $C \in \mathcal{R}^{p\times n}$, $D \in \mathcal{R}^{p\times m}$.

In the remaining part of the paper it will be assumed that the input $w(t) \in \mathcal{R}^m$ is a vector valued signal of random variables, generated by the following linear stochastic filter G with multiplicative noise

$$\begin{array}{rcl} x_f(t+1) & = & \mathcal{A}_f(t)x_f(t) + \mathcal{B}_f(t)v(t) \\ w(t) & = & C_f x_f(t) + D_f v(t),\ t = 0, 1, ... \end{array} \tag{9}$$

where

$$\begin{array}{rcl} \mathcal{A}_f(t) & := & A_{f_0} + \sum_{i=1}^{r} \xi_i(t)A_{f_i} \\ \mathcal{B}_f(t) & := & B_{f_0} + \sum_{i=1}^{r} \xi_i(t)B_{f_i}\,. \end{array}$$

where the order n_f and the matrices $A_{f_i} \in \mathcal{R}^{n_f \times n_f}$, $B_{f_i} \in \mathcal{R}^{n_f \times m}$, $i = 0, 1...., r$, $C_f \in \mathcal{R}^{m \times n_f}$, $D_f \in \mathcal{R}^{m \times m}$ are not prefixed and $v(t) \in \mathcal{R}^m$ are white noise vectors with the properties $E[v(t)] = 0$ and $E\left[v(t)v^T(t)\right] = I_m$, $t = 0, 1, \ldots$. It is assumed that $\{\xi(t)\}_{t \geq 0}$ and $\{v(t)\}_{t \geq 0}$ are independent stochastic processes.

In the forthcoming analysis, we will assume that the components $f_i, i = 1, ..., n$ of $f(x(t))$ depend only on the $i-th$ component x_i of x and that they satisfy the sector conditions $0 \leq x_i f_i(x_i(t)) \leq \sigma_i x_i^2(t)$ which are equivalent to

$$-F_i(x_i(t), f_i) := f_i(x_i(t))(f_i(x_i(t)) - \sigma_i x_i(t)) \leq 0. \tag{10}$$

We shall further assume that

$$\frac{\partial f_i}{\partial x_i}(x_i(t)) \leq \sigma_i, \quad i = 1, ..., n, \tag{11}$$

which although somewhat restrictive is, nevertheless, fulfilled by the usual nonlinearities as saturation, sigmoid, etc., used in the neural networks. The latter assumption allows a useful application of the Mean Value Theorem. Consider $f_i(s)$ where $s \in [a, b]$. It then follows that $\exists c \in [a, b]$, such that $f_i(s) = f_i(a) + \frac{\partial f_i}{\partial x_i}(c)(s - a) \leq f_i(a) + \sigma_i(s - a)$. Therefore, $\int_a^b f_i(s)ds \leq f_i(a)(b - a) + \frac{\sigma_i}{2}(b - a)^2$.

The following definitions of stability and H_2 norm of the nonlinear system (8) are needed in the sequel.

Definition 1. *A stochastic system with multiplicative noise of form (8) with $B_i = 0$, $i = 0, 1, ..., r$ is called exponentially stable in mean square (ESMS) if there exist $\beta \geq 1$ and $\rho \in (0, 1)$ such that $E\left[|\Phi(t, s)x(s)|^2\right] \leq \beta\rho^{t-s}E\left[|x(s)|^2\right]$ for all $t \geq s \geq 0$ and for all $x(s) \in \mathcal{R}^n$ satisfying (10), where $\Phi(t, s)$ denotes the fundamental matrix solution of (8).*

Definition 2. *The H_2-type norm of the ESMS system (8) is defined as*

$$\|F\|_2 = \left[\lim_{\ell \to \infty} \frac{1}{\ell} \sum_{t=0}^{\ell} E\left[y^T(t)y(t)\right]\right]^{\frac{1}{2}},$$

where $\{y(t)\}_{t \in \mathcal{Z}_+}$ is the output of the system (8) with zero initial conditions generated by the sequence $\{w(t)\}_{t \in \mathcal{Z}_+}$ of independent random vectors with the property that $E[w(t)] = 0$ and $E\left[w(t)w^T(t)\right] = I_m$, $\{w(t)\}_{t \in \mathcal{Z}_+}$ being assumed independent of the stochastic process $\{\xi(t)\}_{t \in \mathcal{Z}_+}$.

Similar definitions with the above ones may be used for the case of linear systems corrupted with multiplicative noise as are the filters G of form (9). In this latest case, the following result provides a procedure to compute the H_2 type norm of an ESMS system of form ([18]).

Lemma 1 *The H_2 type norm of the ESMS system (9) is given by*

$$\|G\|_2 = \left(Tr\left(C_f Y C_f^T + D_f D_f^T\right)\right)^{\frac{1}{2}},$$

where $Y \geq 0$ is the solution of the generalized Lyapunov equation

$$Y = \sum_{i=0}^{r} \left(A_{f_i} Y A_{f_i}^T + B_{f_i} B_{f_i}^T\right).$$

The problem analyzed in the following section is to determine conditions under which the following inequality holds

$$\sup_{G \in \mathcal{G}_a} \frac{\|FG\|_2}{\|G\|_2} < \gamma$$

for a given $\gamma > 0$, with the ESMS systems F and G having the state space equations (8) and (9) respectively.

3 Bounded Real Lemma type Result

Introduce the Lyapunov-Krasovskii–type function:

$$V\left(x\left(t\right)\right) = x^T(t) X x\left(t\right) + 2 \sum_{k=1}^{n} \lambda_k \int_0^{x_k(t)} f_k\left(s\right) ds$$

where $X > 0$ is a positive definite matrix and $\lambda_k \geq 0$. By the definition of the H_2-type norm, it follows that the condition $\sup_{G \in \mathcal{G}_a} \frac{\|FG\|_2}{\|G\|_2} < \gamma$ is equivalent with the condition

$$\lim_{\ell \to \infty} \frac{1}{\ell} \sum_{t=0}^{\ell} E\left[|y(t)|^2 - \gamma^2 |w(t)|^2\right] < 0 \tag{12}$$

for all $w(t)$ generated by filters $G \in \mathcal{G}_a$.

Using (8) it follows that

$$\begin{array}{l}
V\left(x\left(t+1\right)\right)-V\left(x\left(t\right)\right)\\
=x^T(t+1)Xx(t+1)-x^T(t)Xx(t)\\
+2\sum_{k=1}^{n}\lambda_k\int_{x_k(t)}^{x_k(t+1)}f_k\left(x(s)\right)ds\\
=\left(\mathcal{A}(t)x(t)+\mathcal{B}(t)w(t)+\mathcal{F}f(x(t))\right)^T X\\
\times\left(\mathcal{A}(t)x(t)+\mathcal{B}(t)w(t)+\mathcal{F}f(x(t))\right)\\
-x^T(t)Xx(t)-y^T(t)y(t)+x^T(t)C^TCx(t)\\
+x^T(t)C^TDw(t)+w^T(t)D^TCx(t)+w^T(t)D^TDw(t)+\\
2\sum_{k=1}^{n}\lambda_k\int_{x_k(t)}^{x_k(t+1)}f_k\left(x(s)\right)ds
\end{array}$$

where we added the zero term $y^T(t)y(t)-(Cx(t)+Dw(t))^T(Cx(t)+Dw(t))$. Collecting terms we readily obtain

$$\begin{array}{l}
y^T(t)y(t)=x^T(t)\left(\mathcal{A}(t)^TX\mathcal{A}(t)-X+C^TC\right)x(t)\\
+w^T(t)\left(D^TD+\mathcal{B}(t)^TX\mathcal{B}(t)\right)w(t)\\
+w^T(t)\left(D^TC+\mathcal{B}(t)^TX\mathcal{A}(t)\right)x(t)\\
+x^T(t)\left(C^TD+\mathcal{A}(t)^TX\mathcal{B}(t)\right)w(t)\\
+f^T(x(t))\mathcal{F}^TX\mathcal{F}f(x(t))+x^T(t)\mathcal{A}(t)^TX\mathcal{F}f(x(t))\\
+f^T(x(t))\mathcal{F}^TX\mathcal{A}(t)x(t)\\
+w^T(t)\mathcal{B}^TX\mathcal{F}f(x(t))+f^T(x(t))\mathcal{F}^TX\mathcal{B}(t)x(t)\\
+x^T(t)Xx(t)-x^T(t+1)Xx(t+1)\\
+2\sum_{k=1}^{n}\lambda_k\int_{x_k(t)}^{x_k(t+1}f_k\left(x(s)\right)ds\,.
\end{array}$$

Noting that the properties of the random sequences $\{\xi_i(t)\}_{t\geq 0}$, $i=1,...,r$ imply
$E\left[\mathcal{A}^TX\mathcal{A}\right]=\sum_{i=0}^{r}A_i^TXA_i$, $E\left[\mathcal{B}^TX\mathcal{B}\right]=\sum_{i=0}^{r}B_i^TXB_i$ and $E\left[\mathcal{A}^TX\mathcal{B}\right]=\sum_{i=0}^{r}A_i^TXB_i$, it follows from the above equation that

$$\begin{array}{l}
E\left[y^T(t)y(t)\right]=E\left[x^T(t)\left(\sum_{i=0}^{r}A_i^TXA_i-X+C^TC\right)x(t)\right.\\
+w^T(t)\left(D^TD+\sum_{i=0}^{r}B_i^TXB_i\right)w(t)\\
+w^T(t)\left(D^TC+\sum_{i=0}^{r}B_i^TXA_i\right)x(t)\\
+x^T(t)\left(C^TD+\sum_{i=0}^{r}A_i^TXB_i\right)w(t)\\
+f^T(x(t))\mathcal{F}^TX\mathcal{F}f(x(t))+x^T(t)A_0^TX\mathcal{F}f(x(t))\\
+f^T(x(t))\mathcal{F}^TXA_0x(t)+w^T(t)B_0^TX\mathcal{F}f(x(t))\\
\left.+f^T(x(t))\mathcal{F}^TXB_0w(t)+V(x(t))-V(x(t+1))\right]\\
+2E\sum_{k=1}^{n}\lambda_k\int_{x_k(t)}^{x_k(t+1)}f_k\left(x(s)\right)ds\,.
\end{array}$$

However,

$$
\begin{aligned}
&2\sum_{k=1}^{n}\lambda_k\int_{x_k(t)}^{x_k(t+1)} f_k\left(x(s)\right)ds \leq 2\sum_{k=1}^{n} f_k(x_k(t))(x_k(t+1)-x_k(t))\lambda_k \\
&+\sigma_k(x_k(t+1)-x_k(t))^2\lambda_k \qquad (13)\\
&=2f^T(x(t))\Lambda(x(t+1)-x(t))+(x(t+1)-x(t))^T S\Lambda(x(t+1)-x(t))
\end{aligned}
$$

where $S := diag\,(\sigma_1,\ldots,\sigma_n)$ and $\Lambda := diag\,(\lambda_1,\ldots,\lambda_n)$ Therefore,

$$
\begin{aligned}
&E\left[y^T(t)y(t)\right] \leq E\left[x^T(t)\left(\sum_{i=0}^{r} A_i^T X A_i - X + C^T C\right)x(t)\right.\\
&+w^T(t)\left(D^T D+\sum_{i=0}^{r} B_i^T X B_i - \frac{1}{q}I\right)w(t)\ +w^T(t)\left(D^T C+\sum_{i=0}^{r} B_i^T X A_i\right)x(t)\\
&+x^T(t)\left(C^T D+\sum_{i=0}^{r} A_i^T X B_i\right)w(t)\ +f^T(x(t))\mathcal{F}^T X\mathcal{F}f(x(t))+x^T(t)A_0^T X\mathcal{F}f(t)\\
&+f^T(x(t))\mathcal{F}^T X A_0 x(t)\ +w^T(t)B_0^T X\mathcal{F}f(x(t))+f^T(t)\mathcal{F}^T X B_0 w(t)\\
&+V(x(t))-V(x(t+1))]+E\left[2f^T(x(t))\Lambda(x(t+1)-x(t))\right.\\
&\left.+(x(t+1)-x(t))^T S\Lambda(x(t+1)-x(t))\ +\frac{1}{q}w^T(t)w(t)\right] \qquad (14)
\end{aligned}
$$

where we added and subtracted $\frac{1}{q}w^T(t)w(t)$. Denoting $\bar{A}_0 = A_0 - I$ and $\bar{A}_i = A_i$ for $i = 1, ..., r$, from the above equation it follows that

$$
\begin{aligned}
&E\left[y^T(t)y(t)\right]\\
&\leq E\left[x^T(t)\left(\textstyle\sum_{i=0}^{r} A_i^T X A_i - X + C^T C\right)x(t)\right.\\
&+w^T(t)\left(D^T D+\textstyle\sum_{i=0}^{r} B_i^T X B_i - \frac{1}{q}I\right)w(t)\\
&+w^T(t)\left(D^T C+\textstyle\sum_{i=0}^{r} B_i^T X A_i\right)x(t)\\
&+x^T(t)\left(C^T D+\textstyle\sum_{i=0}^{r} A_i^T X B_i\right)w(t)\\
&+V(x(t))-V(x(t+1))+\tfrac{1}{q}w^T(t)w(t)\\
&+x^T(t)\left(\textstyle\sum_{i=0}^{r}\bar{A}_i^T S\Lambda\bar{A}_i\right)x(t)\\
&+w^T(t)\left(\textstyle\sum_{i=0}^{r} B_i^T S\Lambda B_i\right)w(t) \qquad (15)\\
&+f^T(x(t))\left(\lambda\mathcal{F}+\mathcal{F}^T\Lambda+\mathcal{F}^T S\Lambda\mathcal{F}+\mathcal{F}^T X\mathcal{F}\right)f(x(t))\\
&+x^T(t)\left(\textstyle\sum_{i=0}^{r}\bar{A}_i^T S\Lambda B_i\right)w(t)\\
&+w^T(t)\left(\textstyle\sum_{i=0}^{r} B_i^T\Lambda S\bar{A}_i\right)x(t)\\
&+x^T(t)\left(A_0^T\Lambda+\bar{A}_0^T S\Lambda\mathcal{F}+A_0^T X\mathcal{F}\right)f(x(t))\\
&+f^T(x(t))\left(\Lambda\mathcal{F}+\mathcal{F}^T\Lambda S\bar{A}_0+\mathcal{F}^T X A_0\right)x(t)\\
&+w^T(t)\left[B_0^T\Lambda+B_0^T S\Lambda\mathcal{F}+B_0^T X\mathcal{F}\right]f(x(t))\\
&\left.+f^T(x(t))\left(\Lambda B_0+\mathcal{F}^T\Lambda S B_0+\mathcal{F}^T X B_0\right)w(t)\right].
\end{aligned}
$$

Defining

$$\chi(t) := [x^T(t), w^T(t), f^T(t)]^T \tag{16}$$

one obtains from (15) that

$$\begin{array}{l} E\left[|y(t)|^2 - \gamma^2 |w(t)|^2\right] \\ \leq E\left[V(x(t)) - V(x(t+1)) + \left(\frac{1}{q} - \gamma^2\right) w^T(t) w(t)\right] \\ +E[\chi^T(t)\Theta\chi(t)] \end{array} \tag{17}$$

where $\Theta = \Theta^T$ having the block elements

$$\begin{array}{l} \Theta_{11} = \sum_{i=0}^{r} A_i^T X A_i - X + C^T C + \sum_{i=0}^{r} \bar{A}_i^T S \Lambda \bar{A}_i \\ \Theta_{12} = C^T D + \sum_{i=0}^{r} A_i^T X B_i + \sum_{i=0}^{r} \bar{A}_i^T S \Lambda B_i \\ \Theta_{13} = A_0^T \Lambda + A_0^T X \mathcal{F} + \bar{A}_0^T S \Lambda \mathcal{F} \\ \Theta_{22} = D^T D + \sum_{i=0}^{r} B_i^T X B_i - \frac{1}{q} I \ + \sum_{i=0}^{r} B_i^T S \Lambda B_i \\ \Theta_{23} = B_0^T \Lambda + B_0^T S \Lambda \mathcal{F} + B_0^T X \mathcal{F} \\ \Theta_{33} = \Lambda \mathcal{F} + \mathcal{F}^T \Lambda + \mathcal{F}^T S \Lambda F + \mathcal{F}^T X \mathcal{F}. \end{array} \tag{18}$$

The inequality (17) latter may be, however, written as $F_0(\chi) \geq 0$ where F_0 is a quadratic function with respect to its arguments. According with the $\mathcal{S}-$ procedure based method (see e.g. ([20])), these conditions subject to the sector constraints (10) are fulfilled if there exist $\tau_i \geq 0, i = 1, \ldots, n$ such that $F_0(\chi) - \sum_{k=1}^{n} \tau_k F_k(x, f) \geq 0$ for all L_2 bounded inputs $w(t)$. Denoting $T := diag(\tau_1, \ldots, \tau_n)$ we, therefore, obtain that (17) is fulfilled if the following holds:

$$\begin{array}{l} E\left[|y(t)|^2 - \gamma^2 |w(t)|^2\right] \\ \leq E\left[V(x(t)) - V(x(t+1)) + \left(\frac{1}{q} - \gamma^2\right) w^T(t) w(t)\right] \\ +E[\chi^T(t)\mathcal{L}\chi(t)] \end{array} \tag{19}$$

where

$$\mathcal{L} = \mathcal{L}^T, \mathcal{L} = [\mathcal{L}_{ij}]_{i,j=1,2,3} \tag{20}$$

with

$$\begin{array}{l} \mathcal{L}_{11} = \sum_{i=0}^{r} A_i^T X A_i - X + C^T C + \sum_{i=0}^{r} \bar{A}_i^T S \Lambda \bar{A}_i \\ \mathcal{L}_{12} = C^T D + \sum_{i=0}^{r} A_i^T X B_i + \sum_{i=0}^{r} \bar{A}_i^T S \Lambda B_i \\ \mathcal{L}_{13} = A_0^T \Lambda + A_0^T X \mathcal{F} + \bar{A}_0^T S \Lambda \mathcal{F} + \frac{1}{2} S T \\ \mathcal{L}_{22} = D^T D + \sum_{i=0}^{r} B_i^T X B_i - \frac{1}{q} I \ + \sum_{i=0}^{r} B_i^T S \Lambda B_i \\ \mathcal{L}_{23} = B_0^T \Lambda + B_0^T S \Lambda \mathcal{F} + B_0^T X \mathcal{F} \\ \mathcal{L}_{33} = \Lambda \mathcal{F} + \mathcal{F}^T \Lambda + \mathcal{F}^T S \Lambda \mathcal{F} + \mathcal{F}^T X \mathcal{F} - T \end{array} \tag{21}$$

where we have used the fact that

$$\begin{array}{l}\sum_{k=1}^{n}\tau_k F_k = \sum_{k=1}^{n}(\tau_k f_k \sigma_k x_k - \tau_k f_k^2)\\ = -f^T T f + \frac{1}{2}x^T S T f + \frac{1}{2} f^T T S x.\end{array} \tag{22}$$

Since the systems (8) and(9) are ESMS,

$$\lim_{\ell\to\infty}\frac{1}{\ell}E\left[V(x(0)) - V(x(\ell))\right] = 0,$$

and, therefore, collecting terms and adding and subtracting $\gamma^2 w^T(t)w(t)$, the following relation is obtained:

$$\begin{array}{l}\lim_{\ell\to\infty}\frac{1}{\ell}\sum_{t=0}^{\ell}E\left[|y(t)|^2 - \gamma^2|w(t)|^2\right]\\ \leq \left(\frac{1}{q}-\gamma^2\right)\lim_{\ell\to\infty}\frac{1}{\ell}\sum_{t=0}^{\ell}E\left[w^T(t)w(t)\right]\\ +\lim_{\ell\to\infty}\frac{1}{\ell}\sum_{t=0}^{\ell}E[\chi^T(t)\mathcal{L}\chi(t)]\end{array} \tag{23}$$

We are now in position to present our main result.

Theorem 1 *The system (8), (9) is stochastically stable and its anisotropic norm is less than $\gamma > 0$ if there exist a $q \in \left(0, \min\left(\gamma^{-2}, \|F\|_\infty^{-2}\right)\right)$, a symmetric matrix $X > 0$, and diagonal matrices $\Lambda > 0, T > 0$, satisfying $\mathcal{L} < 0$ with $\mathcal{L}$ defined by (20), (21) and*

$$\det\left(\frac{1}{q}-\gamma^2\right)\Psi_q^{-1} \leq e^{-2a} \tag{24}$$

$$\Psi_q := \frac{1}{q}I - \sum_{i=0}^{r}B_i^T X B_i - D^T D - \sum_{i=0}^{r}B_i^T S\Lambda B_i > 0. \tag{25}$$

Proof : We further denote $\eta = [x^T(t) f^T(x(t))]^T$, $\mathcal{M}_{11} = \begin{bmatrix}\mathcal{L}_{11} & \mathcal{L}_{13}\\ \mathcal{L}_{31} & \mathcal{L}_{33}\end{bmatrix}$, $\mathcal{M}_{12} = \begin{bmatrix}\mathcal{L}_{12}\\ \mathcal{L}_{32}\end{bmatrix}$, $\mathcal{M}_{21} = \mathcal{M}_{12}^T$ and , $\mathcal{M}_{22} = \mathcal{L}_{22} = -\Psi_q$. Therefore, based on (23) it follows that

$$\begin{array}{l}\lim_{\ell\to\infty}\frac{1}{\ell}\sum_{t=0}^{\ell}E\left[|y(t)|^2 - \gamma^2|w(t)|^2\right]\\ \leq \left(\frac{1}{q}-\gamma^2\right)\lim_{\ell\to\infty}\frac{1}{\ell}\sum_{t=0}^{\ell}E\left[w^T(t)w(t)\right]\\ +\lim_{\ell\to\infty}\frac{1}{\ell}\sum_{t=0}^{\ell}\{E[\eta^T(t)\mathcal{M}_{11}\eta(t)]\\ +E[\eta^T(t)\mathcal{M}_{12}w(t)] + E[w^T(t)\mathcal{M}_{21}\eta(t)]\\ E[w^T(t)\mathcal{M}_{22}w(t)]\} < 0\end{array} \tag{26}$$

Completing the right hand side to squares we readily obtain

$$\begin{array}{l}\lim_{\ell\to\infty}\frac{1}{\ell}\sum_{t=0}^{\ell}E\left[|y(t)|^2-\gamma^2|w(t)|^2\right]\\ \leq\left(\frac{1}{q}-\gamma^2\right)\lim_{\ell\to\infty}\frac{1}{\ell}\sum_{t=0}^{\ell}E\left[w^T(t)w(t)\right]\\ -\lim_{\ell\to\infty}\frac{1}{\ell}\sum_{t=0}^{\ell}E\left[w(t)-w^+(t)\right]^T\Psi_q\left[w(t)-w^+(t)\right]\\ +\lim_{\ell\to\infty}\frac{1}{\ell}\sum_{t=0}^{\ell}E\left[\eta^T(t)\left(\mathcal{M}_{11}-\mathcal{M}_{12}\mathcal{M}_{22}^{-1}\mathcal{M}_{21}\right)\eta(t)\right]\end{array} \tag{27}$$

where $w^+(t):=\Psi_q^{-1}\mathcal{M}_{12}^T\eta(t)$. Then, adding and subtracting $\Psi_q^{-1/2}v(t)$ to $w^+(t)$ and using the properties of $\{v(t)\}_{t\geq 0}$ one obtains

$$\begin{array}{l}\lim_{\ell\to\infty}\frac{1}{\ell}\sum_{t=0}^{\ell}E\left[|y(t)|^2-\gamma^2|w(t)|^2\right]+2Tr\,D_f\Psi_q^{\frac{1}{2}}-m\\ \leq\left(\frac{1}{q}-\gamma^2\right)\lim_{\ell\to\infty}\frac{1}{\ell}\sum_{t=0}^{\ell}E\left[w^T(t)w(t)\right]\\ -\lim_{\ell\to\infty}\frac{1}{\ell}\sum_{t=0}^{\ell}E\,\mathcal{P}(t)\\ +\lim_{\ell\to\infty}\frac{1}{\ell}\sum_{t=0}^{\ell}E\left[\eta^T(t)\left(\mathcal{M}_{11}-\mathcal{M}_{12}\mathcal{M}_{22}^{-1}\mathcal{M}_{21}\right)\eta(t)\right]\end{array} \tag{28}$$

where we denoted

$$\mathcal{P}(t):=\left[w(t)-w^*(t)\right]^T\Psi_q\left[w(t)-w^*(t)\right]$$

with $w^*(t):=\Psi_q^{-1}\mathcal{M}_{12}^T\eta(t)+\Psi_q^{-1/2}v(t)$. It follows that $\mathcal{P}(t)\geq 0$ and $\mathcal{P}(t)=0$ for $w(t)=w^*(t)$. This latest case corresponds to the situation when $f_i(x_i)=0, i=1,...,n$, the filter G has the state $x_f(t)$ equal to the state $x(t)$ of F so that

$$\begin{array}{rcl}C_f&=&\Psi_q^{-1}\mathcal{L}_{12}{}^T\\ D_f&=&\Psi_q^{-\frac{1}{2}}.\end{array} \tag{29}$$

Note that we could, therefore, initially take in (9) and order n rather than n_f for the filter G, without loss of generality.

Based on the expression (29) of D_f and since $x_f(t)=x(t)$, from the second equation in (9) it follows that

$$E\left[\tilde{w}(0)\tilde{w}^T(0)\right]=\Psi_q^{-1}. \tag{30}$$

Further it will be shown that under the condition (24) from the statement, for all ESMS filters $G\in\mathcal{G}_a$ having $D_f=\Psi_q^{-\frac{1}{2}}$ the following condition is accomplished

$$-m+\left(\frac{1}{q}-\gamma^2\right)\|G\|_2^2\leq 0. \tag{31}$$

Indeed, since $G \in \mathcal{G}_a$ and since $D_f = \Psi_q^{-\frac{1}{2}}$ it follows from (2) that

$$\det \frac{m\Psi_q^{-1}}{\|G\|_2^2} \geq e^{-2a}. \tag{32}$$

Taking into account (24) and the above inequality it follows that

$$\det \frac{m\Psi_q^{-1}}{\|G\|_2^2} \geq \det \left(\frac{1}{q} - \gamma^2\right) \Psi_q^{-1}$$

from which one directly obtains (31). Using (31), (28), and the equation for D_f in (29), it follows that $||FG||_2/||G||_2 \leq \gamma$ provided that $\mathcal{M}_{11} - \mathcal{M}_{12}\mathcal{M}_{22}^{-1}\mathcal{M}_{21} < 0$.

Let us consider now the more general case, for which D_f does not necessarily satisfy (29), for a certain filter $G \in \mathcal{G}_a$ satisfying, therefore, the condition

$$-\frac{1}{2} \ln \det \frac{mD_f D_f^T}{\|G\|_2^2} \leq a\,. \tag{33}$$

From the above condition and from the assumption (24) it follows that

$$\det \left(\frac{1}{q} - \gamma^2\right) \Psi_q^{-1} \leq \det \frac{mD_f D_f^T}{\|G\|_2^2}. \tag{34}$$

Assume that D_f is positive semidefinite. This is not a restrictive assumption since $E[y^T(t)y(t)]$ and $E[w^T(t)w(t)]$ depend by (8), (9) and Lemma 1 on $D_f D_f^T$ and therefore the computations remain the same if the arbitrary matrix D_f is replaced by the semidefinite matrix $\tilde{D_f}$ given by the Cholesky factorization of $D_f D_f^T$.

Using the property $\det(A) \leq (Tr(A)/m)^m$ for any $A \geq 0$ ([21]), from the above inequality one obtains

$$Tr\left(D_f \Psi_q^{\frac{1}{2}}\right) \geq \left(\frac{1}{q} - \gamma^2\right)^{\frac{1}{2}} m^{\frac{1}{2}} \|G\|_2 \tag{35}$$

and thus

$$\begin{aligned} &\left(\tfrac{1}{q} - \gamma^2\right) \|G\|_2^2 - 2Tr\left(D_f \Psi_q^{\frac{1}{2}}\right) + m \\ &\leq \left(\tfrac{1}{q} - \gamma^2\right) \|G\|_2^2 - 2\left(\tfrac{1}{q} - \gamma^2\right)^{\frac{1}{2}} m^{\frac{1}{2}} \|G\|_2 + m \\ &= \left(\left(\tfrac{1}{q} - \gamma^2\right)^{\frac{1}{2}} \|G\|_2 - m^{\frac{1}{2}}\right)^2. \end{aligned} \tag{36}$$

From the above inequality it follows that if

$$\left(\frac{1}{q} - \gamma^2\right) \|G\|_2^2 = m, \tag{37}$$

the left hand side of (36) is nonpositive and, therefore, from (28) it follows that $\|FG\|_2/\|G\|_2 \leq \gamma$ since $\|FG\|_2/\|G\|_2$ and $\bar{A}(G)$ are invariant under scalar scaling of G, and and since $\mathcal{M}_{11} - \mathcal{M}_{12}\mathcal{M}_{22}^{-1}\mathcal{M}_{21} < 0$ iff $\mathcal{L} < 0$.

4 An Example

Consider the short period dynamics of an air vehicle at a selected operating point,

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} -1 & 20 \\ -2 & -10 \end{bmatrix} \begin{bmatrix} \alpha \\ q \end{bmatrix} + \begin{bmatrix} 0.2 \\ 2 \end{bmatrix} tanh(\delta)$$

where α is the angle of attack, q is the pitch rate and δ is the elevon angle commanded by the servo system modelled as:

$$\dot{\delta} = -100(\delta - \delta_c)$$

and where the control is given by $\delta_c = (0.9044+\nu)\alpha - 0.6252q + w$. The signal ν represents white noise corrupting gain, due to noise in gain scheduling parameter (e.g. dynamic pressure). The servo is subject to soft saturation due to the effect of aerodynamic hinge moments at large elevon angles. Our aim is to analyze the effect of the measurement noise w on the pitch rate $y = q$ in the sense of the anisotropic norm. Augmenting the state vector to be $x = col\{\alpha, q, \delta\}$ and taking a sample time of $T = 0.1$ sec and a zero-order hold, the discrete-time version of the above system is given by (8) where:

$$A_0 = \begin{bmatrix} 0.9881 & 0.1892 & 0 \\ -0.0189 & 0.9030 & 0 \\ 0.5724 & -0.3087 & 0.3679 \end{bmatrix}, B = \begin{bmatrix} 0.0039 \\ 0.0190 \\ -0.0033 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, D = 0, F = \begin{bmatrix} 0 & 0 & 0.0039 \\ 0 & 0 & 0.0190 \\ 0 & 0 & -0.0033 \end{bmatrix}$$

and where $f_i(x_i) = tanh(x_i)$. Therefore, $\sigma_i = 1$. The above gain noise ν is represented by the following matrix

$$A_1 = \begin{bmatrix} 0.15 & 0 & 0 \\ 0 & 0.15 & 0 \\ -0.009482 & 0 & 0.05182 \end{bmatrix}$$

Note that $f_i, i = 1, 2$ are multiplied in (8) by zero and, therefore, do not play any role. The noise w has been simulated as the output of a low-pass filter with a standard white noise input sequence.

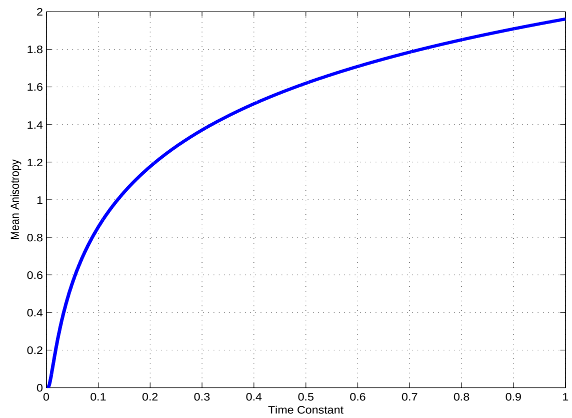

Figure 1: Mean Anistoropy of w

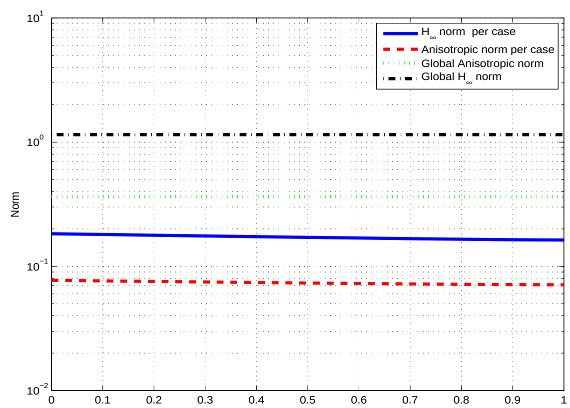

Figure 2: Comparison of H_∞ and Anisotropic norms

The mean anisotropy of the noise is depicted in Fig. 1 as a function of the filter time constant. A time constant of $0.0085\,\text{sec}$. has been taken, leading to the anisotropy of $a = 0.05$. In Fig. 2. the H_∞-norm (solid blue line) is depicted, of the system replacing the non-linearity by a series of "scale factors" in $[0, 1]$ which represent the incremental gain of $f_3(x_3) = tanh(x_3)$.

The global H_∞-norm (black dash-dotted line) has been computed using the results of ([14]) and YALMIP ([22]) for a that tends to infinity, serves, as could be expected, as an upper bound to the case-wise H_∞-norms (solid blue line). Similarly the anisotropic norm (dashed red line) for the same "scale factors" replacing the non-linearity are depicted in Fig. 2, along with the anisotropic norm-bound γ (dotted green line) derived from Theorem 1. Indeed the global anisotropic norm bound γ, provides an upper bound for the case-wise scale factors (red dashed line), and is lower than the global H_∞-norm (black dash-dotted line), demonstrating, therefore, the reduced conservatism in the anisotropic-norm with respect to the H_∞-norm. The above system has been simulated, and the results are depicted in Fig. 3, where Fig. 3a shows δ before and after the soft saturation by the $tanh$ function, and Fig. 3b, depicts the pitch rate $y = q$. We note that the ratio 0.07 between the standard deviation of y and w, is less than $\gamma = 0.363$ predicted by Theorem 1.

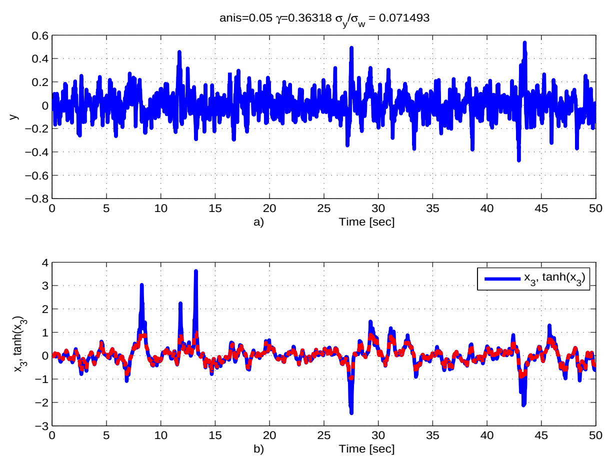

Figure 3: Simulation Results : a) System output b) Control Input

5 Conclusions

A class of stochastic Hopfield networks subject to state-multiplicative noise has been considered. Stochastic stability and disturbance attenuation analysis in an anisotropic-norm setup has been derived. The results can be applied to e.g. a stick balancing related model, inspired by ([3]) and ([23])

which includes such state-multiplicative noise. Such application is left as a topic for future research.

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