Ann. Acad. Rom. Sci.
Ser. Math. Appl.
ISSN 2066-6594 Vol. 10, No. 2/2018

NEWTON TYPE INTERVAL METHODS FOR SOLVING NONLINEAR SCALAR EQUATIONS*

Tonya P. Mateva†

Abstract

The article discusses the Newton's interval method and Ostrowski's interval method. Classical iterative schemes have been modified in intervals and experiments have been performed with the INTLAB program. The results of the proposed modifications are described and compared with Newton's interval method, Ostrowski's interval method and Ostrowski's modified interval method.

MSC: 65H04, 65H05

keywords: Newton's interval method, Ostrowski's interval method, INTLAB, MATLAB

1 Introduction

To solve nonlinear equations iterative schemes are used, the most famous of which is Newton's method. There are many modifications of the Newton's method, which improve the order of convergence [1, 2, 3, 4, 5, 10, 15, 16, 17]. In addition to the classic iteration schemes, interval methods are also used to

*Accepted for publication on March 5, 2018
†tonqmatewa@gmail.com College of Dobrich, Shumen University, Shumen, Bulgaria;
Paper written with financial support of "Information systems and mathematical models - an opportunity for improving the quality of teaching RD-08-107/06.02.2017, paragraph "Research" of Shumen University....

solve nonlinear equations. Interval analysis is formally presented by Moore [7]. In article [6], the authors describe iterative formulas proposed by Ostrowski [8] and based on the Newton's interval method and offer interval modifications of these iterative patterns. In this article, we discuss some modifications of the Newton's method and we present them in an interval form. With them, we make experiments with the computer program Matlab and the INTLAB toolbox to evaluate their performance. New interval schemes are compared to Newton's well-known interval scheme and the schemes described in an article [6]. We look at features reviewed in articles [6, 9] to assess the effectiveness of the modifications we offer. An interval number is a closed set in R that includes the possible range of an unknown real number, where R denotes the set of real numbers. Therefore, a real interval is a set of the form $x = [\underline{x}, \overline{x}]$, where $\underline{x}$ and $\overline{x}$ are the lower and upper bounds (end-points) of the interval number x, respectively. The set of compact real intervals is denoted by $IR = x = [\underline{x}, \overline{x}]\, |\underline{x}, \overline{x} \in R, \underline{x} \leq \overline{x}$. A real number x is identified with a point interval $x = [\underline{x}, \overline{x}]$ and is called degenerated interval. The quality of interval analysis is measured by the width of the interval results, and a sharp enclosure for the exact solution is desirable. The mid-point and the width of an interval x are denoted by $mid(x) = \frac{\underline{x}+\overline{x}}{2}$, and $wid(x) = \underline{x} - \overline{x}$, respectively.
Considering $|x| = \max\{|\underline{x}|, |\overline{x}|\}$ for any $x, y \in IR$ and $a, b \in R$ we can conclude that [7]:

$$wid(ax + by) = |a|wid(x) + |b|wid(y),$$

$$wid(xy) \leq |x|wid(y) + |y|wid(x).$$

Definition 1. *We say that f is an interval extension of f, if for degenerate interval arguments, f agrees with f, i.e.* $f([x, x]) = f(x)$.

Definition 2. *An interval extension f is said to be Lipschitz in* $x^{(0)}$ *if there is a constant L such that* $wid(f(x)) \leq L.wid(x)$ *for every* $x \subseteq x^{(0)}$.

Lemma 1. *[7] If f is a natural interval extension of a real rational function with* $f(x)$ *defined for* $x \subseteq x^{(0)}$*, where x and* $x^{(0)}$ *are intervals, then f is Lipschitz in* $x^{(0)}$*; in other words:* $wid(f(x)) \leq L.wid(x)$.

Definition 3. *An interval sequence* $\{x^{(k)}\}$ *is nested if* $x^{(k+1)} \subseteq x^{(k)}$ *for all k.*

Lemma 2. *([7]) Suppose* $\{x^{(k)}\}$ *is such that there is a real number* $x \in x^{(k)}$ *for all k. Define* $\{y^{(k)}\}$ *by* $y^{(1)} = x^{(1)}$ *and* $y^{(k+1)} = x^{(k+1)} \bigcap y^{(k)}$ *for all* $k = 1, 2, \ldots$ *. Then* $y^{(k)}$ *is nested with limit y, and* $x \in y \subseteq y^{(k)}$ *for all k.*

Lemma 3. *[7] Every nested sequence* $\{x^{(k)}\}$ *converges and has the limit* $\bigcap_{k=1}^{\infty} x^{(k)}$.

2 Background and methodology

2.1 Interval Newton's method

As a basis for our research we use the methods discussed in the article [6]. We look for a root of a real function $f(x)$. Suppose the first derivative of the function $f'(x)$ is continuous in the range $[a, b]$ and $f(a).f(b) < 0$. The most famous and preferred method for solving nonlinear equations is Newton's method. Its interval form is: If $x^{(k)}$ is the interval, where the root is located, then we can narrow the formula interval

$$x^{(k+1)} = x^{(k)} \cap N(x^{(k)}),\ k = 0, 1, 2, ... \tag{1}$$

$$N(x) = mid(x) - \frac{f(mid(x))}{f'(x)},$$

where $mid(x) = \frac{a+b}{2}$ - the middle of interval x. Thus, at each step we get a new, smaller interval and after n in number of steps, we reach the root of the function.
The next two theorems prove the convergence of the method.

Theorem 1. *[7] If an interval* $x^{(0)}$ *contains a zero* x^* *of* $f(x)$*, then so does* $x^{(k)}$ *for all* $k = 0, 1, 2, ...,$ *defined by* (1)*. Furthermore, the intervals* $x^{(k)}$ *form a nested sequence converging to* x^* *if* $0 \notin f'(x^{(0)})$.

Theorem 2. *[7] Given a real rational function f of a single real variable x with rational extensions* f, f' *of* f, f'*, respectively, such that f has a simple zero* x^* *in an interval* $x^{(0)}$ *for which* $f(x^{(0)})$ *is defined and* $f'(x^{(0)})$ *is defined and does not contain zero i.e.* $0 \notin f'(x^{(0)})$*. Then there is a positive real number C such that*

$$wid(x^{(k+1)}) \leq C.(wid(x^{(k)}))^2.$$

If $0 \notin f'(x^{(0)})$, then $0 \notin f'(x^{(k)})$ for all k and mid $(x^{(k)})$ is not contained in $N(x^{(k)})$, unless $f(mid(x^{(k)})) = 0$. So, convergence of the sequence follows [7, 11, 12, 14]. It should be noted that some special cases of (1) have been discussed in [6] in more details.

Ostrowski offers two classic methods for solving nonlinear equations that the authors of the article [6] modify as interval:

2.2 Ostrowski's method [6]

Classic Ostrowski's method:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ S(x_n, y_n) &= y_n - \frac{f(y_n)}{f(x_n) - 2f(y_n)} . \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= S(x_n, y_n). \end{aligned}$$

Interval Ostrowski's method:

We seek a solution of the equation $f(x) = 0$, on interval

$$x^{(k+1)} = x^{(k)} \cap S(x^{(k)}, y^{(k)}),\ k = 0, 1, 2, ..., \tag{2}$$

where

$$\begin{aligned} N(x) &= mid(x) - \frac{f(mid(x))}{f'(x)}, \\ y^{(k)} &= x^{(k)} \cap N(x^{(k)}), \\ \lambda &= \frac{f(mid(x))}{[f(mid(x)) - 2f(mid(y))] .f'(x)}, \\ S(x, y) &= mid(y) - \lambda .f(mid(y)). \end{aligned}$$

The following two theorems prove the conditions for the presence of a root in the selected initial interval and the convergence of the method.

Theorem 3. *Assume $f \in C(x^{(0)})$ and $0 \notin f'(x^{(k)})$ for $k = 0, 1, 2, ...$. If $x^{(0)}$ contains a root x^* of f , then so do all intervals $x^{(k)}$, $k = 1, 2, ...$ Besides, the intervals $x^{(k)}$ form a nested sequence converging to x^*.*

Theorem 4. *Suppose $f \in C(x^{(0)})$ and $0 \notin f'(x^{(k)})$ for $k = 0, 1, 2, ...$.*

1. *If $x^* \in x^{(0)}$ and $S(x^{(k)}, y^{(k)}) \subseteq x^{(k)}$, then $x^{(k)}$ contains exactly one zero of f .*
2. *If $x^{(k)} \cap S(x^{(k)}, y^{(k)}) = \emptyset$, then $x^{(k)}$ does not contain any zero of f .*

Theorems (3) and (4) are valid for all the methods discussed below. The proof can be seen in [10].

Theorem 5. *Assume $f \in C(x^{(0)})$ and $0 \notin f'(x^{(k)})$, and f has a unique simple root $x^* \in x^{(0)}$. Then, if $S(x^{(k)}, y^{(k)}) \subseteq x^{(k)}$, the sequence (2) has convergent rate four, i.e., there exists a constant K such that*

$$wid(x^{(k+1)}) \leq K.(wid(x^{(k)}))^4.$$

The next method is a modification of Ostrowski's method with a higher order of convergence. From the experiments in the article [6] and in the section 4 of this article, it is apparent that it is faster then Ostrowski's method and Newton's method.

2.3 Modified Ostrowski method

Classic modified Ostrowski's method [6]

$$
\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\
\lambda &= \frac{1}{f(x_n) - 2f(y_n)} . \frac{f(x_n)}{f'(x_n)}, \\
z_n &= y_n - \lambda f(y_n), \\
M(x_n, y_n, z_n) &= z_n - \lambda f(z_n), \\
x_{n+1} &= M(x_n, y_n, z_n).
\end{aligned}
$$

Interval modified Ostrowski method: We seek a solution of the equation $f(x) = 0$, on interval $x = [\underline{x}, \overline{x}]$.

$$x^{(k+1)} = x^{(k)} \cap M(x^{(k)}, y^{(k)}, z^{(k)}),\ k = 0, 1, 2, ..., \tag{3}$$

where

$$
\begin{aligned}
N(x) = mid(x) - \frac{f(mid(x))}{f'(x)}, \\
y^{(k)} = x^{(k)} \cap N(x^{(k)}), \\
\lambda = \frac{f(mid(x))}{\left[f(mid(x)) - 2f(mid(y))\right] .f'(x)}, \\
S(x, y) = mid(y) - \lambda.f(mid(y)), \\
z^{(k)} = x^{(k)} \cap S(x^{(k)}, y^{(k)}), \\
M(x, y, z) = mid(z) - \lambda.f(mid(z)).
\end{aligned}
$$

Theorem 6. *Assume $f \in C(x^{(0)})$ and $0 \notin f'(x^{(k)})$, and f has a unique simple root $x^* \in x^{(0)}$. Then, if $M(x^{(k)}, y^{(k)}, z^{(k)}) \subseteq x^{(k)}$, the sequence* (3) *has convergent rate 6, i.e., there exists a constant K such that*

$$wid(x^{(k+1)}) \leq K(wid(x^{(k)}))^6.$$

3 Main results

Interval approach can also be applied to methods that are a modification of the Newton's method. We look at four iterative schemes that we modify in the interval and with them we explore selected functions to find the number of iterations needed to find their roots at a pre-selected interval. In the classic form we select the initial approximations x_n, which satisfy the following conditions:
1. We explore the functions in the interval $[a, b]$, for which $f(a).f(b) < 0$, which ensure that there is a root in the corresponding interval.
2. We require in the corresponding interval, the first and the second derivative to be continuous for $\forall x \in [a, b]$: $f'(x) \neq 0$ and $f''(x) \neq 0$
3. For initial approximation we use this end of the interval for which

$$f(a).f''(a) > 0$$

In the interval form, the initial approximation $x^{(k)}$ is interval, in which f has a simple root x^* and $f'(x^{(k)})$ is defined and does not contain zero i.e $0 \notin f'(x^{(k)}$.

3.1 The method of Weerakoon and Fernando [5] in classic and interval form.

The iterative formula offered by Weerakoon and Fernando in [5] is a modification of the Newton's formula. They approximate the indefinite integral by a trapezoid instead of a rectangle, thus reducing the error in the approximation.

Classic form:

$$x_{n+1} = x_n - \frac{2.f(x_n)}{f'(x_n) + f'(x_n - \frac{f(x_n)}{f'(x_n)})}$$

A detailed description of this method is given in [5].

Interval form:

We seek a solution of the equation $f(x) = 0$, on interval $x = [\underline{x}, \overline{x}]$. Interval extension of the classic form of Weerakoon and Fernando's method is introduced as

$$x^{(k+1)} = x^{(k)} \cap S(x^{(k)}, y^{(k)}),\ k = 0, 1, 2, ..., \tag{4}$$

where

$$N(x) = mid(x) - \frac{f(mid(x))}{f'(x)},$$

$$y^{(k)} = x^{(k)} \cap N(x^{(k)}), \tag{5}$$

$$S(x,y) = mid(x) - \lambda . f(mid(x)), \tag{6}$$

$$\lambda = \frac{2}{f'(x) + f'(y)}.$$

The next theorem proves the convergence of the method:

Theorem 7. *Assume $f \in C(x^{(0)})$ and $0 \notin f'(x^{(k)})$, and f has a unique simple root $x^* \in x^{(0)}$. Then, if $S(x^{(k)}, y^{(k)}) \subseteq x^{(k)}$, the sequence (4) has convergent rate 3, i.e., there exists a constant K such that*

$$wid(x^{(k+1)}) \leq K. \left[\left(wid(x^{(k-1)}) \right)^2 + \left(wid(x^{(k)}) \right)^2 \right] .(wid(x^{(k)})$$

Proof:

By Mean Value Theorem we have

$$f(mid(x^{(k)})) = f'(\xi) \left[mid(x^{(k)}) - x^* \right],$$

where ξ is between $mid(x^{(k)})$ and x^*. Since $S(x^{(k)}, y^{(k)}) \subset x^{(k)}$, thus from (4) and (6) we have

$$x^{(k+1)} = mid(x^{(k)}) - \lambda \left[mid(x^{(k)}) - x^* \right] .f'(\xi), \tag{7}$$

where

$$\lambda = \frac{2}{f'(x^{(k)}) + f'(y^{(k)})}. \tag{8}$$

From (7) we have

$$wid(x^{(k+1)}) = wid(\lambda). \left| mid(x^{(k)}) - x^* \right| . \left| f'(\xi) \right| . \tag{9}$$

Moreover (8) gives

$$wid(\lambda) = wid \left(\frac{2}{f'(x^{(k)}) + f'(y^{(k)})} \right).$$

From Lemma 1 we have

$$wid \left(\frac{2}{f'(x^{(k)}) + f'(y^{(k)})} \right) \leq wid(x^{(k)} + y^{(k)}),$$

$$wid(y^{(k)}) \leq wid(x^{(k)})^2,$$

$$wid(x^{(k)}) \leq \left(wid(x^{(k-1)})\right)^2,$$

$$wid(x^{(k)} + y^{(k)}) \leq \left(wid(x^{(k-1)})\right)^2 + \left(wid(x^{(k)})\right)^2.$$

Furthemore, since $y^{(k)}$ is generated from (5), let

$$\left|f'(\xi)\right| \leq K.$$

And consider (9) and (3.1) we have the following error bound

$$wid(x^{(k+1)}) \leq K.\left[\left(wid(x^{(k-1)})\right)^2 + \left(wid(x^{(k)})\right)^2\right].wid(x^{(k)})$$

Remark 1. By our reasoning, we assume, that

$$wid(x^{(k)} + y^{(k)}) \leq \left(wid(x^{(k-1)})\right)^2 + \left(wid(x^{(k)})\right)^2 \leq \left(wid(x^{(k)})\right)^2,$$

and the local order of convergence of the interval Weerakoon and Fernando's method is 3, but we have no theoretical proof of that.

3.2 The method of Frontini and Sormani/ middle point/ [1] in classic and interval form.

This method is a modification of Newton's method, obtained with a different quadratic formula. The method is with order of convergence 3.
Classic form:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n - \frac{f(x_n)}{2.f'(x_n)})}$$

A detailed description of this method is given in [1].
Interval form:

We seek a solution of the equation $f(x) = 0$, on interval $x = [\underline{x}, \overline{x}]$. Interval extension of the classic form of Frontini and Sormani's method is introduced as

$$x^{(k+1)} = x^{(k)} \cap S(x^{(k)}, y^{(k)}),\ k = 0, 1, 2, ..., \tag{10}$$

where

$$N(x) = mid(x) - \frac{f(mid(x))}{2.f'(x)},$$

$$y^{(k)} = x^{(k)} \cap N(x^{(k)}), \tag{11}$$

$$S(x,y) = mid(x) - \lambda.f(mid(x)), \tag{12}$$
$$\lambda = \frac{1}{f'(y)}.$$

The next theorem proves the convergence of the method:

Theorem 8. *Assume $f \in C(x^{(0)})$ and $0 \notin f'(x^{(k)})$, and f has a unique simple root $x^* \in x^{(0)}$. Then, if $S(x^{(k)}, y^{(k)}) \subseteq x^{(k)}$, the sequence* (10) *has convergent rate 3, i.e., there exists a constant K such that*

$$wid(x^{(k+1)}) \leq K(wid(x^{(k)}))^3.$$

Proof:

By Mean Value Theorem, we have

$$f(mid(x^{(k)})) = f'(\xi)\left[mid(x^{(k)}) - x^*\right],$$

where ξ is between $mid(x^{(k)})$ and x^*. Since $S(x^{(k)}, y^{(k)}) \subset x^{(k)}$, thus from (10) and (12) wee have

$$x^{(k+1)} = mid(x^{(k)}) - \lambda\left[mid(x^{(k)}) - x^*\right].f'(\xi), \tag{13}$$

where

$$\lambda = \frac{1}{f'(y^{(k)})}. \tag{14}$$

From (13) we have

$$wid(x^{(k+1)}) = wid(\lambda).\left|mid(x^{(k)}) - x^*\right|.\left|f'(\xi)\right| \tag{15}$$

Moreover (14) gives

$$wid(\lambda) = wid\left(\frac{1}{f'(y^{(k)})}\right).$$

From Lemma 1 we have

$$wid\left(\frac{1}{f'(y^{(k)})}\right) \leq wid(x^{(k)})^2. \tag{16}$$

Furthemore, since $y^{(k)}$ is generated from (11).

Let

$$\left|f'(\xi)\right| \leq K$$

and consider (15) and (16) we have the following error bound

$$wid(x^{(k+1)}) \leq K.wid(x^{(k)}).\left(wid(x^{(k)})\right)^2 = K.\left(wid(x^{(k)})\right)^3.$$

So the local order of convergence of the interval Frontini and Sormani's method is 3 and the proof is completed.

3.3 Homeier's method [2, 3] in classic and interval form.

Another approach is used by Homeier [2, 3], who use the inverse function $x = f(y)$ instead of $y = f(x)$ from Newtons theorem suggests modification order of convergence 3.

Classic form:

$$x_{n+1} = x_n - \frac{f(x_n)}{2}.\left(\frac{1}{f'(x_n)} + \frac{1}{f'(x_n - \frac{f(x_n)}{f'(x_n)})}\right)$$

A detailed description of this method is given in [2, 3].

Interval form:

We seek a solution of the equation $f(x) = 0$, on interval $x = [\underline{x}, \overline{x}]$. Interval extension of the classic form of Homeier's method is introduced as

$$x^{(k+1)} = x^{(k)} \cap S(x^{(k)}, y^{(k)}),\ k = 0, 1, 2, ..., \tag{17}$$

where

$$N(x) = mid(x) - \frac{f(mid(x))}{f'(x)},$$

$$y^{(k)} = x^{(k)} \cap N(x^{(k)}), \tag{18}$$

$$S(x, y) = mid(x) - \lambda.f(mid(x)), \tag{19}$$

$$\lambda = \frac{1}{2}.\left(\frac{1}{f'(x)} + \frac{1}{f'(y)}\right).$$

The next theorem proves the convergence of the method:

Theorem 9. *Assume $f \in C(x^{(0)})$ and $0 \notin f'(x^{(k)})$, and f has a unique simple root $x^* \in x^{(0)}$. Then, if $S(x^{(k)}, y^{(k)}) \subseteq x^{(k)}$, the sequence* (17) *has convergent rate 3, i.e., there exists a constant K such that*

$$wid(x^{(k+1)}) \leq K(wid(x^{(k)}))^3.$$

Proof:

By Mean Value Theorem we have

$$f(mid(x^{(k)})) = f'(\xi)\left[mid(x^{(k)}) - x^*\right],$$

where ξ is between $mid(x^{(k)})$ and x^*. Since $S(x^{(k)}, y^{(k)}) \subset x^{(k)}$, thus from (17) and (19) wee have

$$x^{(k+1)} = mid(x^{(k)}) - \lambda\left[mid(x^{(k)}) - x^*\right].f'(\xi), \tag{20}$$

where

$$\lambda = \frac{1}{2.(f'(x^{(k)}) + f'(y^{(k)}))}. \tag{21}$$

From (20) we have

$$wid(x^{(k+1)}) = wid(\lambda). \left| mid(y^{(k)}) - x^* \right| . \left| f'(\xi) \right| . \tag{22}$$

Moreover (21) gives

$$wid(\lambda) = wid\left(\frac{1}{2.(f'(x^{(k)}) + f'(y^{(k)})})\right).$$

From Lemma 1 we have

$$wid\left(\frac{1}{f'(y^{(k)})}\right) \leq wid(y^{(k)}) \leq wid(x^{(k)})^2$$

and

$$wid\left(\frac{1}{f'(x^{(k)})}\right) \leq wid(x^{(k)} \leq wid(x^{(k)})^2.$$

Then

$$wid(\lambda) = wid\left(\frac{1}{2.(f'(x^{(k)}) + f'(y^{(k)})}\right) \leq wid(x^{(k)})^2. \tag{23}$$

Furthemore, since $y^{(k)}$ is generated from (18).
Let

$$\left| f'(\xi) \right| \leq K$$

and consider (22) and (23) we have the following error bound

$$wid(x^{(k+1)}) \leq K.wid(x^{(k)}). \left(wid(x^{(k)})\right)^2 = K. \left(wid(x^{(k)})\right)^3.$$

So the local order of convergence of the interval Homeier's method is 3 and the proof is completed.

3.4 Kou's method [4] in classic and interval form.

This method is modification of Weerakoon and Fernando's method approach by using Newton's theorem for the function on a new interval of integration. The method is with cubically convergent. The important characteristic of

the new method is that per iteration it requires two evaluations of the function f, one of the first derivative and no evaluations of the second derivative.
Classic form:

$$x_{n+1} = y_n - \frac{f(y_n)}{f'(x_n)},$$

$$y_n = x_n + \frac{f(x_n)}{f'(x_n)}.$$

A detailed description of this method is given in [4].
Interval form:

We seek a solution of the equation $f(x) = 0$, on interval $x = [\underline{x}, \overline{x}]$. Interval extension of the classic form of Kou's method is introduced as

$$x^{(k+1)} = x^{(k)} \cap S(x^{(k)}, y^{(k)}),\ k = 0, 1, 2, ..., \tag{24}$$

where

$$N(x) = mid(x) - \frac{f(mid(x))}{f'(x)},$$

$$y^{(k)} = x^{(k)} \cap N(x^{(k)}), \tag{25}$$

$$S(x, y) = mid(y) - \lambda.f(mid(y)), \tag{26}$$

$$\lambda = \frac{1}{f'(x)}.$$

The next theorem proves the convergence of the method:

Theorem 10. *Assume $f \in C(x^{(0)})$ and $0 \notin f'(x^{(k)})$, and f has a unique simple root $x^* \in x^{(0)}$. Then, if $S(x^{(k)}, y^{(k)}) \subseteq x^{(k)}$, the sequence (24) has convergent rate 3, i.e., there exists a constant K such that*

$$wid(x^{(k+1)}) \leq K(wid(x^{(k)}))^3.$$

Proof:
By Mean Value Theorem we have

$$f(mid(x^{(k)})) = f'(\xi)\left[x^* - mid(x^{(k)})\right],$$

where ξ is between $mid(x^{(k)})$ and x^*. Since $S(x^{(k)}, y^{(k)}) \subset x^{(k)}$, thus from (24) and (26)

wee have

$$x^{(k+1)} = mid(y^{(k)}) - \lambda\left[x^* - mid(y^{(k)})\right].f'(\xi), \tag{27}$$

where

$$\lambda = \frac{1}{f'(x^{(k)})}. \tag{28}$$

From (27) we have

$$wid(x^{(k+1)}) = wid(\lambda). \left| x^* - mid(y^{(k)}) \right| . \left| f'(\xi) \right| \tag{29}$$

Moreover (28) gives

$$wid(\lambda) = wid \left(\frac{1}{f'(x^{(k)})} \right).$$

From Lemma 1 we have

$$wid \left(\frac{1}{f'(x^{(k)})} \right) \leq wid(x^{(k)}). \tag{30}$$

Furthemore, since $y^{(k)}$ is generated from (25) Theorem 2 leads to

$$\left| x^* - mid(y^{(k)}) \right| \leq wid(y^{(k)}) \leq \left(wid(x^{(k)}) \right)^2. \tag{31}$$

Let

$$\left| f'(\xi) \right| \leq K$$

and consider (29), (30) and (31) we have the following error bound

$$wid(x^{(k+1)}) \leq K.wid(x^{(k)}). \left(wid(x^{(k)}) \right)^2 = K. \left(wid(x^{(k)}) \right)^3.$$

So the local order of convergence of the Kou's method is 3 and the proof is completed.

4 Numerical experiments

All calculations were made using a computer program MATLAB7.6.0(R2008a) and INTLAB toolbox – Application for interval analysis, created by Rump [13]. The desired accuracy is $tol = 1e - 15$. We use the stop criterion $rad(X) = \frac{1}{2}.(b - a) < tol$. We select initial interval $X = [a, b]$, which satisfies the following conditions: 1. $f(a).f(b) < 0$ and 2. $f'(x)$ is continuous in the range $[a, b]$. The column I contains the function, the interval in which we examine it, and the root in that interval. Column II contains the methods, which we compare. Columns III and V contain the intervals, in which we examine the function, and columns IV and VI contain the corresponding number of iterations.

Table 1: Results

I	II	III	IV	V	VI
$f_1(x) = x(x^9 - 1) - 1$	Newton		6		10
	Ostrowski	[1,1.5]	4	[0.8,5.5]	Nan
Root: $x_n = 1.0757661$	Ostr- modif		3		6
	WF		5		9
Space containing the root:	FS		4		7
	Homeier		5		9
$(0.8, \infty)$	Kou		8		16
$f_2(x) = x^2 - \exp(x) - 3x + 2$	Newton		5		4
	Ostrowski	[0,1]	3	[-1,1.5]	3
Root: $x_n = 0.257530$	Ostr- modif		2		3
	WF		4		4
Space containing the root:	FS		3		3
	Homeier		4		4
$(-\infty, +\infty)$	Kou		5		5
$f_3(x) = \exp(-x) + \cos x$	Newton		4		5
	Ostrowski	[1,2]	3	[0.5,2.5]	3
Root: $x_n = 1.7461395$	Ostr- modif		2		3
	WF		4		4
Space containing the root:	FS		3		3
	Homeier		4		4
$(-\infty, 3.1)$	Kou		3		5

Table 2: Results

I	II	III	IV	V	VI
$f_4 = \exp x - 4x^2$	Newton		7		8
	Ostrowski	[1,1.5]	Nan	[0.8,5.5]	Nan
Root: $x_n = 4.306584$	Ostr- modif		Nan		Nan
	WF		6		6
Space containing the root:	FS		4		5
	Homeier		4		6
$(3.3, +\infty)$	Kou		7		7
$f_5 = x^2 - \exp x - 3x + 2$	Newton		4		5
	Ostrowski	[0,0.5]	2	[0,1]	3
Root: $x_n = 0.257530$	Ostr- modif		2		Nan
	WF		3		4
Space containing the root:	FS		2		3
	Homeier		3		4
$(-\infty, +\infty)$	Kou		4		5

Table 3: Results

I	II	III	IV	V	VI
$f_6 = \exp -x + \cos x$	Newton		3		4
	Ostrowski	[1,2]	3	[0.8,2.8]	3
Root: $x_n = 1.746139$	Ostr- modif		1		3
	WF		4		4
Space containing the root:	FS		3		3
	Homeier		4		4
$(-\infty, 3)$	Kou		3		5
$f_7 = x^2 - 3$	Newton		5		5
	Ostrowski	[1,2]	3	[1.1,3]	3
Root: $x_n = 1.404491$	Ostr- modif		1		Nan
	WF		4		4
Space containing the root:	FS		3		3
	Homeier		4		4
$(-\infty, 3)$	Kou		5		6
$f_8(x) = (x+2) * \exp x - 1$	Newton		5		7
	Ostrowski	[-1,0]	3	[-2,5]	Nan
Root: $x_n = -0.442854$	Ostr- modif		Nan		5
	WF		3		6
Space containing the root:	FS		3		5
	Homeier		3		5
$(-2.8, +\infty)$	Kou		5		10
$f_9(x) = x^5 + x^4 + 4 * x^2 - 15$	Newton		6		7
	Ostrowski	[1,2]	3	[0.2,4]	5
Root: $x_n = 1.347428$	Ostr- modif		1		Nan
	WF		5		6
Space containing the root:	FS		3		5
	Homeier		5		7
$(0.1, \infty)$	Kou		6		8
$f_{10}(x) = \cos x - x$	Newton		4		5
	Ostrowski	[0,1]	3	[-1,2]	4
Root: $x_n = 0.739085$	Ostr- modif		1		3
	WF		4		4
Space containing the root:	FS		3		3
	Homeier		4		5
$(-1.5, 4.6)$	Kou		5		6

In article [6] the authors examine the Ostrowski's interval method and Ostrowski's modified interval method. Examined functions f_1, f_2 f_3 are described below in Table 1 and the results compared with the Newton's interval method. The conclusions they have made are that Ostrowski's interval modified method makes the least number of iterations. The functions are examined with the column I. We complement their research with the methods we propose and expand the range, in which we seek root. Indeed, in the first reviewed interval (III), there is the smallest number of iterations. Extending the interval - column (V), it can be seen that the Frontini and Sormani's method makes the same number of iterations as interval modification of Ostrowski. Kou has the highest number of iterations.

5 Concluding remarks

In the article, we propose an interval form of some known formulas for solving nonlinear equations with order of convergence 3. The proposed methods are compared with the known formulas of Newton and Ostrowski. From the research, we can conclude that for a small interval, Ostrowski's modified method makes the least number of iterations compared to all the methods discussed in the article. For a larger interval, Frontini and Sormani's method is the fastest method, and the largest number of iterations are made by Kou's method. We introduced features for which Ostrowski's modified method does not reach the root by doing more than 100 iterations.

References

[1] M. Frontini, E. Sormani, Third-order methods from quadrature formula for solving systems of nonlinear equations. *Appl. Math. Comput.* 149:771-782, 2004

[2] H.H.H. Homeier, On Newton-type methods with cubic convergence, J. *Comput. Appl. Math.* 176:425-432, 2005.

[3] H.H.H. Homeier, A modifed Newton's method with cubic convergence: the multivariate case, J. *Comput. Appl. Math.* 169: 161–169, 2004

[4] J. Kou, Y. Li, X. Wang, A modifcation of Newton's method with third-order convergence. *Appl. Math. Comput.* 181(2):1106–1111, 2006.

[5] S. Weerakoon, G.I. Fernando, A variant of Newton's method with accelerated third-order convergence. *Appl. Math. Lett.* 13:87–93, 2000.

[6] P. Bakhtiari, T. Lotfi, K. Mahdiani and Fazlollah Soleymani. Interval Ostrowski-type methods with quaranteed convergence. *Ann Univ. Ferrara* 59(2):221–234, 2013.

[7] R. E. Moore, R.B Kearfott. *Introduction to Interval Analysis* SIAM, Philadelphia, 2009.

[8] M. Grau, J. L Diaz-Barrero, An improvement to Ostrowski root–finding method. *Appl.Math.Comput.* 173:450–456, 2006.

[9] S. Singh, D. K. Gupta, Higner order interval iterative methods for nonlinear equations. *J.Appl. Math. and Informatics* 33: 61–76, 2015.

[10] T. Lotfi, P. Bakhtiari, K. Mahdiani, M. Salimi, A New Verified Method for Solving Nonlinear Equations. *World Academy of Science, Engineering and Technology* 6:50–53, 2012.

[11] G. Alefeld, G. Mayer, Interval analysis: theory and applications. *J. Comput. Appl. Math.* 121:421–464, 2000.

[12] G. Alefeld, J.Herzberger. *Introduction to Interval Computations.* Academic Press, New York, 1983.

[13] S. Rump. INTLAB–INTerval LABoratory. In:Csendes, T. (ed.) *Developments in Reliable Computing. Kluwer Academic Publishers, Dordecht*, 77–104, 1999.

[14] A. Neumaier. *Interval Methods for Systems of Equations.*, Cambridge University Press, Cambridge, 1990.

[15] J. F. Traub.*Iterative Methods for the Solution of Equations*, Prentice-Hall, Engelwood Cliffs, NJ, 1964.

[16] M. S. Petkovic, B. Neta, L. Petkovic,J. Dzunic. *Multipoint Methods for Solving Nonlinear Equations*, Academic Press. Elsevier, 2013.

[17] M S. Petkovic. *Iterative Methods for Simultaneous Inclusion of Polynomial Zeros*, Springer, 2014.