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ON THE MEAN SQUARE MINIMIZATION OF THE FINAL VALUE OF AN OUTPUT OF A LINEAR STOCHASTIC CONTROLLED SYSTEM*

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Abstract

In this work we deal with the problem of minimization of the mean square of the final value of an output of a controlled stochastic system having the state space representation described by a system of Itô differential equations. We assume that the system is controlled by piecewise constant controls which are non-anticipative with respect to the Brownian motion involved in the mathematical model of the controlled system. We provide explicit formulae of the optimal controls and we show that these are in a state feedback form. For the implementation of these optimal controls we need only the measured values of the states at discrete time instances. The gain matrices of the optimal control are computed based on the solution of a matrix backward differential equation with finite jumps of Riccati type. We also analyze the dependence of the value of the optimal performance with respect to the length of the sampling period.

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1 Introduction

The stochastic linear quadratic optimal control problem is one of the most important optimal control problems and has been playing a central role in modern control theory. Moreover, the sampled-data systems have scored a great success in the past decades. In [6, 7], the authors considered H_2 and LQ robust sampled-data control problems under a unified framework. The problems of stochastic stability and robust control for a class of uncertain sampled-data systems with random jumping parameters described by finite state semi-Markov process are studied in [5], where the design procedure for robust multirate sampled-data control is formulated as linear matrix inequalities.

In this paper, we consider a stochastic system where the state space representation is described by a system of Itô differential equations. We analyze this system under a new class of controls consisting of the piecewise constant stochastic processes. For the implementation of these optimal controls we need only the measured values of the states at discrete time instances. The gain matrices of the optimal control are computed based on the solution of a matrix backward differential equation with finite jumps of Riccati type. We also analyze the dependence of the value of the optimal performance with respect to the length of the sampling period.

2 The problem formulation

2.1 A short discussion on the minimization of the final value of an output

Let us consider the controlled system with the state space representation by:

$$\begin{aligned} dx(t) &= [A_0\, x(t) + B_0\, u(t)]\, dt + [A_1\, x(t) + B_1\, u(t)]\, dw(t) \\ x(0) &= x_0 \\ z(t) &= C\, x(t)\,, \end{aligned} \tag{1}$$

$t \geq 0$, where $z(t) \in \mathbb{R}^{nz}$ is the controlled output, $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ contains the control parameters at instant time t; $\{w(t)\}_{t\geq 0}$ is an one dimensional standard Wiener process (Brownian motion) on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, i.e. $w(0) = 0$ and for $t > 0$, $\mathbb{E}[w(t)] = 0$ and $\mathbb{E}[(w(t) - w(0))^2] = t - s$ for all $t \geq s \geq 0$.

In (1), $(A_k, B_k) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}, k = 0, 1$ are constant matrices. Throughout this work $\mathbb{E}[.]$ stands for the mathematical expectation.

The admissible initial state x_0 is an n-dimensional random vector with the mean $\mathbb{E}[x_0] = \bar{x}_0$ and covariance $\mathbb{E}[x_0 x_0^T] = X_0$. We assume also that the initial state x_0 is independent of the σ-algebras $\mathcal{F}_t, t \geq 0$, where $\mathcal{F}_t = \sigma[w(s), 0 \leq s \leq t]$. We set $\tilde{\mathcal{F}}_t = \sigma[x_0, w(s), 0 \leq s \leq t]$. First we consider a class of the admissible controls $\mathcal{U}_{adm}$ that consists of all stochastic processes $u : [0, \tau] \times \Omega \to \mathbb{R}^m$ with the property that for each $t \in [0, \tau], u(t)$ is $\tilde{\mathcal{F}}_t$-measurable and $\mathbb{E}[|u(t)|^2] < \infty$. Here $\tau > 0$ is fixed.

The optimization problem which we want to solve ask for the minimization of the cost functional $\mathcal{J}(u) := \mathbb{E}[|Cx_u(\tau)|^2]$ over the class of the admissible controls $\mathcal{U}_{adm}$. Here $x_u(.)$ is the solution of the system (1) determined by the input $x_u(.) \in \mathcal{U}_{adm}$.

Let $R(.)$ be the solution of the problem with given final value described by the following Riccati type differential equation:

$$\begin{aligned} -\dot{R}(t) &= A_0^T R(t) + R(t) A_0 + A_1^T R(t) A_1 \\ &\qquad -[R(t)B_0 + A_1^T R(t) B_1]\,[B_1^T R(t) B_1]^\dagger\,[B_0^T R(t) + B_1^T R(t) A_1] \\ R(\tau) &= C^T C\,. \end{aligned} \tag{2}$$

As usually the superscript $\dagger$ denotes the pseudo-inverse of a matrix, [9].

Let us assume that the problem with given final values (2) has a positive semidefinite solution $R(t)$ satisfying the additional condition:

$$[I_m - B_1^T R(t) B_1 (B_1^T R(t) B_1)^\dagger][B_0^T R(t) + B_1^T R(t) A_1] = 0,\ t \in [0, \tau]\,. \tag{3}$$

Applying Itô formula to the function $x^T R(t) x$ one obtains that if (3) is satisfied, then the control

$$\tilde{u}(t) = -(B_1^T R(t) B_1)^\dagger [B_0^T R(t) + B_1^T R(t) A_1] x(t)$$

achieves the minimal value of the performance criterion $\mathcal{J}(u)$ with respect to the class of admissible controls $\mathcal{U}_{adm}$. So, the optimal control problem asking for the minimization of the final value of the mean square of the output $z(t)$ in the class of admissible controls $\mathcal{U}_{adm}$ may be solved under the restrictive condition (3) which have to be satisfied by the solution of the Riccati equation (2).

In this work, we show that a change of the class of admissible controls allows us to solve the problem of minimization of the final value of the output $z(t)$ without any additional condition like (3). The new class of admissible controls will be described in the next subsection.

2.2 The minimization of the final value of an output by piecewise constant controls

Now we consider a new class of admissible controls $\mathcal{U}_{adm}$ consisting of the piecewise constant stochastic processes. More precisely, the set $\mathcal{U}_{adm}$ contains all stochastic processes $u : [0, \tau] \times \Omega \to \mathbb{R}^m$ with the property:

$$u(t) = u_k\,, \quad kh \leq t < (k+1)h\,, \quad k = 0, 1, \ldots, N-1\,, \tag{4}$$

where u_k are m-dimensional random vectors, which are $\tilde{\mathcal{F}}_{kh}$-measurable and $\mathbb{E}[|u_k|^2] < \infty$.

In (4), $h > 0$ and $b > 0$are such that $b = Nh, N \geq 1$ being a natural number. We consider the performance criterion:

$$J(u) = \mathbb{E}[|z_u(Nh)|^2]\,, \tag{5}$$

$z_u(t) = C\,x_u(t)$, where, as before $x_u(t)$ is the solution of the problem with given initial value (1) determined by the control u. The optimal control problem, which we want to solve, can be stated as follows:

OPTIMIZATION PROBLEM:

For a given admissible initial state x_0 find an admissible control u_{opt} with the property that

$$J(u_{opt}) = \min_{u \in \mathcal{U}_{adm}} J(u)\,. \tag{6}$$

2.3 An auxiliary control problem

In this work, we shall derive explicit functional of the optimal control u_{opt}. To this end we shall transform the optimization problem obtained above (in (6)) into a new optimal control problem described by a controlled system with finite jumps. Plugging (4) in (1) one obtains:

$$\begin{aligned} &dx(t) = [A_0\,x(t) + B_0\,u_k]\,dt + [A_1\,x(t) + B_1\,u_k]\,dw(t) \\ &kh \leq t \leq (k+1)h, \quad k = 0, 1, \ldots, N-1 \\ &x(0) = x_0\,. \end{aligned} \tag{7}$$

This system can be written in the form of a system of stochastic differential equations with finite jumps:

$$\begin{aligned} &d\xi(t) = \mathcal{A}_0\,\xi(t)\,dt + \mathcal{A}_1\,\xi(t)\,dw(t) \\ &kh \leq t \leq (k+1)h, \\ &\xi(kh^+) = \mathcal{A}_d\xi(kh) + \mathcal{B}_d u_k\,, \quad k = 0, 1, \ldots, N-1 \\ &\xi(0) = \xi_0 = \begin{pmatrix} x_0^T & u_0^T \end{pmatrix}^T\,, \end{aligned} \tag{8}$$

where $\xi(t) = \left(x^T(t) \quad u^T(t)\right)^T \in \mathbb{R}^{n+m}$,

$$\begin{aligned}
\mathcal{A}_k &= \begin{pmatrix} A_k & B_k \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{(n+m)\times(n+m)}, \; k = 0, 1, \\
\mathcal{A}_d &= \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{(n+m)\times(n+m)}, \\
\mathcal{B}_d &= \begin{pmatrix} 0 \\ I_m \end{pmatrix} \in \mathbb{R}^{(n+m)\times(m)}.
\end{aligned} \tag{9}$$

The class of $\mathcal{U}^d_{adm}$ of admissible controls consists of all finite sequences of m-dimensional random vectors $\mathbf{u} = \{u_0, u_1, \ldots, u_{N-1}\}$, where u_k is $\tilde{\mathcal{F}}_{kh}$-measurable and and $\mathbb{E}[|u_k|^2] < \infty$. Employing (4) we obtain that there exists a one to one correspondence between $\mathcal{U}_{adm}$ and $\mathcal{U}^d_{adm}$. Applying Theorem 5.2.1 from [8] on each interval $[kh, (k+1)h]$, we deduce that for each $\mathbf{u} \in \mathcal{U}^d_{adm}$ the stochastic differential equation with finite jumps (8) has a unique solution $\xi_{\mathbf{u}}(.)$ with the properties:

- $\xi_{\mathbf{u}}(t)$ is $\tilde{\mathcal{F}}_t$-measurable for every $t \in [0, Nh]$,
- $\xi_{\mathbf{u}}(t)$ is left continuous for every $t \in [0, Nh]$,
- $\xi_{\mathbf{u}}(t)$ satisfies the initial condition $\xi_{\mathbf{u}}(0) = \xi_0$.

We associate the performance criterion

$$J_d(\mathbf{u}) = \mathbb{E}[|\mathcal{C}\,\xi_{\mathbf{u}}(Nh)|^2], \tag{10}$$

where $\mathcal{C} = (C \quad 0) \in \mathbb{R}^{nz\times(n+m)}$. One obtains that

$$\inf_{u\in\mathcal{U}_{adm}} J(u) = \inf_{\mathbf{u}\in\mathcal{U}^d_{adm}} J_d(\mathbf{u}).$$

Hence, in order to find the control u_{opt} that solves (6) it is sufficient to find $\mathbf{u} \in \mathcal{U}^d_{adm}$ that satisfies

$$J_d(\tilde{\mathbf{u}}) = min_{\mathbf{u}\in\mathcal{U}^d_{adm}} J_d(\mathbf{u}), , \tag{11}$$

The solution of the optimization problem (11) will be derived in the next section.

3 The solution of the auxiliary optimal control problem

In the developments in this section we need the following extended version of the Schur Lemma.

Lemma 1 *[generalized Schur Lemma [1, 10]] Let (U, V, W) be the given matrices with real entries with compatible dimension such that $U = U^T, W = W^T$. Then the following are equivalent:*

(i) $\begin{pmatrix} U & V \\ V^T & W \end{pmatrix} \geq 0$,

(ii) $W \geq 0\,, U - VW^{\dagger}V^T \geq 0$, *and* $(I - WW^{\dagger})V^T = 0\,.$

Let us consider the backward differential equation with finite jumps of Riccati type on the space $\mathcal{S}_{n+m}$:

$$\begin{aligned} -\dot{P}(t) = &\ \mathcal{A}_0^T P(t) + P(t)\mathcal{A}_0 + \mathcal{A}_1^T P(t)\mathcal{A}_1, kh \leq t < (k+1)h\,, \\ P(kh^-) = &\ \mathcal{A}_d^T P(kh)\mathcal{A}_d \\ &-\mathcal{A}_d^T P(kh)\mathcal{B}_d(\mathcal{B}_d^T P(kh)\mathcal{B}_d)^{\dagger}\mathcal{B}_d^T P(kh)\mathcal{A}_d \\ &\ k = 0, 1, \ldots, N-1 \\ P(Nh^-) = &\ \mathcal{C}^T\mathcal{C}\,. \end{aligned} \tag{12}$$

Throughout the work $\mathcal{S}_d$ stands for the linear space of $d \times d$ symmetric matrices.

Proposition 1 *The problem with given final value (12) has a unique solution $\tilde{P} : [0, Nh] \to \mathcal{S}_{n+m}$ having the properties:*

(i) *$\tilde{P}(t)$ is right continuous and positive semidefinite for all $t \in [0, Nh]$;*

(ii)

$$[I_m - \mathcal{B}_d^T \tilde{P}(kh)\mathcal{B}_d(\mathcal{B}_d^T \tilde{P}(kh)\mathcal{B}_d)^{\dagger}\mathcal{B}_d^T \tilde{P}(kh)\mathcal{A}_d = 0\,, \tag{13}$$

for $0 \leq k \leq N-1$.

Proof: Let $\mathcal{L} : \mathcal{S}_{n+m} \to \mathcal{S}_{n+m}$ be the linear operator defined by

$$\mathcal{L}[X] = \mathcal{A}_0^T\, X + X\,\mathcal{A}_0 + \mathcal{A}_1^T\, X\,\mathcal{A}_1\,.$$

Let $e^{\mathcal{L}t}$ be defined by

$$e^{\mathcal{L}t}[X] = \sum_{j=0}^{\infty} \frac{t^j}{j\,!}\mathcal{L}^j[X]\,, \tag{14}$$

for all $t \geq 0, X \in \mathcal{S}_{n+m}$. applying Corollary 2.2.6 from [3] we deduce that

$$e^{\mathcal{L}t}[X] \geq 0\,, \quad t \geq 0, \text{ if } X \geq 0\,. \tag{15}$$

If $t \in [(N-1)h, Nh]$ we obtain from the first equation of (12) that the solution $\tilde{P}(.)$ is well defined and has the representation:

$$\tilde{P}(t) = e^{\mathcal{L}(Nh-t)}[\mathcal{C}^T\mathcal{C}]\,.$$

Employing (14) we may infer that $\tilde{P}(t) \geq 0$ for all $(N-1)h \leq t \leq Nh$. Let $\begin{pmatrix} \tilde{P}_{11}(t) & \tilde{P}_{12}(t) \\ \tilde{P}_{12}^T(t) & \tilde{P}_{22}(t) \end{pmatrix}$ be the partition of $\tilde{P}(t)$ such that $\tilde{P}_{11}(t) \in \mathcal{S}_n$ and $\tilde{P}_{22}(t) \in \mathcal{S}_m$. By direct calculations one obtains via (9) that

$$\begin{aligned} &\mathcal{A}_d^T\tilde{P}((N-1)h)\mathcal{A}_d - \mathcal{A}_d^T\tilde{P}((N-1)h)\mathcal{B}_d \\ &\times(\mathcal{B}_d^T\tilde{P}((N-1)h)\mathcal{B}_d)^\dagger\,\mathcal{B}_d^T\tilde{P}((N-1)h)\mathcal{A}_d \\ = &\begin{pmatrix} \tilde{P}_{11}((N-1)h) - \tilde{P}_{12}((N-1)h)\,(\tilde{P}_{22}((N-1)h))^\dagger\,\tilde{P}_{12}^T((N-1)h) & 0 \\ 0 & 0 \end{pmatrix} \end{aligned} \tag{16}$$

Invoking the implication (i) $\to$ (ii) from Lemma 1, in the case of the matrix $\tilde{P}((N-1)h)$ we obtain that

$$\tilde{P}_{11}((N-1)h) - \tilde{P}_{12}((N-1)h)\,(\tilde{P}_{22}((N-1)h))^\dagger\,\tilde{P}_{12}^T((N-1)h) \geq 0 \tag{17}$$

and $(I_m - \tilde{P}_{22}((N-1)h)(\tilde{P}_{22}((N-1)h))^\dagger)\,(\tilde{P}_{12}^T((N-1)h) = 0$.

The second equation of (12) written for $k = N-1$ together with (16) and (17) yield

$$\tilde{P}((N-1)h^-) \geq 0\,.$$

thus, we obtain via (15) that $\tilde{P}(t) = e^{\mathcal{L}((N-1)h-t)}[\tilde{P}((N-1)h^-)] \geq 0\,,\ \ t \in [(N-2)h, (N-1)h]$.

Let us assume that for a $k \leq N-1$ we obtained that $\tilde{P}((k+1)h^-) \geq 0$. Then, based on (15) we get

$$\tilde{P}(t) = e^{\mathcal{L}((k+1)h-t)}[\tilde{P}((k+1)h^-)] \geq 0\,, \tag{18}$$

$t \in [kh, (k+1)h]$. Further on, invoking (9) we obtain that

$$\begin{aligned} &\mathcal{A}_d^T\tilde{P}(kh)\mathcal{A}_d - \mathcal{A}_d^T\tilde{P}(kh)\mathcal{B}_d \\ &\times(\mathcal{B}_d^T\tilde{P}(kh)\mathcal{B}_d)^\dagger\,\mathcal{B}_d^T\tilde{P}(kh)\mathcal{A}_d \\ = &\begin{pmatrix} \tilde{P}_{11}(kh) - \tilde{P}_{12}(kh)\,(\tilde{P}_{22}(kh))^\dagger\,\tilde{P}_{12}^T(kh) & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{19}$$

On the other hand, the implication (i) $\to$ (ii) of Lemma 1 applied in the case of the matrix $\tilde{P}(kh) \geq 0$, yields:

$$\tilde{P}_{11}(kh) - \tilde{P}_{12}(kh)\,(\tilde{P}_{22}(kh))^{\dagger}\,\tilde{P}_{12}^{T}(kh) \geq 0 \tag{20}$$

and

$$(I_m - \tilde{P}_{22}(kh)\,(\tilde{P}_{22}(kh))^{\dagger}\,)\,\tilde{P}_{12}^{T}(kh) = 0\,. \tag{21}$$

The second equation of (12) together with (19) and (20) lead to $\tilde{P}(kh^-) \geq 0$. Thus, we may continue obtaining that (18) holds for every $k = 0, 1, \ldots, N-1$. So, we have shown that $\tilde{P}(t)$ is well defined and it is positive semidefinite for any $t \in [0, Nh]$.

In the same time, one obtains inductively, that (21) is true for any $k = 0, 1, \ldots, N-1$. By direct calculations one sees that (21) is just (13). Thus, the proof is complete.

The main result of this section is inclosed in the next theorem. In order to simplify the statement of this theorem we are introducing the notations: $\tilde{\mathbf{u}}_{K\varphi} = (\,\tilde{u}_{K\varphi}(0)\ \tilde{u}_{K\varphi}(1) \ldots \tilde{u}_{K\varphi}(N-1)$, where

$$\tilde{u}_{K\varphi}(k) = \mathbb{F}_K(k)\tilde{\xi}(kh) + [I_m - \mathcal{B}_d^T\tilde{P}(kh)\mathcal{B}_d\,(\mathcal{B}_d^T\tilde{P}(kh)\mathcal{B}_d)^{\dagger}]\,\varphi(k)\,, \tag{22}$$

where

$$\begin{aligned} \mathbb{F}_K(k) = \ & -(\mathcal{B}_d^T\tilde{P}(kh)\mathcal{B}_d)^{\dagger}\,\mathcal{B}_d^T\tilde{P}(kh)\mathcal{A}_d \\ & -[I_m - \mathcal{B}_d^T\tilde{P}(kh)\mathcal{B}_d\,(\mathcal{B}_d^T\tilde{P}(kh)\mathcal{B}_d)^{\dagger}]\,K(k)\,, \end{aligned} \tag{23}$$

$K(k) \in \mathbb{R}^{m\times(n+m)}, \varphi(k) \in \mathbb{R}^m$ are arbitrary and $\tilde{P}(kh)$ are the values of the solution $\tilde{P}(t)$ of the problem with given final value (12).

In (22), $\tilde{\xi}(kh)$ are the values of the solution with given initial values obtained when $\tilde{u}_{k\varphi}(h)$ is plugged in (8), i.e.

$$\begin{aligned} d\xi(t) = \ & \mathcal{A}_0\,\xi(t)\,dt + \mathcal{A}_1\,\xi(t)\,dw(t), \quad kh \leq t \leq (k+1)h, \\ \xi(kh^+) = \ & (\mathcal{A}_d + \mathcal{B}_d\mathbb{F}_K(k))\xi(kh) \\ & +\mathcal{B}_d\left[I_m - \mathcal{B}_d\,\tilde{P}(kh)\mathcal{B}_d\,(\mathcal{B}_d\,\tilde{P}(kh)\mathcal{B}_d)^{\dagger}\right]\varphi(k)\,, \end{aligned} \tag{24}$$

$k = 0, 1, \ldots, N-1, \quad \xi(0) = \xi_0\,.$

Theorem 1 *Given an initial state ξ_0, any control $\tilde{\mathbf{u}}_{K\varphi}$ of type (22)-(23) achieves the minimum of the cost function (10) with respect to the class of admissible controls $\mathcal{U}_{adm}^d$. The minimal value of the performance criteria is*

$$J_d(\tilde{\mathbf{u}}_{K\varphi}) = \mathbb{E}[\xi_0^T\,\tilde{P}(0^-)\,\xi_0]\,. \tag{25}$$

Proof: Applying the Itô formula to the functions $\xi^T\tilde{P}(t)\xi$ of the form $[s_k^1, s_k^2] \subset [kh, (k+1)h]$ and letting $s_k^1 \to kh$ and $s_k^2 \to (k+1)h$ we obtain via the first equation of (12) that

$$\begin{aligned} & \mathbb{E}[\xi_{\mathbf{u}}^T((k+1)h)\tilde{P}((k+1)h^-)\xi_{\mathbf{u}}((k+1)h)] - \mathbb{E}[\xi_{\mathbf{u}}^T(kh^+)\tilde{P}(kh)\xi_{\mathbf{u}}(kh^+)] \\ = \; & \mathbb{E}\left[\int_{kh}^{(k+1)h} \xi_{\mathbf{u}}^T(t)\left(\frac{d}{dt}\tilde{P}(t) + \mathcal{A}_0^T\tilde{P}(t) + \tilde{P}(t)\mathcal{A}_0 + \mathcal{A}_1^T\tilde{P}(t)\mathcal{A}_1\right)\xi_{\mathbf{u}}(t)\,dt\right] \end{aligned}$$

for all $k = 0, 1, \dots, N-1$ and any $\mathbf{u} \neq (u_0, u_1, \dots, u_{N-1}) \in \mathcal{U}_{adm}^d$.

Employing the second equation of (8) we further obtain

$$\begin{aligned} & \mathbb{E}[\xi_{\mathbf{u}}^T((k+1)h)\tilde{P}((k+1)h^-)\xi_{\mathbf{u}}((k+1)h)] \\ = \; & \mathbb{E}[\xi_{\mathbf{u}}^T(kh)\mathcal{A}_d^T\tilde{P}(kh)\mathcal{A}_d\xi_{\mathbf{u}}(kh)] + \mathbb{E}[\xi_{\mathbf{u}}^T(kh)\mathcal{A}_d^T\tilde{P}(kh)\mathcal{B}_d u_k \\ & + u_k^T\mathcal{B}_d^T\tilde{P}(kh)\mathcal{A}_d\xi_{\mathbf{u}}(kh) + u_k^T\mathcal{B}_d^T\tilde{P}(kh)\mathcal{B}_d u_k]\,. \end{aligned} \tag{26}$$

Let

$$\hat{u}_{K\varphi}(k) = [I_m - \mathcal{B}_d^T\tilde{P}(kh)\mathcal{B}_d\,(\mathcal{B}_d^T\tilde{P}(kh)\mathcal{B}_d)^\dagger][K(k)\xi_{\mathbf{u}}(kh) + \varphi(k)] \tag{27}$$

where $K(k) \in \mathbb{R}^{m\times(n+m)}, \varphi(k) \in \mathbb{R}^m$ are arbitrary but fixed.

From (13) and (27) we deduce that

$$\mathcal{A}_d^T\tilde{P}(kh)\mathcal{B}_d\,\hat{u}_{K\varphi}(k) = 0\,. \tag{28}$$

On the other hand, invoking some properties of the pseudo-inverse of a positive semidefinite matrix we may infer that

$$\mathcal{B}_d^T\tilde{P}(kh)\mathcal{B}_d\,\hat{u}_{K\varphi}(k) = 0\,, \quad 0 \le k \le N-1. \tag{29}$$

Further on, (26), (28) and (29) yield:

$$\begin{aligned} & \mathbb{E}[\xi_{\mathbf{u}}^T((k+1)h)\tilde{P}((k+1)h^-)\xi_{\mathbf{u}}((k+1)h)] \\ = \; & \mathbb{E}[\xi_{\mathbf{u}}^T(kh)\mathcal{A}_d^T\tilde{P}(kh)\mathcal{A}_d\xi_{\mathbf{u}}(kh)] + \mathbb{E}[\xi_{\mathbf{u}}^T(kh)\mathcal{A}_d^T\tilde{P}(kh)\mathcal{B}_d(u_k + \hat{u}_{K\varphi}(k)) \\ & + (u_k + \hat{u}_{K\varphi}(k))^T\mathcal{B}_d^T\tilde{P}(kh)\mathcal{A}_d\xi_{\mathbf{u}}(kh) \\ & + (u_k + \hat{u}_{K\varphi}(k))^T\mathcal{B}_d^T\tilde{P}(kh)\mathcal{B}_d(u_k + \hat{u}_{K\varphi}(k))]\,. \end{aligned}$$

Using the second equation in (12) we get:

$$
\begin{aligned}
& \mathbb{E}[\xi_{\mathbf{u}}^T((k+1)h)\tilde{P}((k+1)h^-)\xi_{\mathbf{u}}((k+1)h)] \\
& -\mathbb{E}[\xi_{\mathbf{u}}^T(kh)\tilde{P}(kh^-)\xi_{\mathbf{u}}(kh)] \\
= \; & \mathbb{E}[\xi_{\mathbf{u}}^T(kh)\mathcal{A}_d^T\tilde{P}(kh)\mathcal{B}_d(\mathcal{B}_d^T\tilde{P}(kh)\mathcal{B}_d)^{\dagger}\mathcal{B}_d^T\tilde{P}(kh)\mathcal{A}_d\xi_{\mathbf{u}}(kh) \\
& +\xi_{\mathbf{u}}^T(kh)\mathcal{A}_d^T\tilde{P}(kh)\mathcal{B}_d(u_k+\hat{u}_{K\varphi}(k)) \\
& +(u_k+\hat{u}_{K\varphi}(k))^T\,\mathcal{B}_d^T\tilde{P}(kh)\mathcal{A}_d\xi_{\mathbf{u}}(kh) \\
& +(u_k+\hat{u}_{K\varphi}(k))^T\,\mathcal{B}_d^T\tilde{P}(kh)\mathcal{B}_d(u_k+\hat{u}_{K\varphi}(k))] \\
= \; & \mathbb{E}[(u_b+(\mathcal{B}_d^T\tilde{P}(kh)\mathcal{B}_d)^{\dagger}\,\mathcal{B}_d^T\tilde{P}(kh)\mathcal{A}_d\xi_{\mathbf{u}}(kh)+\hat{u}_{K\varphi}(k))^T \\
& \times\mathcal{B}_d^T\tilde{P}(kh)\mathcal{B}_d \\
& \times(u_b+(\mathcal{B}_d^T\tilde{P}(kh)\mathcal{B}_d)^{\dagger}\,\mathcal{B}_d^T\tilde{P}(kh)\mathcal{A}_d\xi_{\mathbf{u}}(kh)+\hat{u}_{K\varphi}(k))]\,.
\end{aligned}
\tag{30}
$$

To obtain the last equality we have used (13) together with the equality

$$
(\mathcal{B}_d^T\tilde{P}(kh)\mathcal{B}_d)^{\dagger}\,\mathcal{B}_d^T\tilde{P}(kh)\mathcal{B}_d\,(\mathcal{B}_d^T\tilde{P}(kh)\mathcal{B}_d)^{\dagger} \;=\; (\mathcal{B}_d^T\tilde{P}(kh)\mathcal{B}_d)^{\dagger}
$$

which is obtained directly from the definition of the pseudo-unverse. Summing up in (30) for $k=0$ to $N-1$ and taking into account that $\tilde{P}(Nh^-)=\mathcal{C}^T\,\mathcal{C}$, we obtain via (10) that

$$
\begin{aligned}
J_d(\mathbf{u}) = \; & \mathbb{E}[\xi_{\mathbf{u}}^T(0)\tilde{P}(0^-)\xi_{\mathbf{u}}(0)] \\
& +\textstyle\sum_{k=0}^{N-1}\mathbb{E}[(u_b+(\mathcal{B}_d^T\tilde{P}(kh)\mathcal{B}_d)^{\dagger}\,\mathcal{B}_d^T\tilde{P}(kh)\mathcal{A}_d\xi_{\mathbf{u}}(kh)+\hat{u}_{K\varphi}(k))^T \\
& \times\mathcal{B}_d^T\tilde{P}(kh)\mathcal{B}_d \\
& \times(u_b+(\mathcal{B}_d^T\tilde{P}(kh)\mathcal{B}_d)^{\dagger}\,\mathcal{B}_d^T\tilde{P}(kh)\mathcal{A}_d\xi_{\mathbf{u}}(kh)+\hat{u}_{K\varphi}(k))]\,.
\end{aligned}
\tag{31}
$$

for any $\mathbf{u}=(u_0,\,u_1,\ldots,u_{N-1})\in\mathcal{U}_{adm}^d$.

From (22), (23) and (27) together the uniqueness of the solution of the problem(24) one sees that

$$
(\mathcal{B}_d^T\tilde{P}(kh)\mathcal{B}_d)^{\dagger}\;+\mathcal{B}_d^T\tilde{P}(kh)\mathcal{A}_d\xi_{\mathbf{u}}(kh)-\hat{u}_{K\varphi}(k)
$$

reduces to $\tilde{u}_{K\varphi}(k)$ when in (31) $\mathbf{u}$ is replaced by $\tilde{\mathbf{u}}_{K\varphi}$. Thus, we obtain

$$
J_d(\tilde{\mathbf{u}}_{K\varphi})=\mathbb{E}[\xi_0^T\tilde{P}(0^-)\xi_0]\le J_d(\mathbf{u})\,,
$$

for all $\mathbf{u}\in\mathcal{U}_{adm}^d$. Thus, the proof ends.

Remark 1 *From Theorem 1 one sees that the optimal control problem described in (11) has an infinite number of optimal controls.*

3.1 Several procedural issues

To compute any of the optimal controls described in (22)-(23) we need the values $\tilde{P}(kh), k = 0, 1, \ldots, N-1$ of the solution $\tilde{P}(.)$ of the problem with given fiscal values (12). To compute the values of $\tilde{P}(kh)$ let us notice that (18) written for $t = kh$ yields

$$\tilde{P}(kh) = e^{\mathcal{L}h}[\tilde{P}((k+1)h^-)]\,. \tag{32}$$

Using the second equation of (12) we obtain via (32) the following discrete-time backward equation:

$$\begin{aligned}\tilde{P}(kh^-) &= \mathcal{A}_d^T e^{\mathcal{L}h}[\tilde{P}((k+1)h^-)]\mathcal{A}_d - \mathcal{A}_d^T e^{\mathcal{L}h}[\tilde{P}((k+1)h^-)]\mathcal{B}_d \\ &\times(\mathcal{B}_d^T e^{\mathcal{L}h}[\tilde{P}((k+1)h^-)]\mathcal{B}_d)^\dagger \mathcal{B}_d^T e^{\mathcal{L}h}[\tilde{P}((k+1)h^-)]\mathcal{A}_d\,.\end{aligned} \tag{33}$$

Hence, the sequence of the values $\tilde{P}(kh^-), k = N-1, N-2, \ldots, 0$ one obtains recursively using (33) together with $\tilde{P}(kh^-) = \mathcal{C}^T\mathcal{C}$. Finally, the values of $\tilde{P}(kh)$ involved in (22)-(23) are obtained from (32). The values of $e^{\mathcal{L}h}[.]$ involved in (32) and (33) can be computed by truncation of (14), namely,

$$e^{\mathcal{L}h}[X] = \sum_{j=0}^{p} \frac{h^j}{j\,!}\mathcal{L}^j[X]\,. \tag{34}$$

where $p \geq 1$ is sufficiently large such that

$$\lambda_{max}\{\mathcal{L}^{p+1}[X]\}\,\frac{h^{p+1}}{(p+1)!} < tol\,,$$

where tol is a small positive number. Let us recall that $\mathcal{L}^j[X]$ are obtained from

$$\mathcal{L}^j[X] = \mathcal{A}_0^T\mathcal{L}^{j-1}[X] + \mathcal{L}^j[X]\mathcal{A}_0 + \mathcal{A}_1^T\mathcal{L}^{j-1}[X]\mathcal{A}_1\,, \tag{35}$$

with $\mathcal{L}^0[X] = X$.

Remark 2 *In the special case when in (1) we have $A_1 = 0, B_1 = 0$, then $e^{\mathcal{L}h}[X] = e^{\mathcal{A}_0^T h}\,X\,e^{\mathcal{A}_0 h}$. In this case (33) becomes*

$$\begin{aligned}&\tilde{P}(kh^-) = \mathbb{A}^T\tilde{P}((k+1)h^-)\mathbb{A} \\ &-\mathbb{A}^T\tilde{P}((k+1)h^-)\mathbb{B}\,(\mathbb{B}^T\tilde{P}((k+1)h^-)\mathbb{B})^\dagger\mathbb{B}^T\tilde{P}((k+1)h^-)]\mathbb{A}\,,\end{aligned}$$

for $k = N-1, N-2, \ldots, 0$, $\tilde{P}(Nh^-) = \mathcal{C}^T\mathcal{C}$, where

$$\mathbb{A} = e^{\mathcal{A}_0 h}\mathcal{A}_d\,, \qquad \mathbb{B} = e^{\mathcal{A}_0 h}\mathcal{B}_d\,.$$

The values $\tilde{P}(kh)$ involved in the computation of the gain matrices of the optimal controls are obtained now from $\tilde{P}(kh) = (e^{\mathcal{A}_0 h})^T \tilde{P}((k+1)h^-) e^{\mathcal{A}_0 h}$. A reliable approximation of $e^{\mathcal{A}_0 h}$ is obtained computing $\sum_{j=0}^{p} \frac{h^j}{j!}\mathcal{A}_0^j$, where $p \geq 1$ is sufficiently large.

Let us consider again the partition : $\tilde{P}(t) = \begin{pmatrix} \tilde{P}_{11}(t) & \tilde{P}_{12}(t) \\ \tilde{P}_{12}^T(t) & \tilde{P}_{22}(t) \end{pmatrix}$ of the solution of the problem with given final value (12) such that $\tilde{P}_{11}(t) \in \mathcal{S}_n$ and $\tilde{P}_{22}(t) \in \mathcal{S}_m$. By direct calculations one obtains the following partition of (12):

$$\begin{aligned}
-\tfrac{d}{dt}\tilde{P}_{11}(t) &= A_0^T\tilde{P}_{11}(t) + \tilde{P}_{11}(t)A_0 + A_1\tilde{P}_{11}(t)A_1 \\
-\tfrac{d}{dt}\tilde{P}_{12}(t) &= A_0^T\tilde{P}_{12}(t) + \tilde{P}_{11}(t)B_0 + A_1\tilde{P}_{11}(t)B_1 \\
-\tfrac{d}{dt}\tilde{P}_{22}(t) &= B_0^T\tilde{P}_{12}(t) + \tilde{P}_{12}^T(t)B_0 + B_1^T\tilde{P}_{11}(t)B_1 \\
& kh \leq t < (k+1)h\,, \\
\tilde{P}_{11}(kh^-) &= \tilde{P}_{11}(kh) - \tilde{P}_{12}(kh)(\tilde{P}_{22}(kh))^\dagger\tilde{P}_{12}^T(kh) \\
\tilde{P}_{12}(kh^-) = 0\,,\ \tilde{P}_{22}(kh^-) &= 0\,, \quad k = 0, 1, \ldots, N-1\,, \\
\tilde{P}_{11}(Nh^-) = \mathcal{C}^T\mathcal{C}\,, \tilde{P}_{12}(Nh^-) &= 0\,,\ \tilde{P}_{22}(Nh^-) = 0\,.
\end{aligned} \tag{36}$$

The integration of the equation (36) may be used as an alternative procedure to the one described by (32)-(35) to compute $\tilde{P}(kh), 0 \leq k \leq N-1$.

4 The solution of the optimization problem by piecewise constant controls

Let us consider the controls $\tilde{\mathbf{u}}_{K\varphi} = (\,\tilde{u}_{K\varphi}(0),\ \tilde{u}_{K\varphi}(1), \ldots, \tilde{u}_{K\varphi}(N-1)$ of type (22)-(23) in the special case $K(k) = (\hat{K}(k) = 0)$, where $\hat{K}(k) \in \mathbb{R}^{m\times n}$ is an arbitrary matrix. By direct calculation, employing the structure given in (9) for the matrices $\mathcal{A}_d$ and $\mathcal{B}_d$, we deduce that in the special case we have

$$\tilde{u}_{K\varphi}(k) = \Psi_{\hat{K}}(k)\tilde{x}(kh) + [I_m - \tilde{P}_{22}(kh)\,\tilde{P}_{22}^\dagger(kh)]\varphi(k)\,, \tag{37}$$

where

$$\Psi_{\hat{K}}(k) = -\tilde{P}_{22}^\dagger(kh)\,\tilde{P}_{12}^T(kh) + [I_m - \tilde{P}_{22}(kh)\,\tilde{P}_{22}^\dagger(kh)]\hat{K}(k)\,, \tag{38}$$

and $(\tilde{P}_{22}(kh)\,, \tilde{P}_{12}(kh)) \in \mathbb{R}^{m\times m} \times \mathbb{R}^{n\times m}$ being the block component of $\tilde{P}(kh)$. In (37), $\tilde{x}(kh)$ are the values at the instance time kh of the first

n-components of the solution $\tilde{\xi}(kh)$ of the problem with given initial values (24).

Since the controls (37)-(38) depend upon $\hat{K}(k) \in \mathbb{R}^{m \times n}$ and $\varphi(k) \in \mathbb{R}^m$ we shall write in the following, $\tilde{u}_{\hat{K}\varphi}(k)$ and $\tilde{\mathbf{u}}_{\hat{K}\varphi}$ instead of $\tilde{u}_{K\varphi}(k)$ and $\tilde{\mathbf{u}}_{K\varphi}$ all the time when we refer to (37) and (38).

Consider now an arbitrary $u(t) \in \mathcal{U}_{adm}$. We set $\mathbf{u} = (u_0, u_1, \ldots, u_{N-1})$, where u_k are the values involved in the definition of $u(t)$ via (4). It is obvious that $\mathbf{u} \in \mathcal{U}^d_{adm}$. Conversely, if $\mathbf{u} = (u_0, u_1, \ldots, u_{N-1})$ is an element of the set $\mathcal{U}^d_{adm}$, we can associate an unique piecewise constant admissible control (via (4)). In this way, we have shown that there exists an one to one correspondence between the classes of admissible controls $\mathcal{U}_{adm}$ and $\mathcal{U}^d_{adm}$. Now we are in position to state and proof the main result of this work.

Theorem 2 *Given an admissible initial state x_0, the minimum of the cost functional (5) in the class of admissible controls $\mathcal{U}_{adm}$ is achieved by the piecewise constant controls*

$$u_{opt}(t) = \Phi_{\hat{K}}(k)\tilde{x}(kh) + [I_m - \tilde{P}_{22}(kh)\, \tilde{P}^{\dagger}_{22}(kh)]\varphi(k)\,, \tag{39}$$

$kh \leq t < (k+1)h\,,\ k = 0, 1, \ldots, N-1$, where $\Phi_{\hat{K}}(k)$ is described in (38), and $\tilde{x}(kh)$ being the solution of the closed-loop system obtained plugging (39) in (1), $\hat{K} \in \mathbb{R}^{m \times n}$ and $\varphi(k) \in \mathbb{R}^m$ being arbitrary. The minimal value of the performance criterion (5) is given by

$$J(u_{opt}) = Tr[\tilde{P}_{11}(0^-)\, X_0]\,. \tag{40}$$

Proof: Let $u(.) \in \mathcal{U}_{adm}$ be arbitrary and let $\mathbf{u} = (u_0, u_1, \ldots, u_{N-1})$, where u_k are the constant values of $u(.)$ Invoking (4), (5), (7), (8), (10) and Theorem 1, we obtain

$$J(u) = J_d(\mathbf{u}) \geq J_d(\tilde{\mathbf{u}}_{\hat{K}\varphi}) = J(u_{opt})\,.$$

This confirms the optimality of the controls of type (39). The optimal value of the performance (40) one obtains from (25) because $\tilde{P}_{12}(0^-) = 0$, and $\tilde{P}_{22}(0^-) = 0$. The proof ends.

Remark 3 *a) From Theorem 2 one sees that the optimal control problem described by the cost functional (5) in the class of piecewise constant controls of type (4) has an infinite number of optimal controls.*

The family of the optimal controls is parameterized by the sequences $\{\hat{K}(k)\}_{0 \leq k \leq (n-1)} \subset \mathbb{R}^{m \times n}, \{\varphi(k)\}_{0 \leq k \leq (n-1)} \subset \mathbb{R}^m$.

In the special case $\hat{K}(k) = 0, \varphi(k) = 0, 0 \leq k \leq N-1$ *(39) reduces to*

$$u_{opt}(t) = \tilde{F}(k)\tilde{x}(kh)\,, kh < t \leq (k+1)h\,, \tag{41}$$

$k = 0, 1, \ldots, 2N-1$, *where*

$$\tilde{F}(k) = -\tilde{P}_{22}^{\dagger}(kh)\tilde{P}_{12}^{T}(kh)\,, \tag{42}$$

and $x(kh)$ *are the values at instant time* kh *of the solution of the problem with given initial values:*

$$dx(t) = [A_0 x(t) + B_0\tilde{F}(k)x(kh)]dt + [A_1 x(t) + B_1\tilde{F}(k)x(kh)]dw(t) \tag{43}$$

for $kh < t \leq (k+1)h$. $k = 0, 1, \ldots, N-1, x(0) = x_0$.

b) For implementation of an optimal control (37)- (38) or (41)- (42), respectively, we need to measure the state $x(kh)$ *at discrete time instances* $t_k = kh, 0 \leq k \leq N-1$. *For these reasons* $h > 0$ *is named the sampling period.*

5 The dependence of the optimal performance with respect to the sampling period

The decreasing of the length of the sampling period, could provide an improving of the minimal value achieved by the performance index (5).

In order to illustrate this fact let us assume that in (4) h is replaced by $h/2$. Hence, the admissible controls are of the form:

$$u(t) = u_k\,, \quad k\frac{h}{2} \leq t \leq (k+1)\frac{h}{2}, \; k = 0, 1, \ldots, 2N-1\,. \tag{44}$$

The problem with given final values (12) is replaced by

$$\begin{aligned}
-\dot{P}(t) = \;& \mathcal{A}_0^T P(t) + P(t)\mathcal{A}_0 + \mathcal{A}_1^T P(t)\mathcal{A}_1, k\tfrac{h}{2} \leq t < (k+1)\tfrac{h}{2}\,,\\
P(k\tfrac{h}{2}^{-}) = \;& \mathcal{A}_d^T P(k\tfrac{h}{2})\mathcal{A}_d\\
& -\mathcal{A}_d^T P(k\tfrac{h}{2})\mathcal{B}_d(\mathcal{B}_d^T P(k\tfrac{h}{2})\mathcal{B}_d)^{\dagger}\mathcal{B}_d^T P(k\tfrac{h}{2})\mathcal{A}_d\\
& k = 0, 1, \ldots, N-1\\
P(2N\tfrac{h}{2}) = \;& \mathcal{C}^T\mathcal{C}\,.
\end{aligned} \tag{45}$$

Reasoning as in the proof of Proposition 1 one can show that the problem with given final values (45) has a unique solution $\hat{P}(.)$ defined on the whole interval [0, Nh], which is right continuous and positive semidefinite in each

$t \in [0, Nh]$. Applying Theorem 1 in the case when the length of the sampling period is $\frac{h}{2}$ instead of h, we obtain that the minimal value of the cost (5) in the class of admissible controls of type (44) is achieved by the control

$$\hat{u}_{opt}(t) = \hat{F}(k)\hat{x}(k\frac{h}{2}), \quad k\frac{h}{2} \leq t \leq (k+1)\frac{h}{2}, \; k = 0, 1, \ldots, 2N-1, \tag{46}$$

where

$$\hat{F}(k) = -\hat{P}_{22}^{\dagger}(k\frac{h}{2})\hat{P}_{12}^{T}(k\frac{h}{2}), \tag{47}$$

where$(\hat{P}_{12}(k\frac{h}{2}), \hat{P}_{22}(k\frac{h}{2})) \in \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times m}$ being the block components of $\hat{P}(k\frac{h}{2})$. The value of the performance (5) achieved by the control (46) is given by

$$J(\hat{u}_{opt}) = Tr[\hat{P}_{11}(0^-)X_0].$$

Now, we have

Proposition 2 *If $\tilde{P}(.)$ and $\hat{P}(.)$ are the unique solutions of (12) and (45), respectively, then we have $0 \leq \hat{P}(t) \leq \tilde{P}(t)$ for all $t \in [0, Nh]$ and $0 \leq \hat{P}(0^-) \leq \tilde{P}(0^-)$.*

The proof may be done using the positivity properties of the operator $e^{\mathcal{L}t}[.]$ together with Lemma 1 from above and Corollary 4.5 from [4].

Corollary 1 *We have $J(\hat{u}_{opt}) \leq J(u_{opt})$.*

So, decreasing the length of the sampling period one obtains an improving of the value of the optimal performance. However, the implementation of the control (46)-(47) needs a greater number of measurements.

6 Numerical experiments

We execute numerical simulation to compute $u_{opt}(t)$ via (41).

Example 1. A deterministic case. Consider again (43):

$$\begin{aligned}&dx(t) = [A_0x(t) + B_0\tilde{F}(k)x(kh)]dt + [A_1x(t) + B_1\tilde{F}(k)x(kh)]dw(t),\\&kh < t \leq (k+1)h.\; k = 0, 1, \ldots, N-1, x(0) = x_0\end{aligned}$$

in the special case of $A_1 = 0, B_1 = 0$. One may compute the values of the solutions $\tilde{x}(kh), k = 0, 1, \ldots, N$ as follows. To this end, the obtained version of (43) is

$$dx(t) = [A_0x(t) + B_0\tilde{F}(k)x(kh)]dt.$$

It is rewritten as:

$$x((k+1)h) = [e^{A_0 h} + \int_0^h e^{A_0 s} ds B_0 \tilde{F}(k)]x(kh)$$

$k = 0, 1, \ldots, N-1$, where

$$e^{A_0 h} = I_n + A_0 h + A_0^2 h^2/2 + \ldots$$

and

$$\int_0^h e^{A_0 s} ds = hI_n + h^2/2A_0 + h^3/6A_0^2 + \ldots$$

We take $A_0 = np.matrix([[-1.5, 0.17], [0.07, -1.42]])$, and $C = np.matrix$ $([[0.5, 0.004], [0.01, 0.42]])$ and $h = 0.1$. We compute

$$e^{A_0 h} \sim \begin{pmatrix} 0.8608 & 0.0147 \\ 0.0060 & 0.8677 \end{pmatrix}, \quad \int_0^h e^{A_0 s} ds \sim \begin{pmatrix} 0.0861 & 0.0015 \\ 0.0006 & 0.0868 \end{pmatrix}.$$

Further on, we take $B_0 = np.matrix([[1.5, 0.7], [0.3, 0.4]])$, and $x_0 = np.matrix([[0.02], [0.035]])$, and compute $\tilde{F}(k), k = 0, 1, \ldots, N-1$ via (42). Here, the notation "np.matrix" is used to define the corresponding matrices in Python. Experiments in this example are executed in Python.

We compute for $h = 0.1$ and $N = 10 (Nh = \tau = 1)$:

$$x(0) = \begin{pmatrix} 0.02 \\ 0.035 \end{pmatrix}, \; x(h) = \begin{pmatrix} 0.124 \\ 0.265 \end{pmatrix}, \; x(2h) = \begin{pmatrix} 0.009 \\ 0.203 \end{pmatrix}, \; \ldots$$

$$x(8h) = \begin{pmatrix} -0.004 \\ 0.043 \end{pmatrix}, \; x(9h) = \begin{pmatrix} -0.003 \\ 0.033 \end{pmatrix}, \; x(10h) = \begin{pmatrix} -0.002 \\ 0.025 \end{pmatrix}.$$

In addition, we compute for $h = 0.05$ and $N = 20 (Nh = \tau = 1)$

$$x(0) = \begin{pmatrix} 0.02 \\ 0.035 \end{pmatrix}, \; x(h) = \begin{pmatrix} 0.2685 \\ 0.3017 \end{pmatrix}, \; x(2h) = \begin{pmatrix} 0.0819 \\ 0.2622 \end{pmatrix}, \; \ldots$$

$$x(18h) = \begin{pmatrix} -0.0029 \\ 0.0306 \end{pmatrix}, \; x(19h) = \begin{pmatrix} -0.0026 \\ 0.0268 \end{pmatrix}, \; x(20h) = \begin{pmatrix} -0.0023 \\ 0.0234 \end{pmatrix}.$$

Moreover, we apply (41) to find $u_{opt}(t)$, $kh < t \leq (k+1)h$, $k = 0, 1, \ldots, N-1$.

Example 2. A stochastic case with a zero matrix A_1.

The matrices for system (1) are
A_0 =[1.5, 0.17; 0.07, -1.42]; ,
A_1=[0.0, 0; 0, 0.0]; ,
B_0 =[1.5, 0.7; 0.3, 0.4]; ,
B_1=[0.2,0.04; 0.02, 0.01]; , and
C=[0.5, 0.004; 0.01, 0.42]; .

Note that $n = 2$ and $x(t) = (x_1(t)\ x_2(t))^T$. Here we use the Matlab description.

In this example we approximate the deterministic part as in Example 1. For the stochastic part we use Matlab procedures *sde* and *simByEuler*.

Thus we obtain the values of $x(kh), k = 0, 1, \ldots, N, x(0) = x_0[0.04; 0.02]$ and $h = 0.1, (Nh = \tau = 1)$:

$$x(0) = \begin{pmatrix} 0.04 \\ 0.02 \end{pmatrix},\ x(h) = \begin{pmatrix} 0.0339 \\ 0.0141 \end{pmatrix},\ x(2h) = \begin{pmatrix} 0.0286 \\ 0.0081 \end{pmatrix},\ \ldots$$

$$x(8h) = \begin{pmatrix} 0.0057 \\ -0.0514 \end{pmatrix},\ x(9h) = \begin{pmatrix} 0.0026 \\ -0.0715 \end{pmatrix},\ x(10h) = \begin{pmatrix} -0.0003 \\ -0.0978 \end{pmatrix}.$$

Example 3. A stochastic case.

The matrices for system (1) are
A_0=[7.5, 9.7; -0.35, 7.0]; ,
A_1=[3.5, 7.0; 1.4, 4.5]; ,
B_0 =[1.5, 0.7; 0.3, 0.4]; ,
B_1=[0.2,0.04; 0.02, 0.01]; , and
C=[0.5, 0.004; 0.01, 0.42]; .

We take $x(0) = x_0 = [0.04; 0.02]$ and $h = 0.1, (Nh = \tau = 1)$.

We approximate the values of $x(kh), k = 0, 1, \ldots, N$, applying the Matlab procedures *sde* and *simByEuler*.

$$x(0) = \begin{pmatrix} 0.04 \\ 0.02 \end{pmatrix},\ x(h) = \begin{pmatrix} -0.2469 \\ 0.1136 \end{pmatrix},\ x(2h) = \begin{pmatrix} -0.5902 \\ 0.2240 \end{pmatrix},\ \ldots$$

$$x(8h) = \begin{pmatrix} -3.4765 \\ 1.1144 \end{pmatrix},\ x(9h) = \begin{pmatrix} -3.8312 \\ 1.2231 \end{pmatrix},\ x(10h) = \begin{pmatrix} -3.7486 \\ -1.3059 \end{pmatrix}.$$

7 Conclusions

In this paper, the problem of minimization of the mean square of the value of the final instance time of a signal generated by a linear controlled system subject to multiplicative white noise perturbations was solved.

The class of admissible controls consists of all piecewise constant stochastic processes that are adapted to the filtration generated by the Brownian motion affecting the considered controlled system. We have shown that the considered optimal control problem is well possed, but it has an infinite number of optimal controls.

Explicit formulae of the optimal controls were obtained. These formulae are in the affine state feedback form. The gain matrices of the optimal controls are computed based on the unique solution of a backward matrix differential equation with finite jumps.

It remains as an open problem the minimization of the variance of the final value of an output of a linear stochastic system.

Another problem of interest could be the problem of minimization of the mean square of the deviation of the final value of the output of a given system from a target value.

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