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THE CLASSIFICATION OF RIEMANN SURFACES AND CONDITIONS ON THE QUANTUM THEORY *

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Abstract

From the classification of Riemann surfaces, the categories relevant for quasiconformal rigidity, non-renormalization theorems, vanishing flux conditions and the value of the nonperturbative four-dimensional string coupling are described. It is verified that the domain of string perturbation theory should be identified with the class O_G. The restriction to these surfaces is sufficient to induce a reduction of the exceptional group invariance required by the intersection form of a nonsmooth four-manifold that is an embedding space of an infinite-genus surface to the gauge groups of the standard model. The occurrence of condensation of string ground states follows from the structure of the Hilbert spaces on a countable set of ends of noncompact surfaces.

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1 Domain of String Perturbation Theory and Quasiconformal Transformations

The class of surfaces in the universal moduli space of string theory would be required to satisfy the following conditions: (i) computability of the

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correlation functions (ii) invariance under quasiconformal transformations (iii) valid definition of the S-matrix (iv) the values of the string and unified gauge coupling and (v) non-renormalization theorems and vanishing flux conditions.

While conformal transformations define the same Riemann surface, quasiconformal transformations generate a transformation in the Teichmüller space of surface of a fixed genus. Consequently, it may be considered valid to impose the condition of invariance under quasiconformal transformations which do not require more than one copy of moduli space. The class O_{KD} is quasiconformally invariant, whereas O_{KB}, O_{AB} and O_{AD} do not have this invariance [54]. By a theorem of Beurling and Ahlfors, there exists a mapping of the extended complex plane which preserves the unit circle and there is a compact subset of this circle of zero linear measure such that the image has positive linear measure [2]. Furthermore, there are Fuchsian groups uniformizing Type I surfaces with no border arc which are mapped to Fuchsian groups of Type II surfaces under conformal conjugation. The class of infinite-genus O_{AB} and O_{AD} surfaces are not invariant under conformal transformations, and Myrberg's surfaces represent the counterexamples. These results suggests that the class of surfaces identified with the domain of string perturbation theory either must have ideal boundaries of zero harmonic measure or the unit circle except for a set of zero linear measure.

The categories O_G, O_{HD}, O_{HD}^n and U_{HD} are invariant under Royden mappings, which have the property

$$K(T,A)^{-1} log\ mod\ A \leq log\ mod\ TA \leq K(T,A)\ log\ mod\ A \qquad (1)$$

where $K(T,A) \geq 1$ and A is any A-set consisting of a pair (G_1, G_2) where $(\partial G_j) \cap \bar{U}$ is a Jordan arc connecting two boundary points of ∂U and $\bar{G}_2 \cap U \subset G_1$ [54]. Since every quasiconformal transformation is a Royden mapping, O_G, O_{HD}, O_{HD}^n and U_{HD} are quasiconformally invariant.

Furthermore, O_G, O_{HB}, O_{HB}^n and U_{HB} are invariant under Wiener mappings, defined to maintain $f \in \mathbb{N}(\Sigma)$, where $\mathbb{N}(\Sigma)$ consists of functions that are bounded, continuous and harmonizable on Σ. Every Royden mapping is required to be a Wiener mapping for O_{HB}, O_{HB}^n and U_{HB} to be quasiconformally invariant. Royden's algebra $\mathbb{M}(\Sigma)$ has been proven to be a subset of $\mathbb{N}(\Sigma)$ [54], and it remains to be shown that $f \circ T \in \mathbb{N}(\Sigma_1)$ if and only if $f \in \mathbb{N}(\Sigma_2)$ when $T : \Sigma_1 \to \Sigma_2$ is a Royden mapping, given that it is valid for the subsets $\mathbb{M}(\Sigma_1)$ and $\mathbb{M}(\Sigma_2)$. Representing the map of functions by a change of coordinates of the Riemann surface, the action of T can be extended from $\mathbb{M}(\Sigma)$ to $\mathbb{N}(\Sigma)$. Since the change of coordinates preserves continuity and square-integrability of the partial derivatives, it must be generalized such that the inequalities satisfied by superharmonic and subharmonic functions in $G - K_s$, together with the resolutive property of the surface, where G is a normal subregion of Σ and K_s is a compact set, continue to hold. If this property is valid, the result follows.

A comparison of the categories of surfaces with respect to the five criteria is provided in §2 to §5. Several of the criteria are satisfied by different classes of surfaces. Almost all of the conditions are found to be satisfied by O_G and O_{HD}. It will be demonstrated that the entire set is consistent with the closed finite genus surfaces and infinite-genus surfaces in O_G, and this quasiconformally invariant class may be identified with the domain of string perturbation theory, because no new string states are introduced in the S-matrix calculations in contrast with O_{HD}.

This method therefore yields a precise definition of the perturbation series of string theory, described in §6, and generates an addition factor of 2^g in the scattering amplitudes and a coincidence of the effective string coupling with the unified gauge coupling of the minimal supersymmetric extension of the standard model. Furthermore, the nonperturbative effects within this path integral formalism would be related to other classes of infinite-genus surfaces. Tachyon condensation can be described by the collection of ground states in a countable union of Hilbert spaces on the ends of a noncompact surface which may be identified [4][22]. It has been postulated, for example, that instanton effects could be identified with the surfaces not belonging to O_{HD}, since there must exist harmonic functions with finite Dirichlet norm. Given that the surfaces are analytically embedded in space-time, this class can be chosen to be O_{AB}/O_{HD} if the ideal boundary is required to have a vanishing linear measure.

The evaluation of scattering amplitudes in superstring theory has been completed through genus two with a parametrization of the moduli space integral through super-period matrices [24]. Renormalization theorems have been demonstrated at arbitrary genus in the covariant [42] and pure spinor formalisms [7]. Estimates of the amplitudes at higher genus require another coordinatization of the interior moduli space, such as the Schottky group parameters, while the contribution of the compactification divisor also may be bounded [21]. Beyond these computations, the study of nonperturbative effects returns to summation techniques which yield the exponential terms [5], conformal field theories that represent nontrivial exact backgrounds [57], nontrivial tunnneling geometries occurring in solutions to the effective field equations after the imposition of specific boundary conditions [11] and the winding modes of space-time instantons over compactifications [9].

The worldsheet instantons in the literature have been defined by nontrivial configurations such as the worldsheet wrapping around cycles in the compact component of a solution to the ten-dimensional field equations such as a Calabi-Yau manifold [35]. It is not possible to relate this type of instanton with the classification theory of Riemann surfaces, which is background-independent. The analogue for the space-time instanton [29] with a $G_2/SU(3)$ compactification of the ten-dimensional space-time has been found to result in a necessary factor in converting scattering amplitudes in the Euclidean and Lorentzian formalisms [23]. Nevertheless, with a differentiation between terms in the sum over surfaces, the categorization of all of the instanton effects in string theory now would be available.

2 Inclusions of Classes of Riemann Surfaces

Consider the following categories of Riemann surfaces

$$\begin{aligned}
O_G &= \{surfaces\ with\ no\ Green\ function\ with\ a\ single\ source\} \\
&= \{surfaces\ ideal\ boundaries\ of\ zero\ harmonic\ measure\} \\
O_{HD} &= \{surfaces\ with\ no\ non-constant\ harmonic \\
&\quad functions\ of\ finite\ Dirichlet\ norm\} \\
O_{HB} &= \{surfaces\ with\ no\ non-constant\ bounded\ harmonic\ functions\} \\
O_{KD} &= \{surfaces\ with\ no\ non-constant\ harmonic\ functions\ of \\
&\quad finite\ Dirichlet\ norm\ with\ semi-exact\ dual\ differential\} \\
O_{KB} &= \{surfaces\ with\ no\ non-constant\ bounded\ harmonic\} \\
&\quad functions\ with\ semi-exact\ dual\ differential\} \\
O_{AD} &= \{surfaces\ with\ no\ non-constant\ analytic \\
&\quad functions\ of\ finite\ Dirichlet\ norm\} \\
O_{AB} &= \{surfaces\ with\ no\ non-constant\ bounded\ analytic\ functions\}
\end{aligned} \tag{2}$$

where a differential ω is semi-exact when $\int_\gamma \omega = 0$ for every dividing cycle. Then the following inclusions hold:

$$\begin{array}{ccccccc}
O_G \subset & O_{HB} & \subset & O_{KB} & \subset & O_{AB} \\
& \cap & & \cap & & \cap \\
& O_{HD} & \subset & O_{KD} & \subset & O_{AD}.
\end{array} \tag{3}$$

An example of a surface which belongs to the space $O_{HD} - O_{HB}$ is Toki's surface [54].

Type I and II surfaces are characterized by the divergence and convergence of the Poincare series of the uniformizing Fuchsian group. Every surface in the class O_G is a Type I surface, and every Type II surface is conformally equivalent to a fundamental domain with a border arc in the hyperbolic disk.

A Type I surface is uniformized by Fuchsian group Γ which has a limit set $\Lambda(\Gamma)$ given by $\partial\Delta$. Since the border arc is $(\partial\Delta - \Lambda(\Gamma))/\Gamma$, it is not present for a Type I surface.

Let

$$\begin{aligned} O^n_{HD} &= \{\text{surfaces with dim HD}(\Sigma) \leq n\} \\ O^n_{HB} &= \{\text{surfaces with dim HB}(\Sigma) \leq n\} \end{aligned} \tag{4}$$

such that $O^1_{HD} = O_{HD}$ and $O^1_{HB} = O_{HB}$. Any surface which belongs to O^n_{HD} and not O_G has infinite genus. A set of harmonic functions, subject to conditions similar to effective closure in O_G, can be defined on regular O^n_{HD} surfaces of finite type. However, there is a surface in $O^n_{HD} \cap P_G$ which does not admit harmonic functions satisfying an integral condition including a Green function [6]. Furthermore, all O^n_{HD} surfaces can be labelled as Type I [54]. The quasiconformal invariance of O_G is consistent with the category of stable Type I groups [43]. Equivalence under conformal conjugation to Type II groups would be possible only for Type I surfaces in the complement of O_G.

If

$$\begin{aligned} U_{HD} &= \{surfaces\ with\ a\ non-constant\ HD-minimal\ function\} \\ U_{HB} &= \{surfaces\ with\ a\ non-constant\ HB-minimal\ function\} \end{aligned} \tag{5}$$

then there are the inclusions

$$\begin{array}{ccccccc} O_{HB} & \subset & O^n_{HB} & \subset & U_{HB} \cup O_G & \subset & O_{AB} \\ \cap & & \cap & & & & \cap \\ O_{HD} & \subset & O^n_{HD} & \subset & U_{\tilde{HD}} \cup O_G & \subset & O_{AD}. \end{array} \tag{6}$$

There is a Type II surface which is an element of U_{HB} [54]. Since $U_{HB} \cup O_G \subset U_{\tilde{HD}} \cup O_G$, Type I surfaces define a category between O^n_{HB} and $U_{HB} \cup O_G$.

Since the category of P_G surfaces represent the complement of the class O_G, it has a non-null overlap with Type I, and these surfaces must have ideal boundaries with non-zero harmonic measure. The distinction between surfaces with ideal boundaries of different dimension and measure is relevant for the universal moduli space of string theory, and this problem shall be considered in more detail in this paper.

3 Characterization of Spheres with an Infinite Number of Handles and the Computability of Superstring Amplitudes

Computability of N-point superstring amplitudes depends on the parameterization of supermoduli space. Uniformization by super-Schottky groups, in particular, yields the Neveu-Schwarz amplitudes after integration over the multipliers and odd Grassmann variables and a sum over 2^g spin structures together with the remaining even spin structures [16] [25]. The measure for the Ramond sector has been derived [13][30] [50] and the sum can be extended to the other spin structures through modular transformations.

For the region of supermoduli space which has a global slice, the measure has the property of holomorphic factorizability and can be given defined in terms of the super-Schottky group parameters with $3g-3$ even and $2g-2$ odd dimensions. The $3g-3$ multipliers represent surfaces that can be uniformized by the Schottky groups.

Expansions of the correlation functions on the Riemann surface in automorphic forms with respect to the Schottky group are valid when the series converge [8][14]. Since the expressions also can be defined by theta functions, it is possible to analytically continue the equations into the region of moduli parameter space for which the series diverge. Theta functions, in particular, can be constructed on surfaces with an exhaustion function of finite charge. These surfaces are parabolic and satisfy a set of geometric hypotheses [28].

The following condition is imposed on the handles of surfaces which satisfy the geometric hypotheses. For all $j \geq g+1$, A_j is the homology class represented by the oriented loops $\phi_j(\{(\sqrt{t_j}e^{i\theta}, \sqrt{t_j}e^{-i\theta})|\ 0 \leq \theta \leq 2\pi\}) \quad 0 < t_j < \frac{1}{2}$, where ϕ_j is a map from the handle to the punctured disk, and for all $\beta > 0$, $\sum_{j \geq g+1} t_j^{\beta} < \infty$ [28]. The sizes of the handles then would be decreasing exponentially.

The surfaces also can be embedded in a connected space, constructed with a chosen class and by a method analagous to the definition of universal moduli space. The set of infinite-genus surfaces with nodes should be dense in $\bar{M}_\infty$ since adding or subtracting a handle or splitting a finite-genus component of the surface is a small effect. If the sizes of the handles decrease such that in the intrinsic metric, the thickness of each handle is bounded below, then the surface is not arbitrarily close to a surface with nodes. However, if the sizes of the handles decrease more rapidly, then the distance in moduli space based on the intrinsic metric will tend to zero, and the original surface is arbitrarily close to a surface with nodes. The condition $\sum_g r_g^2 < \infty$ implies that $r_g \sim \frac{1}{g^{\frac{1}{2}+\epsilon}}$ in the Euclidean metric. The restriction on the sizes of the handle is less restrictive than the condition in the geometric hypotheses. Given that the areas in the intrinsic metric are multiplied by a factor proportional to n for a surface with g handles, it follows that $r_g^{int.} \sim \frac{1}{g^{\epsilon}}$. This property would be valid in the moduli space of effectively closed infinite-genus surfaces.

By the retrosection theorem, any closed Riemann surface of finite genus can be uniformized by a Schottky group which maps a set of g Jordan curves into another set of curves of this kind [10] [36]. However, any fractional linear transformation $T_n z = \frac{\alpha_n z+\beta_n}{\gamma_n z+\delta_n}$ has the isometric circle $|\gamma_n z + \delta_n| = 1$ which mapped under T_n to another isometric circle $| - \gamma_n z + \alpha_n| = 1$. Therefore, the group is classical provided that $\alpha_n,\ \gamma_n,\ \delta_n \neq 0$. Furthermore, every closed finite-genus surface is uniformized by a Fuchsian group, from which a Fuchsian Schottky group can be constructed. The Fuchsian Schottky group may be regarded as a classical Schottky group, perhaps with a different set of generators [10]. Therefore, classical Schottky groups can be used to uniformize every closed finite-genus surface, and the existence of nonclassical Schottky groups [41] [59] does not affect this result.

Theorem 3.1. *Spheres with an infinite number of handles, which represent effectively closed surfaces in the domain of string perturbation theory, belong to the class of O_G surfaces.*

Proof. Spheres with an infinite number of handles, that can be uniformized by infinitely-generated groups of Schottky type, are also uniformized by Fuchsian groups with divergent Poincare series [14][17]. These surfaces are therefore classified as Type I. A normal operator maps a function to a harmonic function on a region R_1 in a Riemann surface, such that $Lf|_{\alpha_1} = f$ and $\int_{\alpha_1} *dLf = 0$, where α_1 bounds R_1, and $min\ f \leq Lf \leq max\ f$. The principal function p is defined by the normal operator L by $L(p-s)|_{\alpha_1} = p-s$, where $s = ln|z-\zeta|$ is the dependence near the singularity, R_1 is located away from the singularity ζ and a curve α_1 with $p|_{\alpha_1} = const.$ [53]. Furthermore, by a theorem on principal functions with pairs of logarithmic singularities, if p_0 has zero normal derivative and p_1 has vanishing flux at the accumulation point of the handles, and $p_0 - p_1$ is constant, the surface belongs to the class

O_{HD} [3]. With sources at z_R and z_S, the Green function is

$$G(z_P; z_R, z_S) = \sum_\alpha ln\left|\frac{z_P - V_\alpha z_R}{z_P - V_\alpha z_S}\right| \qquad (7)$$
$$- \frac{1}{2\pi}\sum_{m,n=1} Re(v_n(z_P))(Im\ \tau)^{-1} Re(v_n(z_R) - v_n(z_S))$$

where $v_n(z) = \sum_\alpha{}^{(n)} ln\left(\frac{z - V_\alpha \xi_{1n}}{z - V_\alpha \xi_{2n}}\right)$, with ξ_{1n}, ξ_{2n} being the fixed points of the generator T_n and the sum $\sum_\alpha{}^{(n)}$ includes V_α such that T_n and T_n^{-1} is not the right-most member. Adding the harmonic function $\sum_n h_n(z, \bar{z})$, with

$$h_n(z, \bar{z}) = \sum_\alpha \sum_p Re[v_n(V_\alpha z) - v_n(V_\alpha z_0)] b_p (V_\alpha z - V_\alpha z_0)^p \qquad (8)$$

the normal derivative in the limit $z \to \infty$ equals

$$\frac{1}{2}\sum_n a_n e^{i\theta} h_n'(z) + c.c. \qquad (9)$$

$$= \frac{1}{2}\sum_n a_n e^{i\theta} \sum_p \sum_\gamma{}^{(\beta)} \sum_\beta{}^{(n)} \frac{(\xi_{1n} - \xi_{2n})\gamma_\beta^{2-2p} b_p \left(\frac{\alpha_\gamma}{\gamma_\gamma} - V_\gamma' z_0\right)^p}{\left(\frac{\alpha_\gamma}{\gamma_\gamma} - \xi_{1n}\right)\left(\frac{\alpha_\gamma}{\gamma_\gamma} - \xi_{2n}\right)\left(\frac{\alpha_\gamma}{\gamma_\gamma} + \frac{\delta_\beta}{\gamma_\beta}\right)^{p-2}\left(V_\gamma' z_0 + \frac{\delta_\beta}{\gamma_\beta}\right)^p} \qquad (10)$$

$$+ \frac{1}{2}\sum_n a_n e^{i\theta} \sum_p \sum_\gamma{}^{(\beta)} \sum_\beta{}^{(n)} \sum_\delta{}^{(n)} ln\left|\frac{\frac{\alpha_\gamma}{\gamma_\gamma} - V_\delta \xi_{1n}}{\frac{\alpha_\gamma}{\gamma_\gamma} - V_\delta \xi_{2n}} \frac{V_\gamma' z_0 - V_\delta \xi_{2n}}{V_\gamma' z_0 - V_\delta \xi_{1n}}\right|$$
$$\cdot\, p b_p \left[\frac{\left(\frac{\alpha_\gamma}{\gamma_\gamma} - V_\gamma' z_0\right)}{\left(\frac{\alpha_\gamma}{\gamma_\gamma} + \frac{\delta_\beta}{\gamma_\beta}\right)\left(V_\gamma' z_0 + \frac{\delta_\beta}{\gamma_\beta}\right)}\right]^{p-1} \gamma_\beta^{2-2p} + c.c.$$

The flux is $2\pi \sum_n a_n c_{1n}$, where

$$c_{1n} = \sum_\gamma{}^{(\beta)} \sum_\beta{}^{(n)} \frac{\gamma_\beta{}^{-2p}}{\left(z_0 + \frac{\delta_\beta}{\gamma_\beta}\right)^p} \left(\frac{\alpha_\gamma}{\gamma_\gamma} + \frac{\delta_\beta}{\gamma_\beta}\right)^{-(p-1)} \left(V_\gamma' z_0 + \frac{\delta_\beta}{\gamma_\beta}\right)^{-p} \qquad (11)$$
$$\cdot \frac{1}{\left(\frac{\alpha_\gamma}{\gamma_\gamma} - \xi_{1n}\right)} \frac{1}{\left(\frac{\alpha_\gamma}{\gamma_\gamma} - \xi_{2n}\right)}$$
$$\left[\frac{\beta_\gamma - \delta_\gamma}{\gamma_\gamma}\left[2 - \left(\frac{\alpha_\gamma}{\gamma_\gamma} + \frac{\delta_\beta}{\gamma_\beta}\right)\left[\frac{1}{\left(\frac{\alpha_\gamma}{\gamma_\gamma} - \xi_{1n}\right)} + \frac{1}{\left(\frac{\alpha_\gamma}{\gamma_\gamma} - \xi_{2n}\right)}\right]\right]\right.$$
$$\left. - p\left[\left(z_0 + \frac{\delta_\beta}{\gamma_\beta}\right)\left(\frac{\alpha_\gamma}{\gamma_\gamma} + \frac{\delta_\beta}{\gamma_\beta}\right) + \frac{\beta_\gamma - \gamma_\gamma}{\delta_\gamma}\right]\right].$$

Since

$$2-\left(\frac{\alpha_\gamma}{\gamma_\gamma}+\frac{\delta_\beta}{\gamma_\beta}\right)\left[\frac{1}{\left(\frac{\alpha_\gamma}{\gamma_\gamma}-\xi_{1n}\right)}+\frac{1}{\left(\frac{\alpha_\gamma}{\gamma_\gamma}-\xi_{2n}\right)}\right] \tag{12}$$
$$=-\frac{\left(\xi_{1n}+\frac{\delta_\beta}{\gamma_\beta}\right)}{\left(\frac{\alpha_\gamma}{\delta_\gamma}-\xi_{1n}\right)}-\frac{\left(\xi_{2n}+\frac{\delta_\beta}{\gamma_\beta}\right)}{\left(\frac{\alpha_\gamma}{\gamma_\gamma}-\xi_{2n}\right)},$$

$\frac{\alpha_\gamma}{\gamma_\gamma}\to\infty$ and the ratio tends to zero. However, when $\left|\frac{\alpha_\gamma}{\gamma_\gamma}-\xi_{1n}\right|$, $\left|\frac{\alpha_\gamma}{\gamma_\gamma}-\xi_{2n}\right|$ are bounded and $\frac{\delta_\beta}{\gamma_\beta}\to\infty$, this quotient has unbounded magnitude. Therefore, $|c_{1n}|=\infty$ for all n, and the flux vanishes only if a_n, $n=1,2,3,...$, equals zero [15]. Consequently, the difference between the two principal functions equals a constant, and the surfaces represent a subset of O_{HD}.

Since the boundary consists of an accumulation point, the harmonic measure will vanish. Then

$$\int_{\partial\Sigma}\sqrt{h(z)}\partial_i G(z,\bar{z})dz^i=0 \tag{13}$$

where h is the determinant of metric on the Riemann surface, and the equation

$$\triangle G(z,z')=\frac{1}{\sqrt{h(z)}}\delta(z,z') \tag{14}$$

has no solutions. This property will hold regardless of the convergence of the Poincare series of the Schottky group and when there are a finite number of endpoints. While it might be possible for the ideal boundary point to have positive harmonic measure [49][51], such that a Green function with a single source in the interior of the surface exists, it can be verified any harmonic automorphic function on the plane, which equals zero on a circle of radius R, also vanishes at $z=\infty$. The limit set of a Schottky group has positive logarithmic capacity [8] and positive Hausdorff dimension. The ideal boundary, defined by the factorization of the set of ordinary points by the group, must have zero capacity, linear measure zero and Hausdorff dimension determined by the exponent of convergence of the Poincare series. Then the harmonic measure of the ideal boundary must vanish. While the algebraic limit of a sequence of Schottky groups may have a fundamental domain not bounded entirely by circles [41], the example of the infinitely-generated group of Schottky type shows that the surface which may be uniformized by the limiting group belongs to O_G [17][37]. □

The class of spheres with an arbitrary number of nonoverlapping handles would belong to the intersection $Type\ I \cap O_{HD}$ [14] [15][17], and it is a quasiconformally invariant proper subset of this set. The set of Type I surfaces, however, cannot be selected as a proper domain for perturbative calculations because there are Fuchsian groups of the first kind that are conformally conjugate to Fuchsian groups of the second kind [43]. The countability of the basis of the modular group of a Riemann surface has been shown to be connected to the category of O_G surfaces [44]. This property is sufficient for the evaluation of moduli space integrals based on coordinates defined by a uniformizing group parameterization.

4 Non-Renormalization Theorems and Surfaces of Infinite Genus

Non-renormalization theorems have been used to prove heuristically finiteness of arbitrary-genus superstring amplitudes and the fixed values of parameters in the effective field theory [26][30][42]. These theorems are valid for closed finite-genus surfaces because contour integrals arising from BRST transformations can be set to zero as there is no boundary. At infinite-genus, the vanishing of the contour integral would follow immediately if the ideal boundary has zero linear measure, thereby including the O_G surfaces.

The class of functions $KD(\Sigma)$ is defined to be the space of harmonic functions u with finite Dirichlet norm and semi-exact differential $*du$ [54]. An O_{KD} surface, which is Type I, cannot support harmonic functions in the contour integrals. Since the function u must have singularities, the integral of $*du$ over a dividing cycle equals

$$\sum_i \int_{\gamma_i} *du = \int_{idl.bdy.} *du_h \tag{15}$$

where $\{\gamma_i\}$ surrounds each of the singularities of u and u_h is the harmonic part of the function, and it vanishes when the function tends asymptotically to a constant.

Vertex operators represent the contributions of the external lines of string states in a genus-g scattering amplitude. The external lines for strings are semi-infinite cylinders that can be conformally mapped to a punctured disk on the surface. In a supersymmetric theory, the fermion and boson vertex operators are related by commutators with the BRST charge Q_{BRST} through

$V_F = \int_C [Q_{BRST}, V_B]$ [30][42], where the contour surrounds the location of the vertex operator V_B at the puncture on the Riemann surface. Given a vertex operator V located at z_0, the analogue of the above integral would be

$$\int_{\gamma_0} [Q_{BRST}, V] = \int_{\gamma} [Q_{BRST}, V] = \int_{ideal\ bdy} [Q_{BRST}, V]_h \qquad (16)$$

with γ_0 surrounding z_0 and γ being a homologous dividing cycle. This integral will vanish if $[Q_{BRST}, V]_h \to 0$, near the ideal boundary. Vertex operators have singularities on the Riemann surface, and based on the BRST charge and the form of the correlation functions [30], the commutator would vanish at the ideal boundary and the non-renormalization theorem would remain valid for this class.

5 Other Categories of Surfaces and String Diagrams

The class of infinite-genus hyperelliptic surface has been shown to be maximal [52]. The space of hyperelliptic surfaces may be regarded as a subspace of the class of surfaces satisfying the condition of conformal rigidity that arises in the domain of string perturbation theory. While not all hyperelliptic surfaces satisfy the criteria for surfaces with exhaustions of finite charge, conditions on the branch points can be be given such that the geometric hypotheses hold [28]. The joining of the branch points in the second Riemann sheet generates the handles, and therefore, the hyperelliptic surfaces are contained in the space defined by attaching handles to a sphere. These surfaces are uniformized by groups of Schottky type even when there an infinite number of handles. Consequently, the space of hyperelliptic surfaces would be contained in O_G. A regularization is required for the definition of the function corresponding to the hyperelliptic equation [39]. This function must be used for each accumulation of the branch points and does not exist for an infinite number of accumulation points.

A phase transition occurs for the Coulomb gas system modelling the free-fermion theory with branch point operators in the space of hyperelliptic surfaces [39]. However, not all of the fields in string theory would have this interpretation, and furthermore, the integration region must be enlarged to include the $3g-3$-dimensional moduli space at genus g.

The O' and O'' classes are defined by constraints on the extremal length of geodesics [1] [37]. These classes are both contained in O_G. It is possible that these conditions are related to the properties of closed string field theory [60].

6 The Value of the Four-Dimensional String Coupling

An infinite number of accumulation points is not allowed for O_G surfaces, and this property is required for the determination of the nonperturbative four-dimensional string coupling. The coefficient of the integral for the graviton amplitude is

$$C_g^{II} = \left(\frac{1}{4}\right)^g (g_D)^{2g-2} \left(\frac{1}{2\pi}\right)^{\frac{Dg}{2}+3g-3} \cdot (\alpha')_c^{-2+g\frac{4-D}{2}} \tag{17}$$

where α'_c is a parameter defined for closed strings [17]. The relation between α'_c and α' for open strings can be deduced by deriving a formula for the tension. Suppose that the length of the open string is equal to the radius of a circular closed string configuration. The average longitudinal length of half of the closed string would be twice that of the radius after inclusion of the oscillations. With a uniform distribution of mass along the string, the closed string tension would be

$$T_{closed} = \frac{2\pi[m_P]c^2}{(4\ell_P)^2} = \frac{\pi}{8} T_{open}. \tag{18}$$

It follows that $G_N \to 16\alpha'$ and $g_4^2 \to \frac{16\pi G_N}{\alpha'} = 256\pi$ and

$$C_g^{II} \to \frac{1}{(16\sqrt{\pi})^2(2\pi)^3\alpha'^2} \left(\frac{2}{\pi^4}\right)^g. \tag{19}$$

Theorem 6.1. *A factor of 2^g from the inclusion of surfaces in the class O_G is introduced in the string perturbation series increasing the effective coupling to $\frac{4}{\pi^4}$ at finite genus. This category of surfaces would have a set of ends represented by $E(p_1p_2...)$, which is restricted by the condition $lim_{\nu\to\infty}\langle p_\nu\rangle = 1$ with the validity of another constraint, $\langle p_\nu\rangle \sim e^{\frac{2^\nu}{\nu(ln\ \nu)^\alpha}}$, $\alpha \le 1$, to be determined, since the upper bound for the Robin constant is infinite. The non-zero value of the capacity for surfaces satisfying the second relation is established. The inequalities for the Robin constant are sufficient to conclude that the surface does not belong to O_G if p_ν is a bounded function such that $p_\nu > 1+\epsilon$, where $\epsilon > 0$ is constant, for all ν.*

Proof. The set of ends of an O_G surface can be derived from a Cantor set $\cap E(p_1 p_2 ...)$ with the limiting value $lim_{\nu\to\infty} p_\nu \to 1$ [17][54] or $p_{\nu_0} = 1$ for some ν_0 for the capacity of the boundary to be zero. □

Lemma 6.1. *When the values of p_ν are bounded, it is necessary for $1 \le \langle p_\nu \rangle < \frac{1}{1-e^{-2^\nu}}$ when ν is sufficiently large, where $\langle p_\nu \rangle$ represents an average over sufficiently many subdivisions of the Cantor set.*

Proof. It is known that the harmonic measure of the ideal boundary is

$$\prod_\nu \left(1 - \frac{1}{p_\nu}\right)^{2^{-\nu}} = 0. \tag{20}$$

or equivalently

$$lim_{\nu\to\infty} \left(1 - \frac{1}{p_\nu}\right)^{-2^{-\nu}} = \epsilon \tag{21}$$

with $0 \le \epsilon < 1$, such that the terms in the product at finite ν are bounded. Since $lim_{w\to 0} e^{f(w) ln\ w} \to 0$ only if $f(w) \to 0$ more slowly than $\left|\frac{1}{ln\ w}\right|$. In Eq.(6.5), this dependence would imply that $2^{-\nu}$ tends to zero more slowly than $\left|\frac{1}{ln\left(1-\frac{1}{p_\nu}\right)}\right|$ and

$$\left| ln\left(1 - \frac{1}{p_\nu}\right)\right| > 2^\nu \qquad for\ sufficiently\ large\ \nu. \tag{22}$$

Then

$$1 \le p_\nu < \frac{1}{1-e^{-2^\nu}} \qquad for\ sufficiently\ large\ \nu. \tag{23}$$

In this range, again $p_{\nu_0} = 1$ for some ν_0, and there is no further subdivision of the subintervals beyond ν_0. For infinite-genus surfaces in this class, either the contribution of thin ends to the harmonic measure is zero or the size of the ends vanish in this limit □

The top limit could be defined to be an ordinal other than ω. For example, if it is the next ordinal, $\omega + 1$, after 2^ω subdivisions, $p_{\omega+1}$ can be chosen to be 1. Equivalently, the infimum of the harmonic measures of sets of $2^{\aleph_0}$ points in the real interval or circle may be used to establish this result. The maximal cardinality of the endpoints of accumulating handles on a sphere likewise would equal the maximal cardinality of the of endpoints in the Cantor set for surfaces in the class O_G. The Heins problem consists of the enumeration of the harmonic dimensions of all O_G surfaces [47][48], and it has been found that an upper bound for the cardinality of set of ends would be $2^{\aleph_0}$. If the continuum hypothesis is valid, there is no set with a cardinality betweeh $\aleph_0$ and $2^{\aleph_0}$, and the multiplication of the coefficient of the genus-g amplitude by 2^g would occur for other classes of surfaces.

Lemma 6.2. *The harmonic measure of the ideal boundary of a sphere with accumulating sequences of infinite number of handles may be zero if the cardinality of the set of endpoints is* $\aleph_0$*.*

Proof. The maximum cardinality of the set of ends of surfaces in the class O_G would be bounded number of endpoints of accumulating handles on a sphere by Theorem 1. When the accumulation points can be chosen to be located on the equator, the number of thin ends will be determined by the number of endpoints that are allowed on this circle. An infinite number of points placed on a compact circle would be spaced by arbitrarily infinitesimal distances, although the complement of this set in S^2 remains Hausdorff if the cardinality is less than or equal to $\aleph_0$. Suppose that every rational point on the equator is an accumulation point of handles on the sphere that is removed from the manifold. Between any two different irrational points in the angular range $[0, 2\pi]$, α_1 and α_2, there exists a rational number, since $lim_{n\to\infty}|x_n - \alpha_1| = 0$ for a Cauchy sequence of fractions $\{x_n\}$ converging to α_1. For a sufficiently large integer N_0, such that $|x_n - \alpha_1| < |\alpha_2 - \alpha_1|$ for $n > N_0$. When the accumulation points of handles on a sphere are located on the equator only, both hemispheres with handles are available for neighbourhoods of points on the equator. Although there would be handles arbitrarily closed to the equator, the neighbourhood of an irrational point could be defined to include an area on the handle. Then the intersection of the boundary of a neighbourhood of α_1 may be chosen to be an irrational number α_3 such that

$$|x_{N_1} - \alpha_1| < |\alpha_3 - \alpha_1| < |x_{N_2} - \alpha_1| \qquad N_1,\ N_2 > 0.$$

The boundary of a neighbourhood of α_2 also can be selected to contain α_4, where

$$|x_{N_2} - \alpha_1| < |\alpha_4 - \alpha_1| < |x_{N_3} - \alpha_1| \qquad N_3 > N_0.$$

Consequently, two separate points on the equator and the two hemispheres will have nonoverlapping neighbourhoods, and the manifold is a Hausdorff space. It remains to be established if the handles on the spheres accumulating to the endpoints on the equator can be arranged on the unit sphere. A finite number of handles of finite width would be located at the same longitude θ_b. However, an infinite number of rational points may be placed in the range $[0, 2\pi]$. The integers $\mathbb{Z}$ are identified with the diagonal in a two-dimensional array of numerators and denominators representing $\mathbb{Q}$. Consider instead the sequence of fractions $\left\{\frac{m}{m+1},\ m \in \mathbb{N}\right\}$ and angles $\left\{2\pi\frac{m}{m+1}\right\}$. Each of the points is distinct, and the set has cardinality $\aleph_0$. Furthermore, the thickness of the handle can be set equal to an infinitesimal value since the embedding in higher dimensions allows another angular variable representing the orientation such that the handles do not intersect. Given two arbitrarily close rational numbers a_1 and a_2, the distances can be multiplied by $\left\{\frac{1}{n^\alpha},\ n \in \mathbb{Z}\right\}$ yielding a sequence tending towards an endpoint a_3. Together with the infinitesimally thickened handles, a sequence of handles accumulating towards a_3 has been constructed. The initial angle θ_b can be selected to be a value bounded away from zero, and the distance between longitudes may decrease as $\frac{1}{\zeta(\beta)}\frac{1}{n^\beta},\ \beta > 1$. The set of endpoints would have cardinality $\aleph_0$.

Harmonic measures of subsets of $2^{\aleph_0}$ points in the real interval or circle may be evaluated by determining the rapidity with which the harmonic measure of each end, defined by the dependence $p_\nu = \frac{1}{1-e^{-2^\nu}}$, tends to zero as $\nu \to \infty$. □

Lemma 6.3. *The capacity of the ideal boundary of a Riemann surface with a Cantor set of ends will not vanish for unbounded* p_ν *if* $p_\nu \sim \frac{e^{2^\nu}}{\nu(ln\ \nu)^\alpha}$ *and* $\alpha \leq 1$.

Proof. Since the Robin constant r_n for the set $\cap E(p_1p_2...p_n)$ satisfies the inequalities [54]

$$\frac{log\ 4}{2^n} + log\left[\prod_{\nu=1}^{n}\left(1 - \frac{1}{p_\nu}\right)^{-2^{-\nu}}\right] + \left(1 - \frac{1}{2^n}\right) log\ 2 \tag{24}$$

$$\leq r_n \leq \frac{log\ 4}{2^n} + log\left[\prod_{\nu=1}^{n}\left(1 - \frac{1}{p_\nu}\right)^{-2^{-\nu}}\right] + \left(1 - \frac{1}{2^n}\right) log\ 2 + \sum_{\nu=1}^{n}\frac{log\ p_\nu}{2^\nu}$$

and the capacity of the Cantor set $\cap E(p_1p_2...)$ equals $\frac{1}{r_\infty}$, where $r_\infty = lim_{n\to\infty} r_n$.

The sum $\sum_{\nu=1}^{n}\frac{log\ p_\nu}{2^\nu}$ will diverge if $p_\nu \geq e^{\frac{2^\nu}{\nu^k}}$ where $k \leq 1$. Suppose $p_\nu \sim e^{\frac{2^\nu}{\nu ln\ \nu}}$. Then

$$\sum_{\nu=1}^{\infty}\frac{ln\ p_\nu}{2^\nu} = \sum_{\nu=1}^{\infty}\frac{1}{\nu\ ln\ \nu} \tag{25}$$

and the upper bound for r_∞ is infinite. For $p_\nu \sim e^{\frac{2^\nu}{\nu(ln\ \nu)^\alpha}}$,

$$\sum_{\nu=1}^{\infty} \frac{ln\ p_\nu}{2^\nu} = \sum_{\nu=1}^{\infty} \frac{1}{\nu\ (ln\ \nu)^\alpha} \tag{26}$$

and, since $\int_1^\infty \frac{dx}{x\ (ln\ x)^\alpha} = \int_0^\infty \frac{dw}{w} = w^{1-\alpha}\Big|_0^\infty$ where $w = ln\ x$, which is infinite for $\alpha < 1$, the sum again diverges.

The capacity of the ideal boundary β is

$$\begin{aligned} c_\beta &= e^{-k_\beta} \\ k_\beta &= \frac{1}{2\pi}\int_\beta s_\beta * ds_\beta \\ s_\beta &= lim_{n\to\infty} s_n \\ s_n &= -ln|z| + \varphi_n(z). \end{aligned} \tag{27}$$

where $\varphi_n(z)$ is a harmonic function in the subregion R_n of an end Ω. The choice of the sign of $s_n(z)$ in Eq.(6.13) is preferred, since, setting $s_n(z) = ln|z| + \varphi_n(z)$, it can tend to a constant as $z \to \infty$ only if $\varphi_n(z) \to -ln|z| + C_n$. However, this harmonic function is not allowed because the inequality $min_{z\in\beta_n}\varphi_n(z) \geq min_{z\in\beta_{n-1}}\varphi_n(z)$ does not hold.

For the set of ends given by the partition function, it would be

$$\begin{aligned} &\frac{1}{2\pi}\cdot 2\pi\left[exp\left(\frac{2^\nu}{\nu(ln\ \nu)^\alpha}\right) - 2^\nu\right] exp\left(-\frac{2^\nu}{\nu(ln\ \nu)^\alpha}\right) \sum_{\sharp \text{ of connected subintervals } \{i\}} \frac{s_{\beta_{\{i\}}}}{2^\nu} \\ &= \langle s_\beta\rangle \left[exp\left(\frac{2^\nu}{\nu(ln\ \nu)^\alpha}\right) - 2^\nu\right] exp\left(-\frac{2^\nu}{\nu(ln\ \nu)^\alpha}\right) \to \langle s_\beta\rangle \quad as\ \nu\to\infty \end{aligned} \tag{28}$$

with

$$\langle s_\beta\rangle = \frac{1}{2^\nu} \sum_{\sharp \text{ of connected subintervals } \{i\}} s_{\beta_{\{i\}}}. \tag{29}$$

The function s_β tends to a constant if $\varphi_n(z) \to ln|z| + C_n$ as $n \to \infty$ in the vicinity of β_n. If $\varphi_n(z)$ increased more rapidly than $ln\ |z|$, the identity

$$lim_{n\to\infty}\int_{\partial\Omega_1} s_n * ds_n = lim_{n\to\infty}\int_{\partial\Omega_2} \cap\beta_n s_n * ds_n + lim_{n\to\infty} D_{R_n}(s_n), \tag{30}$$

where R_n is a subregion of an end Ω, $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$, $\partial R_n = \partial\Omega_1 \cup \partial\Omega_2 \cap \beta_n$, with $lim_{n\to\infty}\beta_n = \beta$, while $D_{R_n}(s_n)$ is the Dirichlet norm of s_n in R_n, will not be satisfied because $\int_{\partial\Omega_1} s_n * ds_n$ is finite and $lim_{n\to\infty} D_{R_n}(s_n)$ is infinite. The function $\varphi_n(z)$ can be chosen to be sufficiently positive such that divergences do not cancel in the formula. It follows that $lim_{n\to\infty} s_n$ has a finite value on the ideal boundary, which would have a non-zero capacity.

When $\alpha > 1$, $\int_2^\infty \frac{dx}{x(ln\ x)^\alpha} = \frac{(ln\ 2)^{1-\alpha}}{\alpha-1}$ and the finiteness of r_∞ is determined by
$log\left[\prod_{\nu=1}^\infty \left(1-\frac{1}{p_\nu}\right)^{-2^{-\nu}}\right]$ when $1 \le B_1 < p_1 < B_2 < \infty$ and $p_\nu \sim e^{\frac{2^\nu}{\nu(ln\ \nu)^\alpha}}$ for $\nu \ge 2$. Then

$$\begin{aligned} log\left[\prod_{\nu=1}^\infty \left(1-\frac{1}{p_\nu}\right)^{-2^{-\nu}}\right] &= log\left[\left(1-\frac{1}{p_1}\right)^{-\frac{1}{2}}\right] + log\left[\prod_{\nu=2}^\infty \left(1-\frac{\nu(ln\ \nu)^\alpha}{2^\nu}\right)^{-2^{-\nu}}\right] \qquad (31) \\ &> -\frac{1}{2} log\left(1-\frac{1}{B_2}\right) - \sum_{\nu=2}^\infty \frac{1}{2^\nu} log\left(1-\frac{\nu(ln\ \nu)^\alpha}{2^\nu}\right) \\ &> -\frac{1}{2} log\left(1-\frac{1}{B_2}\right) + \sum_{\nu=2}^\infty \frac{1}{2^\nu}\left[\left(\frac{\nu(ln\ \nu)^\alpha}{2^\nu}\right) + \frac{1}{2}\left(\frac{\nu(ln\ \nu)^\alpha}{2^\nu}\right)^2 + ...\right] \\ &= -\frac{1}{2} log\left(1-\frac{1}{B_2}\right) + \sum_{\nu=2}^\infty \frac{\nu(ln\ \nu)^\alpha}{2^{2\nu}} + \sum_{\nu=2}^\infty \frac{\nu^2(ln\ \nu)^{2\alpha}}{2^{3\nu+1}} + ... \end{aligned}$$

which is finite. If $\alpha > 1$, the class of surfaces with the Cantor set of ends defined by $1 \le B_1 < p_1 < B_2 < \infty$ and $p_\nu \sim \frac{e^{2^\nu}}{\nu(ln\ \nu)^\alpha}$ will have non-zero capacity.

The deletion of central subinterval in the Cantor sets yields a continued branching of boundary components at a finite stage. Because the subintervals decrease quickly, the effect will be a surface with extremely thin ends. Nevertheless, by the bounds for $\langle p_\nu \rangle$, it would follow that $\langle p_\infty \rangle = 1$ for an O_G surface. The density of the branching decreases to zero, which consistent with a countable number of ends. The multiplicative factor at infinite genus therefore does not necessarily equal $lim_{g\to\infty} 2^g$ for the class of O_G surfaces. Nevertheless, the splitting of the strings introduces a compensating factor of 2^g at finite genus. There are non-perturbative corrections to this leading-order estimate of $\frac{4}{\pi^4}$, resulting in the shift $\kappa_{str} \to e^{A_0^{open-closed\ coupling}(open\ superstring)} \kappa_{str}$, where A_0 is the disk amplitude representing the coupling of the open superstring to the closed superstring, which yields the value $\kappa_{str}^{eff.} = \frac{1}{24.3497748}$ coinciding approximately with the extrapolated unified gauge coupling [18].

For other classes of Riemann surfaces, the ideal boundary will have non-zero capacity. Conditions for the surface not to belong to O_G are

$$\prod_{\nu=1}^{\infty}\left(1-\frac{1}{p_\nu}\right)^{2^{-\nu}} > 0 \qquad (32)$$
$$0 < \sum_{\nu=1}^{\infty}\frac{log\ p_\nu}{2^\nu} < \infty.$$

The product is

$$B\prod_{\nu=1}^{\infty}\left(1-\frac{1}{\langle p_\nu\rangle}\right)^{-2^{-\nu}} = BB'\left(1-\frac{1}{\langle p_\infty\rangle}\right)^{\sum_\nu 2^{-\nu}} = BB'\left(1-\frac{1}{\langle p_\infty\rangle}\right) \qquad (33)$$

where $B, B' > 0$, and for each $\epsilon > 0$, there exists an integer N such that $|\langle p_N\rangle - \langle p_\infty\rangle| < \epsilon$. Similarly,

$$\sum_{\nu=1}^{\infty}\frac{log\ p_\nu}{2^\nu} = C\sum_{\nu=1}^{\infty}\frac{log\langle p_\infty\rangle}{2^\nu} = C\ log\langle p_\infty\rangle \qquad C > 0.$$

Therefore, it is sufficient for $\langle p_\infty\rangle$ to be greater than 1 for a P_G surface. Given that p_ν never equals 1 at finite ν, the description in terms of the bifurcation of branches at each order would imply that there is no effective closure of the ends, or that the ends are not sufficiently fine, and additional final states can exist in the sum over surfaces. □

Each of the components of the compactification divisor contribute to the scattering amplitude, and the number is given by compositions of the genus [20] when the order of the degeneration limits is respected. The multiplicative factor for both the bosonic string and superstring could be affected by the genus dependence. The bosonic amplitudes add constructively, and an increase of the upper limit for the regularized integral of $(2g)!$ would yield a multiplicative factor off $\sum_{g_1+...+g_n=g}\frac{(2g_1)!...(2g_n)!}{(2g)!}$ if the integration in the neighbourhood of the compactification divisor required a product of integrals over the genus-g_k moduli spaces for $k = 1, ..., n$, which might be anticipated by the factorization of the partition function [31]. With the quotient by a factorial function of the genus for superstring amplitudes, the effect of the summation over these components may be bounded within integration over the fundamental domain of the supermodular group [21] under this condition. The neighbourhood of the compactification divisor, however, exists in the genus-g moduli space, and instead, it can be evaluated through an integration over the Teichmüller spaces at genus g_k followed by a quotient for each of the degeneration limits, the number of compositions would be sufficient for the multiplicative factor of 2^g at finite genus. The required coefficient in the limit of infinite genus also has been derived in Theorem 6.1 through the cardinality of the set of ends. The coincidence of the value of the effective string coupling and the unified gauge coupling reflects the symmetries of the standard model, which might be derived from a reduction of an exceptional group invariance of the intersection form of four-manifolds that are embedding spaces of infinite-genus surfaces. The theoretical prediction of the gauge group would represent a condition on the class of surfaces in the path integral for string theory.

Consider the resolution of rational double points representing ADE surface singularities [45] [55]. These rational singularities would be located generally in the interior of the surface. There exist transformations, however, which can map interior singularities to cusps of the surface [40] and the equivalence of the two methods would be established. When these rational singularities occur on an infinite-genus surface, the cardinality of the set that can be mapped to the cusps may be determined.

Theorem 6.2. *The number of ends of an infinite-genus surface, that may be conformally dilated to be nonsingular regions, is countably infinite. The reduction of the exceptional group invariance, required by the intersection form of an embedding four-manifold, to classical Lie groups will occur at the ends of O_G surfaces.*

Proof. The condition for the resolution of a rational singularity is the invariance of the genus of the surface. The resolution of the cusp singularity of a parabolic infinite-genus surface would not affect its infinite value. Even though this process does not change the infinite value assigned to the genus, it remains to be determined whether the genus of the singularity vanishes.

Suppose that V is a two-dimensional complex analytic manifold with an isolated singularity. Let $\mathcal{V}$ be the germ of this space, $\mathcal{S}$ be the germ of complex numbers at the origin, $\mathcal{W}$ be the three-dimensional complex analytic space and $f : \mathcal{W} \to \mathcal{S}$ be a flat map with $f^{-1}(s)$ be nonsingular for $s \neq 0$. The resolution of the singularity $\pi : \tilde{V} \to V$. Defining $V' = f^{-1}(\delta)D_\epsilon$, where D_ϵ is a disk of radius ϵ, the genus of the singularity V is

$$p = dim_C H^1(\tilde{V}, \mathcal{O}_{\tilde{V}}) = \frac{1}{2}(\sigma_0 + \sigma_+), \tag{34}$$

with σ_+ and σ_0 equal to the number of positive and zero eigenvalues of the intersection pairing of V' [27].

Parabolic translation surfaces of infinite genus are uniformized by Fuchsian groups of the first kind [56], with the connection points of two affine diffeomorphisms being rational points [32]. Since a rectifiable Jordan curve will encompass an infinite number of minimal surfaces [34], these local regions of zero mean curvature will exist on the infinite-genus surface can be chosen to be the initial location of the rational singularities. There may be a countably infinite number of geodesics along the ends of the surface.

The endpoint of a sequence of accumulating handles on a sphere may be conformally dilated without changing the genus. It is feasible then to map the rational singularities to the ends of the surface, in the neighbourhoods of the accumulation points. The irreducible components of the minimal resolution will have a negative-definite intersection form, and $\sigma_0 = \sigma_+ = 0$. With the genus of the singularity vanishing, the ADE surface singularities will be rational. These singularities will be included amongst the countably infinite number of ends of a sphere with infinite sequences of accumulating handles belonging to the class of O_G surfaces by Lemma 2. □

The gauge symmetry of the Lagrangian of a unified theory of the elementary particle interactions in the higher-dimensional ambient space has been postulated to be $G_2 \times SU(2) \times U(1)$ [19]. Given a gauge invariance in this ambient space, the matching with the groups A_n, D_n, E_6, E_7 or E_8, G_2 must be broken to a subgroup that belongs to one of these sequences. Therefore, G_2 is reduced to $SU(3)$ and the residual symmetry, $SU(3) \times SU(2) \times U(1)$ coincides with that of the bosonic sector of the minimal supersymmetric standard model that yields the unification of the gauge couplings.

7 Conclusion

The consistency of the perturbative expansion initially would require the domain of string perturbation theory to be identified with O_G rather than O_{HD} or O_{KD}, even though the latter satisfy the first and fifth criteria in the introduction repsectively. Amongst the surfaces not in O_G are the sets O_{HD}/O_G and the complement of O_{HD}. Although O_{HD} is a quasiconformally invariant class of Riemann surfaces larger than O_G, the counting of the surfaces and the cardinality of the set of ends could affect the term at infinite genus. A different cardinality could change this result even if it leaves the effective string coupling unaltered. The third condition then may be considered. The ideal boundary can have non-zero harmonic measure, and it would serve as a second source for the Laplace equation and an additional contribution to the Green function. However, the existence of the Green function would be sufficient to interpret the ideal boundary as a source of point-like particle or string states. Furthermore, the Hilbert spaces of a countable set of ends of a noncompact surface can be identified and produce tachyon condensation. By contrast with O_G, there would be a combination of point-particles sources and tachyon condensation at the boundaries of O_{HD}/O_G surfaces.

The complement to O_{HD} would include the surfaces that may be regarded as finite-action solutions to the Euclidean continuation of the equations derived from the string action. Amongst these surfaces is the class O^n_{HD} for $n \geq 2$. Although the second condition of quasiconformal invariance is valid also for O^n_{HD} surfaces, compatibility with a restriction of the class of functions in the amplitude, required for the expantion of the S-matrix, would have to be verified. The computability of n-point amplitudes corresponding to surfaces in O^n_{HD} is not immediate, since there are additional boundary terms in the bilinear relations beyond O_{HD}, which may affect the existence of the inverse of the imaginary part of the period matrix. The occurrence of a surface in O^n_{HD}, for some $n \geq 2$, which does not admit functions satisfying a finiteness of vanishing condition for an integral containing the Green function might prevent the use of the entire class O^n_{HD} for string amplitudes.

The nonperturbative effects in string theory are described currently through the addition of boundaries to the worldsheets representing coupling to open strings. The flux of energy through surfaces not in O_G, beyond the domain of string perturbation theory, could be considered in connection with interactions mediated by particles of non-zero momentum. The physical criteria, however, will be relevant for these classes of surfaces in the string theory formalism.

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